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MATRIX TRANSFORMATIONS AND TOEPLITZ DUALS OF GENERALIZED ORLICZ HILBERT SEQUENCE SPACES

Kuldip Raj, Ayhan Esi and Charu Sharma

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Abstract In this article, we study some sequence spaces originated with an infinite Hilbert matrix and a Musielak-Orlicz function. Some topological and algebraic properties of new formed spaces are discuss. The α-, β- and γ- duals are also determine. Finally, an attempt is made to characterize some matrix transformations between these spaces. Hilbert matrix is used for authentication, confidentiality to study cryptographic methods and frequently used in the security systems also.

1 Introduction and Preliminaries

The Hilbert matrix has played a prominent role in the structure theory of several branches of mathematics. Indeed, it serves as one of the most vivid examples for many unusual aspects in operator theory (see [4, 9]). In linear algebra, a Hilbert matrix introduced by Hilbert in (1894), is a square matrix with entries being the unit fractions $H = (h_{ij}) = \frac{1}{i+j-1}$ for each $i, j \in \mathbb{N}$. Let us consider the matrix H as follows:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & ... \\ 1/2 & 1/3 & 1/4 & ... & \\ 1/3 & 1/4 & ... & & \\ 1/4 & ... & & & \\ . & . & & & \\ . & . & & & \\ . & . & & & \end{pmatrix}$$

which is called an infinite Hilbert matrix. An inequality of Hilbert [9] asserts that the matrix H determines a bounded linear operator on Hilbert space of square summable complex sequences. For Example, 5×5 Hilbert matrix is as follows:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{pmatrix}$$

The Hilbert matrix can be regarded as derived from the integral

$$H = (h_{ij}) = \int_0^1 x^{i+j-2} dx.$$

It arises in the least squares approximation of arbitrary functions by polynomials.

Let w be the space of all real or complex sequences. We shall write c, c_0 and l_∞ for the sequence spaces of all convergent, null and bounded sequences, respectively. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively. Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex entries, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{A_n(x)\}$ is in Y, where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

converges for each $n \in \mathbb{N}$. By (X, Y) we denote the class of all matrices A such that $A : X \to Y$.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

which is also a sequence space. A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. For the triangle matrices A, B and a sequence x, $A(Bx) = (AB)x$ holds. We remark that a triangle matrix A uniquely has an inverse $A^{-1} = B$ and the matrix B is also a triangle.

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. For example, the space $l_p (1 \leq p < \infty)$ is a BK-space with $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$ and c_0, c and l_∞ are BK-spaces with $\|x\|_\infty = \sup_k |x_k|$.

A sequence (b_n) in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e.,

$$\lim_m \left\| x - \sum_{n=0}^{m} \alpha_n b_n \right\| = 0.$$

The notion of difference operator in the sequence spaces was firstly introduced by Kızmaz [12]. The idea of difference sequence spaces of Kızmaz was further generalized by Et and Çolak [8]. Later concept have been studied by Bektaş et al. [6] and Et et al. [7]. Now, the difference matrix $\Delta = \delta_{nk}$ defined by

$$\delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \leq k \leq n) \\ 0, & (0 < n-1 \text{ or } n > k). \end{cases}$$

The difference operator order m is defined by $\Delta^m : w \to w$, $(\Delta^1 x)_k = (x_k - x_{k-1})$ and $\Delta^m x = (\Delta^1 x)_k \circ (\Delta^{m-1} x)_k$ for $m \geq 2$.

The triangle matrix $\Delta^{(m)} = \delta_{nk}^{(m)}$ defined by

$$\delta_{nk}^{(m)} = \begin{cases} (-1)^{n-k} \begin{pmatrix} m \\ n-k \end{pmatrix}, & (\max\{0, n-m\} \leq k \leq n) \\ 0, & (0 \leq k < \max\{0, n-m\} \text{ or } n > k) \end{cases}$$

for all $k, n \in \mathbb{N}$ and for any fixed $m \in \mathbb{N}$.
The infinite Hilbert matrix is defined by $H = (h_{ij}) = \left(\frac{1}{i+j-1}\right)$ for each $i, j \in \mathbb{N}$. The inverse of Hilbert matrix is defined by

$$H^{-1} = \left(h_{ij}^{-1}\right) = (-1)^{i+j}(i+j-1) \begin{pmatrix} n+i-1 \\ n-j \end{pmatrix} \begin{pmatrix} n+j-1 \\ n-i \end{pmatrix} \begin{pmatrix} i+j-1 \\ i-1 \end{pmatrix}^2$$

for all $i, j, n \in \mathbb{N}$. In [18], Polat and [11] Kirisci and Polat have defined some new sequence spaces by using Hilbert matrix. Let h_c, h_0 and h_∞ be convergent Hilbert, null convergent Hilbert and bounded Hilbert sequence spaces, respectively.

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to construct the following sequence space

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

The space ℓ_M with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}$$

becomes a Banach space which is called an Orlicz sequence space. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [15, 14]). For more details about sequence spaces see ([16, 20, 19]) and references therein.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

$(P1)$ $p(x) \geq 0$ for all $x \in X$,

$(P2)$ $p(-x) = p(x)$ for all $x \in X$,

$(P3)$ $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,

$(P4)$ if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [22, Theorem 10.4.2, pp. 183]).

Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of positive real numbers and $H = (h_{ij})$ be an infinite Hilbert matrix. In the present paper we defined the following sequence spaces:

$h_c\big(\Delta^{(m)}, \mathcal{M}, u, p\big) =$

$$\Big\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} \text{ exists, for some } \rho > 0\Big\},$$

$h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big) =$

$$\Big\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} = 0, \text{ for some } \rho > 0\Big\}$$

and

$h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big) =$

$$\Big\{x = (x_k) \in w : \sup_{n} \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If $M_k(x) = x$, for all $k \in \mathbb{N}$. Then above sequence spaces reduces to $h_c\big(\Delta^{(m)}, u, p\big)$, $h_0\big(\Delta^{(m)}, u, p\big)$ and $h_\infty\big(\Delta^{(m)}, u, p\big)$.

By taking $(p_k) = 1$ and $(u_k) = 1$, for all $k \in \mathbb{N}$, then we get the sequence spaces $h_c\big(\Delta^{(m)}, \mathcal{M}\big)$, $h_0\big(\Delta^{(m)}, \mathcal{M}\big)$ and $h_\infty\big(\Delta^{(m)}, \mathcal{M}\big)$.

We define the sequence $y = (y_n)$ which will be frequently used, as the $H\Delta^{(m)}$-transform of a sequence as below:

$$\begin{aligned}(y_n) &= \big(H\Delta^{(m)} x\big)^{(\mathcal{M},u,p)} \\ &= \sum_{k=1}^{n} \frac{1}{n+k-1}\left[M_k\left(\frac{|u_k \sum_{i=k}^{n} (-1)^{i-k}\binom{m}{i-k} x_k|}{\rho}\right)\right]^{p_k}\end{aligned} \tag{1.3}$$

for each $k, m, n \in \mathbb{N}$.

The following inequality will be used throughout the paper. If $0 < p_k \leq \sup p_k = R$, $K = \max(1, 2^{R-1})$, then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \quad (1.4)$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.
The main purpose of this paper is to study and introduced some sequence spaces with Hilbert matrix and a Musielak-Orlicz function. We shall study some topological and algebraic properties of these sequence spaces. We shall determine the α-, β- and γ- duals of the spaces. Finally, we also made an attempt to characterize some matrix transformations between these spaces.

2 Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. Then $h_c\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$, $h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ and $h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We shall prove the assertion for $h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ only and others can be proved similarly. Let $x = (x_k), y = (y_k) \in h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho_1}\Big)\Big]^{p_k} < \infty, \quad \text{for some } \rho_1 > 0$$

and

$$\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}y_k|}{\rho_2}\Big)\Big]^{p_k} < \infty, \quad \text{for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is a non-decreasing and convex so by using inequality (1.4), we have

$$\begin{aligned}
&\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}(\alpha x_k + \beta y_k)|}{\rho_3}\Big)\Big]^{p_k} \\
&= \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}\alpha x_k|}{\rho_3}\Big) + \Big(\frac{|u_k\Delta^{(m)}\beta y_k|}{\rho_3}\Big)\Big]^{p_k} \\
&\leq K \sup_n \frac{1}{2^{p_k}} \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho_1}\Big)\Big]^{p_k} \\
&+ K \sup_n \frac{1}{2^{p_k}} \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}y_k|}{\rho_2}\Big)\Big]^{p_k} \\
&\leq K \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho_1}\Big)\Big]^{p_k} \\
&+ K \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}y_k|}{\rho_2}\Big)\Big]^{p_k} \\
&< \infty.
\end{aligned}$$

Thus, $\alpha x + \beta y \in h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$. This proves that $h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ is a linear space. Similarly, we can prove that $h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ and $h_c\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ are also linear spaces. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. Then $h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$ is paranormed space with the paranorm,*

$$g(x) = \inf\bigg\{(\rho)^{\frac{p_k}{G}} : \Big(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho}\Big)\Big]^{p_k}\Big)^{\frac{1}{G}} \leq 1, \textit{ for some } \rho > 0\bigg\},$$

where $0 \leq p_k \leq \sup p_k = R$, *and* $G = \max(1, R)$.

Proof. (i) Clearly $g(x) \geq 0$ for $x = (x_k) \in h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$. Since $M_k(0) = 0$, we get $g(0) = 0$.
(ii) $g(-x) = g(x)$.
(iii) Let $x = (x_k)$ and $y = (y_k) \in h_\infty\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$, then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho_1}\Big)\Big]^{p_k} \leq 1$$

and

$$\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} y_k|}{\rho_2}\Big)\Big]^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} (x_k + y_k)|}{\rho}\Big)\Big]^{p_k}$$

$$\begin{aligned}
&= \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} (x_k + y_k)|}{\rho_1+\rho_2}\Big)\Big]^{p_k}\\
&\leq \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho_1+\rho_2}\Big)\Big]^{p_k}\\
&+ \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} y_k)|}{\rho_1+\rho_2}\Big)\Big]^{p_k}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big) \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho_1}\Big)\Big]^{p_k}\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big) \sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} y_k|}{\rho_2}\Big)\Big]^{p_k}\\
&\leq 1
\end{aligned}$$

and thus,

$$\begin{aligned}
g(x+y) &= \inf\Big\{(\rho)^{\frac{p_k}{G}} : \Big(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} (x_k + y_x)|}{\rho}\Big)\Big]^{p_k}\Big)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0\Big\}\\
&\leq \inf\Big\{(\rho_1)^{\frac{p_k}{G}} : \Big(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho_1}\Big)\Big]^{p_k}\Big)^{\frac{1}{G}} \leq 1, \text{ for some } \rho_1 > 0\Big\}\\
&+ \inf\Big\{(\rho_2)^{\frac{p_k}{G}} : \Big(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} y_x|}{\rho_2}\Big)\Big]^{p_k}\Big)^{\frac{1}{G}} \leq 1, \text{ for some } \rho_2 > 0\Big\}.
\end{aligned}$$

Therefore, $g(x+y) \leq g(x)+g(y)$. Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$\begin{aligned}
g(\lambda x) &= \inf\Big\{(\rho)^{\frac{p_k}{G}} : \Big(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} (\lambda x_k)|}{\rho}\Big)\Big]^{p_k}\Big)^{\frac{1}{G}} \leq 1 \text{ for some } \rho > 0\Big\}\\
&= \inf\Big\{(|\lambda| t)^{\frac{p_k}{G}} : \Big(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{t}\Big)\Big]^{p_k}\Big)^{\frac{1}{G}} \leq 1, \text{ for some } t > 0\Big\},
\end{aligned}$$

where $t = \frac{\rho}{|\lambda|} > 0$. Since $|\lambda|^{p_k} \le \max(1, |\lambda|^{\sup p_k})$, we have

$$g(\lambda x) \le \max(1, |\lambda|^{\sup p_k}).$$
$$\inf\Big\{(t)^{\frac{p_k}{G}} : \Big(\sup_n \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{t}\Big)\Big]^{p_k}\Big)^{\frac{1}{G}} \le 1, \text{ for some } t > 0\Big\}.$$

So, the fact that the scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $u = (u_k)$ be a sequence of positive real numbers. If $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers with $0 \le p_k \le q_k < \infty$ for each k, then $h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big) \subseteq h_0\big(\Delta^{(m)}, \mathcal{M}, u, q\big)$.*

Proof. Let $x \in h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$. Then

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

This implies that

$$\frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} < 1,$$

for sufficiently large values of k. Since M_k is increasing and $p_k \le q_k$, we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{q_k} &\le \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} \\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Hence, $x \in h_0\big(\Delta^{(m)}, \mathcal{M}, u, q\big)$. This completes the proof. □

Theorem 2.4. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $\varrho = \lim_{t\to\infty} \frac{M_k(t)}{t} > 0$. Then $h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big) \subseteq h_0\big(\Delta^{(m)}, u, p\big)$.*

Proof. In order to prove that $h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big) \subseteq h_0\big(\Delta^{(m)}, u, p\big)$. Let $\varrho > 0$. By definition of ϱ, we have $M_k(t) \ge \varrho(t)$, for all $t > 0$. Since $\varrho > 0$, we have $t \le \frac{1}{\varrho} M_k(t)$ for all $t > 0$.

Let $x = (x_k) \in h_0\big(\Delta^{(m)}, \mathcal{M}, u, p\big)$. Thus, we have

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\Big[\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big]^{p_k} \le \sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k},$$

which implies that $x = (x_k) \in h_0\big(\Delta^{(m)}, u, p\big)$. This completes the proof. □

Theorem 2.5. *Let $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ are sequences of Orlicz functions, then*

$$h_0\big(\Delta^{(m)}, \mathcal{M}', u, p\big) \cap h_0\big(\Delta^{(m)}, \mathcal{M}'', u, p\big) \subseteq h_0\big(\Delta^{(m)}, (\mathcal{M}' + \mathcal{M}''), u, p\big).$$

Proof. Let $x = (x_k) \in h_0\big(\Delta^{(m)}, \mathcal{M}', u, p\big) \cap h_0\big(\Delta^{(m)}, \mathcal{M}'', u, p\big)$. Therefore,

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M'_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} \longrightarrow 0 \text{ as } n \to \infty$$

and

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\Big[M''_k\Big(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\Big)\Big]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

Then, we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{n+k-1}\Big[(M'_k+M''_k)\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho}\Big)\Big]^{p_k} \\ &\leq K\Big\{\sum_{k=1}^{n}\frac{1}{n+k-1}\Big[M'_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho}\Big)\Big]^{p_k}\Big\} \\ &+ K\Big\{\sum_{k=1}^{n}\frac{1}{n+k-1}\Big[M''_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho}\Big)\Big]^{p_k}\Big\} \\ &\longrightarrow 0 \text{ as } n\to\infty.\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} \frac{1}{n+k-1}\Big[(M'_k+M''_k)\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho}\Big)\Big]^{p_k} \longrightarrow 0 \text{ as } n\to\infty.$$

Therefore, $x=(x_k)\in h_0\big(\Delta^{(m)},(\mathcal{M}'+\mathcal{M}''),u,p\big)$ and this completes the proof. □

Theorem 2.6. *Let $\mathcal{M}'=(M'_k)$ and $\mathcal{M}''=(M''_k)$ be two sequences of Orlicz functions, then*

$$h_0\big(\Delta^{(m)},\mathcal{M}',u,p\big)\subseteq h_0\big(\Delta^{(m)},\mathcal{M}'\circ\mathcal{M}'',u,p\big).$$

Proof. Let $x=(x_k)\in h_0\big(\Delta^{(m)},\mathcal{M}',u,p\big)$. Then we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\Big[M'_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho}\Big)\Big]^{p_k}=0.$$

Let $\epsilon>0$ and choose $\delta>0$ with $0<\delta<1$ such that $M_k(t)<\epsilon$, for $0\leq t\leq\delta$.

Write $y_k=\frac{1}{n+k-1}\Big[M'_k\Big(\frac{|u_k\Delta^{(m)}x_k|}{\rho}\Big)\Big]$ and consider

$$\sum_{k=1}^{n}[M_k(y_k)]^{p_k}=\sum_1[M_k(y_k)]^{p_k}+\sum_2[M_k(y_k)]^{p_k},$$

where the first summation is over $y_k\leq\delta$ and second summation is over $y_k>\delta$. Since M_k is continuous, we have

$$\sum_1[M_k(y_k)]^{p_k}<\epsilon^H \tag{2.1}$$

and for $y_k>\delta$, we use the fact that

$$y_k<\frac{y_k}{\delta}\leq 1+\frac{y_k}{\delta}.$$

By the definition, we have for $y_k>\delta$

$$M_k(y_k)<2M_k(1)\frac{y_k}{\delta}.$$

Hence,

$$\sum_1[M_k(y_k)]^{p_k}\leq\max\Big(1,(2M_k(1)\delta^{-1})^H\Big)\sum_1[y_k]^{p_k}. \tag{2.2}$$

From equation (2.3) and (2.4), we have

$$h_0\big(\Delta^{(m)},\mathcal{M}',u,p\big)\subseteq h_0\big(\Delta^{(m)},\mathcal{M}'\circ\mathcal{M}'',u,p\big).$$

This completes the proof. □

Theorem 2.7. *The Hilbert sequence spaces $h_c\big(\Delta^{(m)},\mathcal{M},u,p\big)$, $h_0\big(\Delta^{(m)},\mathcal{M},u,p\big)$ and $h_\infty\big(\Delta^{(m)},\mathcal{M},u,p\big)$ are isometrically isomorphic to the space c, c_0 and l_∞ respectively, that is, $h_c\big(\Delta^{(m)},\mathcal{M},u,p\big)\cong c$, $h_0\big(\Delta^{(m)},\mathcal{M},u,p\big)\cong c_0$ and $h_\infty\big(\Delta^{(m)},\mathcal{M},u,p\big)\cong l_\infty$.*

Proof. We only consider the case $h_0(\Delta^{(m)}, \mathcal{M}, u, p) \cong c_0$ and others will follow similarly. To show the theorem, we must show the existence of linear bijection between the space $h_0(\Delta^{(m)}, \mathcal{M}, u, p)$ and c_0. For this, we consider the transformation T defined with the notation (1.3), from $h_0(\Delta^{(m)}, \mathcal{M}, u, p)$ to c_0 by $x \to y = Tx$. The linearity of T is obvious. Further, it is trivial that $x = \theta = (0, 0, 0...)$ whenever $Tx = \theta$ and hence T is injective. Next, let $y = (y_n) \in c_0$ and defined the sequence $x = (x_n)$ by

$$x_n = \sum_{k=1}^{n} \left[\sum_{i=k}^{n} \begin{pmatrix} m+n-i-1 \\ i-k \end{pmatrix} h_{ik}^{-1} \right] y_k,$$

where h_{ik}^{-1} is defined by (1.3). Then,

$$\begin{aligned} \lim_{n\to\infty} \left(H\Delta^{(m)}x\right)_n^{\mathcal{M},u,p} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k-1} \left[M_k\left(\frac{|u_k \Delta^{(m)} x_k|}{\rho}\right)\right]^{p_k} \\ &= \sum_{k=1}^{n} \frac{1}{n+k-1} \left[M_k\left(\frac{|u_k \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}|}{\rho}\right)\right]^{p_k} \\ &= \sum_{k=1}^{n} \frac{1}{n+k-1} \left[M_k\left(\frac{|u_k \sum_{i=k}^{n} (-1)^{i-k} \binom{m}{i-k} x_k|}{\rho}\right)\right]^{p_k} \\ &= \lim_{n\to\infty} y_n = 0. \end{aligned}$$

Thus, $x \in h_0(\Delta^{(m)}, \mathcal{M}, u, p)$. Consequently, it is clear that T is surjective. Because of the fact that is linear bijection, $h_0(\Delta^{(m)}, \mathcal{M}, u, p)$ and c_0 are linearly isomorphic. This completes the proof. □

Remark 2.8. It is well known that the spaces c, c_0 and l_∞ are BK-spaces. Let us considering the fact that $\Delta^{(m)}$ is a triangle, we can say that the Hilbert sequence spaces $h_c(\Delta^{(m)}, \mathcal{M}, u, p)$, $h_0(\Delta^{(m)}, \mathcal{M}, u, p)$ and $h_\infty(\Delta^{(m)}, \mathcal{M}, u, p)$ are BK-spaces with the norm defined by

$$\begin{aligned} \|x\|_{\Delta}^{\mathcal{M},u,p} &= \|H\Delta^{(m)}x\|_{\infty}^{\mathcal{M},u,p} \\ &= \sup_n \left| \sum_{k=1}^{n} \frac{1}{n+k-1} \left[M_k\left(\frac{|u_k \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}|}{\rho}\right)\right]^{p_k} \right|. \end{aligned} \tag{2.3}$$

Corollary 2.9. *Define the sequence* $b^{(k)} = \left(b_n^{(k)}(\Delta^{(m)}, \mathcal{M}, u, p)\right)_{n\in\mathbb{N}}$ *by*

$$b_n^{(k)}(\Delta^{(m)}, \mathcal{M}, u, p) = \begin{cases} \sum_{k=1}^{n} \left[M_k\left(\frac{\left|u_k \sum_{i=k}^{n} \binom{m+n-i-1}{n-i} h_{ik}^{-1}\right|}{\rho}\right)\right]^{p_k}, & (n \geq k) \\ 0, & (n < k) \end{cases}$$

for every fixed $k \in \mathbb{N}$. *the following statements hold:*
(i) *The sequence* $b_n^{(k)}(\Delta^{(m)}, \mathcal{M}, u, p)$ *is a basis for the space* $h_0(\Delta^{(m)}, \mathcal{M}, u, p)$ *and every* $x \in h_0(\Delta^{(m)}, \mathcal{M}, u, p)$ *has a unique representation of the form*

$$x = \sum_k \left(H\Delta^{(m)}x\right)_k^{\mathcal{M},u,p} b^{(k)}.$$

(ii) *The set* $\{t, b^{(1)}, b^{(2)}, ...\}$ *is a basis for the space* $h_c(\Delta^{(m)}, \mathcal{M}, u, p)$ *and every* $x \in h_c(\Delta^{(m)}, \mathcal{M}, u, p)$ *has a unique representation of the form*

$$x = st + \sum_k \left[\left(H\Delta^{(m)}x\right)_k^{\mathcal{M},u,p} - s\right] b^{(k)},$$

where $t = t_n(\Delta^{(m)}, \mathcal{M}, u, p) = \sum_{k=1}^{n} \left[M_k\left(\frac{\left|u_k \sum_{i=k}^{n} \binom{m+n-i-1}{n-i} h_{ik}^{-1}\right|}{\rho}\right)\right]^{p_k}$ *for all* $k \in \mathbb{N}$ *and* $s = \lim_{k\to\infty} \left(H\Delta^{(m)}x\right)_k^{\mathcal{M},u,p}$.

Corollary 2.10. *The Hilbert sequence spaces* $h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)$ *and* $h_c\big(\Delta^{(m)}\mathcal{M},u,p\big)$ *are separable.*

3 Characterizations of Matrix Transformation and α-, β- and γ- duals

Let $A=(a_{nk})$ be an infinite matrix of complex numbers, X and Y be subsets of sequence space w. Let $x=(x_k)$ and $y=(y_k)$ be two sequences. Thus, we can write $xy=(x_k y_k)$, $x^{-1}*Y=\{a\in w: ax\in Y\}$ and $M(X,Y)=\cap_{x\in X}x^{-1}*Y=\{a\in w: ax\in Y \text{ for all } x\in X\}$ for the multiplier space of X and Y . In the special cases of $Y=\{l_1,cs,bs\}$, we write $x^\alpha = x^{-1}*l_1,\ x^\beta=x^{-1}*cs,\ x^\gamma=x^{-1}*bs$ and $X^\alpha=M(X,l_1),\ X^\beta=M(X,cs),\ X^\gamma=M(X,bs)$ for the $\alpha-$ dual, $\beta-$ dual, $\gamma-$ dual of X. By $A_n=(a_{nk})$ we denote the sequence in the $n^{th}-$ row of A, and we write $A_n(x)=\displaystyle\sum_{k=1}^{\infty}a_{nk}x_k\ \ \forall n\in\mathbb{N}$ and $A(x)=(A_n(x))$, provided $A_n\in x^\beta$ for all n.

We shall begin with the lemmas due to Stieglitz and Tietz [21] which will be used in the computation of the β and γ-duals of the Hilbert sequence spaces.

Lemma 3.1. *[5] Let* X,Y *be any two sequence spaces.* $A\in(X:Y_T)$ *if and only if* $TA\in(X:Y)$, *where* A *an infinite matrix and* T *a triangle matrix.*

Lemma 3.2. *Let* $A=(a_{nk})$ *be an infinite matrix. Then* $A\in(c_0:c)$ *if and only if*

$$\sup_n\sum_k|a_{nk}| \quad < \quad \infty \tag{3.1}$$

$$\lim_n a_{nk} \ \ \textit{exists for all } k. \tag{3.2}$$

Lemma 3.3. (i) *Let* $A=(a_{nk})$ *be an infinite matrix. Then* $A\in(c_0:l_\infty)$ *if and only if* (3.1) *holds.*
(ii) $A\in(c_0:cs)$ *if and only if*

$$\sup_m\sum_k\Big|\sum_{n=0}^{m}\Big|<\infty$$

$$\sum_n a_{nk} \textit{convergent for all } k.$$

(iii) $A\in(c_0:bs)$ *if and only if* (3.3) *holds.*

Lemma 3.4. *Let* $A=(a_{nk})$ *be an infinite matrix. Then* $A\in(l_\infty:c)$ *if and only if* (3.2) *holds and*

$$\lim_n\sum_k|a_{nk}| \quad = \quad \sum_k\big|\lim_{n\to\infty}a_{nk}\big|. \tag{3.3}$$

Lemma 3.5. *[2] Let* $U=(u_{nk})$ *be an infinite matrix of complex numbers for all* $n,k\in\mathbb{N}$. *Let* $B^U=(b_{nk})$ *be defined via a sequence* $a=(a_k)\in w$ *and inverse of the triangle matrix* $U=(u_{nk})$ *by*

$$b_{nk}=\sum_{j=k}^{n}a_j v_{jk}$$

for all $k,n\in\mathbb{N}$. *Then,*

$$X_U^\beta=\{a=(a_k)\in w: B^U\in(X:c)\}$$

and

$$X_U^\gamma=\{a=(a_k)\in w: B^U\in(X:l_\infty)\}.$$

Theorem 3.6. *The β- and γ duals of the Hilbert sequence spaces defined as*

$$\begin{aligned}
\Big[h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\beta &= \{a=(a_k)\in w : V\in (c_0:c)\},\\
\Big[h_c\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\beta &= \{a=(a_k)\in w : V\in (c:c)\},\\
\Big[h_\infty\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\beta &= \{a=(a_k)\in w : V\in (l_\infty:c)\},\\
\Big[h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\gamma &= \{a=(a_k)\in w : V\in (c_0:l_\infty)\},\\
\Big[h_c\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\gamma &= \{a=(a_k)\in w : V\in (c:l_\infty)\},\\
\Big[h_\infty\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\gamma &= \{a=(a_k)\in w : V\in (l_\infty:l_\infty)\}.
\end{aligned}$$

Proof. We shall only compute the β- and γ- duals of $h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)$ sequence space. Let h^{-1}_{nk} is defined by (1.3). Let us take any $a=(a_k)\in w$. We define the matrix $V=(v_{nk})$ by

$$v_{nk} = \sum_{k=1}^{n}\left[M_k\left(\frac{u_k|\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}h^{-1}_{ik}a_n|}{\rho}\right)\right]^{p_k}. \tag{3.4}$$

Consider the equation

$$\begin{aligned}
\sum_{k=1}^{n}a_kx_k &= \sum_{k=1}^{n}\left[M_k\left(\frac{u_k|\sum_{i=1}^{k}\{\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}h^{-1}_{ij}\}a_ky_i|}{\rho}\right)\right]^{p_k} \qquad (3.5)\\
&= \sum_{k=1}^{n}\left[M_k\left(\frac{u_k|\sum_{i=1}^{k}\{\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}h^{-1}_{ij}a_i\}y_k|}{\rho}\right)\right]^{p_k}\\
&= (Vy)_n.
\end{aligned}$$

Using (3.7), we have $ax=(a_kx_k)\in cs$ or bs whenever $x=(x_k)\in h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)$ if and only if $Vy\in c$ or l_∞ whenever $y=(y_k)\in c_0$. Then, from Lemma 3.1 and Lemma 3.5, we obtain that $a=(a_k)\in\Big[h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\beta$ or $a=(a_k)\in\Big[h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\gamma$ if and only if $V\in(c_0:c)$ or $V\in(c_0:l_\infty)$, which is required result. □

Therefore, the β- and γ-duals of Hilbert sequence spaces will help in the characterization of matrix transformations. Let X and Y be arbitrary subsets of w. We shall show that the characterization of the classes $(X:Y_T)$ and $(X_T:Y)$ can be reduced to (X,Y), where T is a triangle. Since if the sequence spaces $h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)$ and c_0 are linearly isomorphic, then the equivalence class $x\in h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)\Leftrightarrow y\in c_0$ holds. Then, by using the Lemmas 3.1 and 3.5, we have

Theorem 3.7. *Let us consider the infinite matrices $A=(a_{nk})$ and $D=(d_{nk})$. These matrices get associated with each other by the relations:*

$$d_{nk}=\sum_{k=1}^{n}\left[M_k\left(\frac{u_k|\sum_{j=k}^{\infty}\binom{m+n-j-1}{n-j}h^{-1}_{jk}a_{nj}|}{\rho}\right)\right]^{p_k}$$

for all $k,m,n\in\mathbb{N}$. Then the following statements are true:

(i) $A\in\big(h_0\big(\Delta^{(m)}\mathcal{M},u,p\big):Y\big)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in\Big[h_0\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\beta$ for all $n\in\mathbb{N}$ and $D\in(c_0:Y)$, where Y be any sequence space;

(ii) $A\in\big(h_c\big(\Delta^{(m)}\mathcal{M},u,p\big):Y\big)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in\Big[h_c\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\beta$ for all $n\in\mathbb{N}$ and $D\in(c:Y)$, where Y be any sequence space;

(iii) $A\in\big(h_\infty\big(\Delta^{(m)}\mathcal{M},u,p\big):Y\big)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}}\in\Big[h_\infty\big(\Delta^{(m)}\mathcal{M},u,p\big)\Big]^\beta$ for all $n\in\mathbb{N}$ and $D\in(l_\infty:Y)$, where Y be any sequence space.

Proof. We suppose that the relation (3.8) holds between $A = (a_{nk})$ and $D = (d_{nk})$. Since the spaces $h_0(\Delta^{(m)}\mathcal{M}, u, p)$ and c_0 are linearly isomorphic. Let $A \in \big(h_0(\Delta^{(m)}\mathcal{M}, u, p) : Y\big)$ and $y = (y_k) \in c_0$. Then $DH\Delta^{(m)}$ exists and $(a_{nk}) \in \Big[h_0(\Delta^{(m)}\mathcal{M}, u, p)\Big]^{\beta}$ for all $k \in \mathbb{N}$, it means that $(d_{nk}) \in c_0$ for all $k, n \in \mathbb{N}$. Hence, Dy exists for each $y \in c_0$. thus, if we take $m \to \infty$ in the equality

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m} \left[M_k \left(\frac{|u_k[\sum_{i=1}^{k}\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}h_{ij}^{-1}]a_{nk}|}{\rho} \right) \right]^{p_k} = \sum_k d_{nk}y_k$$

for all $m, n \in \mathbb{N}$ which conclude that $D \in (c_0 : Y)$. On the contrary, let $(a_{nk})_{k\in\mathbb{N}} \in \Big[h_0(\Delta^{(m)}\mathcal{M}, u, p)\Big]^{\beta}$ for each $n \in \mathbb{N}$ and $D \in (c_0 : Y)$ and $x = (x_k) \in h_0(\Delta^{(m)}\mathcal{M}, u, p)$. Then it is clear that Ax exists. Thus, we attain from the following equality for all $n \in \mathbb{N}$

$$\sum_k d_{nk}y_k = \sum_k a_{nk}x_k,$$

as $m \to \infty$ that $Ax = Dy$ and it is easy to show that $A \in \big(h_0(\Delta^{(m)}\mathcal{M}, u, p) : Y\big)$. This completes the proof. □

Theorem 3.8. *Let us assume that the components of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the following relation*

$$(e_{nk}) = \sum_{k=1}^{n}\sum_{j=k}^{n} \frac{1}{n+j-1} \left[M_k \left(\frac{|u_k \sum_{j=k}^{n}(-1)^{j-k}\binom{m}{j-k}a_{jk}|}{\rho} \right) \right]^{p_k},$$

for all $m, n \in \mathbb{N}$ and X be any given sequence space. Then, the following statements are true:
(i) $A = (a_{nk}) \in \big(X : h_0(\Delta^{(m)}\mathcal{M}, u, p)\big)$ if and only if $E \in (X : c_0)$;
(ii) $A = (a_{nk}) \in \big(X : h_c(\Delta^{(m)}\mathcal{M}, u, p)\big)$ if and only if $E \in (X : c)$;
(iii) $A = (a_{nk}) \in \big(X : h_\infty(\Delta^{(m)}\mathcal{M}, u, p)\big)$ if and only if $E \in (X : l_\infty)$.

Proof. Let us suppose that $z = (z_k) \in X$. Using the relation (3.9), we have

$$\sum_{k=1}^{m}(e_{nk}) = \sum_{k=1}^{m} \left[\sum_{k=1}^{n}\sum_{j=k}^{n} \frac{1}{n+j-1} \left[M_k \left(\frac{|u_k[\sum_{j=k}^{n}(-1)^{j-k}\binom{m}{j-k}a_{jk}]z_k|}{\rho} \right) \right]^{p_k} \right], \tag{3.6}$$

for all $m, n \in \mathbb{N}$. Then, for $m \to \infty$ equation (3.10) gives us that $(Ez)_n = \{H\Delta^{(m)}(Az)\}_n$. Thus, we can obtain that $Az \in h_0(\Delta^{(m)}\mathcal{M}, u, p)$ if and only if $Ez \in c_0$. This completes the proof. □

Now, we give some conditions:

$$\lim_k a_{nk} = 0 \text{ for all } n, \tag{3.7}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0, \tag{3.8}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0, \tag{3.9}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = 0, \tag{3.10}$$

$$\sup_n \sum_k |a_{nk} - a_{n,k+1}| < 0, \tag{3.11}$$

$$\lim_k (a_{nk} - a_{n,k+1}) \text{ exists for all } k, \tag{3.12}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = \sum_k \big| \lim_{n\to\infty} (a_{nk} - a_{n,k+1})\big|, \tag{3.13}$$

$$\sup_n \big| \lim_k a_{nk} \big| < \infty, \tag{3.14}$$

$$\sum_n \sum_k a_{nk} \quad \text{convergent}, \tag{3.15}$$

$$\lim_m \sum_k \big| \sum_{n=m}^{\infty} a_{nk} \big| = 0. \tag{3.16}$$

Corollary 3.9. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*
(a) $A = (a_{nk}) \in \big(h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big), l_\infty\big)$ if and only if (3.1) holds with d_{nk} instead of a_{nk};
(b) $A = (a_{nk}) \in \big(h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big), bs\big)$ if and only if (3.3) holds with d_{nk} instead of a_{nk};
(c) $A = (a_{nk}) \in \big(h_c\big(\Delta^{(m)}\mathcal{M}, u, p\big), cs\big)$ if and only if (3.3), (3.4) and (3.19) hold with d_{nk} instead of a_{nk};
(d) $A = (a_{nk}) \in \big(h_\infty\big(\Delta^{(m)}\mathcal{M}, u, p\big), c\big)$ if and only if (3.2) and (3.5) hold with d_{nk} instead of a_{nk};
(e) $A = (a_{nk}) \in \big(h_\infty\big(\Delta^{(m)}\mathcal{M}, u, p\big), cs\big)$ if and only if (3.20) holds with d_{nk} instead of a_{nk}.

Corollary 3.10. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*
(a) $A = (a_{nk}) \in \big(l_\infty, h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big)\big)$ if and only if (3.13) holds with e_{nk} instead of a_{nk};
(b) $A = (a_{nk}) \in \big(bs, h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big)\big)$ if and only if (3.11) and (3.14) hold with e_{nk} instead of a_{nk};
(c) $A = (a_{nk}) \in \big(bs, h_c\big(\Delta^{(m)}\mathcal{M}, u, p\big)\big)$ if and only if (3.11), (3.16) and (3.17) hold with e_{nk} instead of a_{nk};
(d) $A = (a_{nk}) \in \big(cs, h_c\big(\Delta^{(m)}\mathcal{M}, u, p\big)\big)$ if and only if (3.15) and (3.2) hold with e_{nk} instead of a_{nk};
(e) $A = (a_{nk}) \in \big(bs, h_\infty\big(\Delta^{(m)}\mathcal{M}, u, p\big)\big)$ if and only if (3.11) and (3.15) hold with e_{nk} instead of a_{nk};
(f) $A = (a_{nk}) \in \big(cs, h_\infty\big(\Delta^{(m)}\mathcal{M}, u, p\big)\big)$ if and only if (3.15) and (3.18) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$, (3.12) and (3.13) also hold with $\alpha = 0$ with $a(n, k)$ instead of a_{nk}.

4 Examples

If we take any sequence spaces X and Y in Theorem 3.7 and 3.8, then we have several consequences. For example, if we choose l_∞ and the spaces which are isomorphic to l_∞ instead of Y in Theorem 3.7, we obtain the following examples:

Example 4.1. The Taylor sequence space t^r_∞ is defined by $t^r_\infty = \Big\{x \in w : \sup_{n\in\mathbb{N}} \big| \sum_{k=n}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} (1-r)^{n+1} r^{k-n} x_k \big| < \infty \Big\}$ [10]. Let us consider the infinite matrix $A = (a_{nk})$ and define the matrix $P = (p_{nk})$ as

$$p_{nk} = \sum_{k=1}^{n} \Big[M_k \Big(\frac{|u_k \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} a_{jk}|}{\rho} \Big) \Big]^{p_k} \quad (k, n \in \mathbb{N}).$$

If we replace the entries of the matrix A by those of the matrix C in Theorem 3.7. Then we get the necessary and sufficient conditions for the class $\big(h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big) : t^r_\infty\big)$.

Example 4.2. The Euler sequence space e^r_∞ is defined by $e^r_\infty = \Big\{x \in w : \sup_{n\in\mathbb{N}} \big|\sum_{k=0}^{n} \binom{n}{k} (1-n)^{n-k} r^k x_k\big| < \infty\Big\}$ ([3] and [1]). We consider the infinite matrix $A = (a_{nk})$ and define the matrix $C = (c_{nk})$ by

$$c_{nk} = \sum_{k=1}^{n} \Big[M_k\Big(\frac{|u_k \sum_{j=1}^{n}\binom{n}{j}(1-r)^{n-j} r^j a_{jk}|}{\rho}\Big)\Big]^{p_k} \quad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $\big(h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big) : e^r_\infty\big)$ in Theorem 3.7, then we replace the entries of the matrix A by those of the matrix C.

Example 4.3. Let $T_n = \sum_{k=0}^{n} t_k$ and $A = (a_{nk})$ be an infinite matrix. We define the matrix $G = (g_{nk})$ by is defined by

$$g_{nk} = \frac{1}{T_n} \sum_{k=1}^{n} \Big[M_k\Big(\frac{|u_k \sum_{j=1}^{n} t_j a_{jk}|}{\rho}\Big)\Big]^{p_k} \quad (k, n \in \mathbb{N}).$$

Then the necessary and sufficient conditions in order for $A \in \big(h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big) : r^t_\infty\big)$ are obtained by replacing the entries of the matrix A by those of the matrix G in Theorem 3.7, where r^t_∞ is the space of all sequences whose R^t-transforms is in the space $\l_\infty$ [15].

Example 4.4. If we take $t = e$ in the space r^t_∞, then this space become to the Cesaro sequence space of non-absolute type X_∞ [17]. As a special case, Example 4.3 includes the characterization of the class $\big(h_0\big(\Delta^{(m)}\mathcal{M}, u, p\big) : r^t_\infty\big)$.

5 Conclusion

Hilbert matrix introduced in 1894 by Hilbert. Hilbert matrices whose entries are specified as machine-precision numbers are difficult to invert using numerical techniques. The applications of Hilbert matrix can be found in image processing and cryptography. In the present paper, we have studied sequence spaces with the Hilbert matrix, Orlicz function and difference operator of order m. We have calculated dual spaces and characterize matrix transformation between new formed sequence spaces. In our further study, we shall plotted the images of these spaces by using Mathematica. One can investigate different applications of the cryptography also.

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Author information

Kuldip Raj, Department of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K, India.
E-mail: `kuldipraj68@gmail.com`

Ayhan Esi, Malatya Turgut Ozal University, Department of Basic Engineering Sciences, 44040, Malatya, Turkey.
E-mail: `aesi23@hotmail.com`

Charu Sharma, Department of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K, India.
E-mail: `charu145.cs@gmail.com`

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