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PSEUDO-POWERFUL IDEALS IN AN INTEGRAL DOMAIN

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Abstract Let R be an integral domain with quotient field K. In this paper, we introduce and investigate a new class of ideals that is closely related to the class of pseudo-strongly prime ideals in the sense of Badawi [6]. We define an ideal I of R to be *pseudo-powerful* if whenever $xyI \subseteq I$ with $x, y \in K$, we have either $x^n \in R$ or $y^n I \subseteq R$ for some $n \geq 1$.

Introduction

Let R be an integral domain with quotient field K. We start by recalling some background material. Hedstrom and Houston [12] defined a prime ideal P of R to be *strongly prime* if whenever $xy \in P$ with $x, y \in K$, we have either $x \in P$ or $y \in P$. If each prime ideal P of R is strongly prime, then R is called a *pseudo-valuation domain* (*PVD*). For an extensive study of strongly prime ideals, see Jahani-Nezhad's paper [13]; for a survey on PVDs, see Badawi's paper [5]. D. D. Anderson and D. F. Anderson [1] defined a radical ideal I of R to be *strongly radical* if whenever $x \in K$ satisfies $x^n \in I$ for some $n \geq 1$, we have $x \in I$. Following Sato and Sugatani [14], an integral domain R is called *rooty* if each radical ideal of R is strongly radical (equivalently, each prime ideal of R is strongly radical [4, Theorem 1.8]). As a generalization of the concept of strongly prime, Badawi and Houston [7] defined an ideal I of R to be *powerful* if whenever $xy \in I$ with $x, y \in K$, we have either $x \in R$ or $y \in R$. They showed that a prime ideal P is strongly prime if and only if it is powerful. As another generalization of the notion of strongly prime, Badawi [6] defined a prime ideal P of R to be *pseudo-strongly prime* if whenever $xyP \subseteq P$ with $x, y \in K$, we have either $x^n \in R$ or $y^n P \subseteq P$ for some $n \geq 1$. If each prime ideal P of R is pseudo-strongly prime, then R is called a *pseudo-almost valuation domain* (*PAVD*). Note that, a strongly prime ideal is strongly radical and pseudo-strongly prime; hence, a PVD is a rooty PAVD.

In this paper, we define an ideal I of R to be *pseudo-powerful* if whenever $xyI \subseteq I$ with x, $y \in K$, we have either $x^n \in R$ or $y^n I \subseteq R$ for some $n \geq 1$. It is easy to see that R itself is pseudo-powerful if and only if R is an almost valuation domain (recall from [2] that an integral domain R is said to be an almost valuation domain (AVD) if for every nonzero $x \in K$, there exists an $n = n(x) \geq 1$ with either $x^n \in R$ or $x^{-n} \in R$).

This paper consists of one section in which we present some basic properties of pseudo-powerful ideals. Among other things, we show that a prime ideal P of R is pseudo-strongly prime if and only if P is pseudo-powerful if and only if $P : P$ is an AVD, and if x is a nonunit of $P : P$, then $x^n \in R$ for some $n \geq 1$ (see Proposition 1.5). We also show that a pseudo-powerful ideal is contained in any two incomparable prime ideals; so in any maximal ideal (see Proposition 1.2). We show that the radical $\sqrt{I}$ of a proper pseudo-powerful ideal I of R is prime (see Proposition 1.10); as a consequence, we show that either $I \subseteq P$ or $P \subseteq \sqrt{I}$ for any prime ideal P of R (see Proposition 1.14). We also show that a proper pseudo-powerful ideal I of R is powerful when R is rooty (see Proposition 1.7); from which it follows that a rooty PAVD is a PVD (see Corollary 1.9). We also consider the stability of pseudo-powerful ideals under passage to homomorphic images and overrings (see Propositions 1.15 and 1.17). Our results generalize the work of Badawi [7] on powerful ideals.

Throughout this paper, R will be an integral domain with quotient field K and $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \geq 0\}$ for I an ideal of R. An overring of R is a subring of K containing

R. In particular, if I is an ideal of R, then $I : I = \{x \in K \mid xI \subseteq I\}$ is an overring of R. For any undefined terminology, see [11].

1 Definitions and properties

In this section, we introduce the notion of pseudo-powerful ideals and provide their properties. If R is an integral domain and K its quotient field, then an ideal I of R is called *pseudo-powerful* if whenever $xyI \subseteq I$ with $x, y \in K$, we have either $x^n \in R$ or $y^nI \subseteq R$ for some $n \geq 1$. It is easy to see that R itself is pseudo-powerful if and only if R is an almost valuation domain (recall from [2] that an integral domain R is said to be an almost valuation domain (AVD) if for every nonzero $x \in K$, there exists an $n = n(x) \geq 1$ with either $x^n \in R$ or $x^{-n} \in R$). Another way to look at the definition of pseudo-powerful ideals is the following.

For a subset S of R, we define

$$E(S) := \{x \in K \mid x^n \notin S \text{ for each } n \geq 1\}.$$

Lemma 1.1. *Let R be an integral domain and I be an ideal of R. Then, the following assertions are equivalent.*

(i) I is a pseudo-powerful ideal of R.

(ii) For every $x \in E(R)$, $x^{-n}I \subseteq R$ for some $n \geq 1$.

(iii) For all elements a, b of R, there is an $n \geq 1$ such that either $a^n \mid b^n$ in R or $b^n \mid a^nc$ in R for every element c of I.

Proof. (1) ⇒ (2) Let $x \in E(R)$. As $x \cdot x^{-1}I = I$ and I is pseudo-powerful in R, $x^{-n}I \subseteq R$ for some $n \geq 1$, as desired.

(2) ⇒ (1) To show that I is pseudo-powerful in R, let $xyI \subseteq I$ with $x, y \in K$. If $x^n \in R$ for some $n \geq 1$, we are done. Assume that $x \in E(R)$. By hypothesis, $x^{-n}I \subseteq R$ for some $n \geq 1$. Then $y^nI = x^{-n} \cdot (xy)^nI \subseteq x^{-n}I \subseteq R$, as desired.

(2) ⇔ (3) This is straightforward. □

Proposition 1.2. *Let R be an integral domain and I be a proper pseudo-powerful ideal of R. If P and Q are two incomparable prime ideals of R, then $I \subseteq P \cap Q$. In particular, I is contained in every maximal ideal of R.*

Proof. Choose $a \in P \setminus Q$ and $b \in Q \setminus P$. Then $a/b \in E(R)$. But I is pseudo-powerful in R, so $(b/a)^nI \subseteq R$ for some $n \geq 1$, by Lemma 1.1. Therefore $b^nI \subseteq a^nR \subseteq P$. As P is prime in R and $b^n \notin P$, $I \subseteq P$. By symmetry, $I \subseteq Q$, as desired. The "in particular" statement follows from the fact that two distinct maximal ideals are not comparable. □

It was shown [7, Theorem 1.5 (3)] that if I is a powerful ideal of R, then the prime ideals of R contained in $\sqrt{I}$ are linearly ordered. This fact remains valid for pseudo-powerful ideals as the following result shows.

Corollary 1.3. *Let R be an integral domain and I be a proper pseudo-powerful ideal of R. Then the set of prime ideals of R that are contained in $\sqrt{I}$ is linearly ordered by inclusion.*

Proof. Let P, Q be two prime ideals of R properly contained in $\sqrt{I}$. We show that P and Q are comparable. Deny. Then, by Proposition 1.2, $I \subseteq P \cap Q$. Therefore $\sqrt{I} \subseteq P \cap Q$, a contradiction. □

It is known that if $J \subseteq I$ are ideals of R such that I is powerful, then J is also powerful [7, Proposition 1.4]. The following result generalizes this fact.

Proposition 1.4. *Let R be an integral domain. If I is a pseudo-powerful ideal of R, then any ideal $J \subseteq I$ of R is also pseudo-powerful.*

Proof. This is straightforward by Lemma 1.1 (2). □

Let R be an integral domain and K be the quotient field of R. Recall from [6] that, a prime ideal of R is called *pseudo-strongly prime* if whenever $xyP \subseteq P$ with $x, y \in K$, we have $x^n \in R$ or $y^n P \in P$ for some $n \geq 1$. If each prime ideal P of R is pseudo-strongly prime, then R is called a *pseudo-almost valuation domain* (PAVD). The following is a generalization of [7, Proposition 1.3].

Proposition 1.5. *Let R be an integral domain and P be a prime ideal of R. The following assertions are equivalent:*

(i) P is pseudo-strongly prime in R.

(ii) P is pseudo-powerful in R.

(iii) $P : P$ is an AVD, and if x is a nonunit of $P : P$, then $x^n \in R$ for some $n \geq 1$.

Proof. (1) $\Rightarrow$ (2) This is straightforward.

(2) $\Rightarrow$ (1) Let $x \in E(R)$. Then, as P is pseudo-powerful in R, $x^{-n}P \subseteq R$ for some $n \geq 1$, by Lemma 1.1. But $x^{2n} \in E(R)$, so that $x^{-2nm}P \subseteq R$ for some $m \geq 1$, again by Lemma 1.1. Therefore $(x^{-n}P)^{2m} = x^{-2nm}P^{2m} \subseteq P$. Hence $x^{-n}P \subseteq P$; so P is pseudo-strongly prime in R by [6, Lemma 2.1], as desired.

(1) $\Rightarrow$ (3) To show that $V := P : P$ is an AVD, let $x \in E(V)$. Then $x \in E(R)$. Therefore, by [6, Lemma 2.1], $x^{-n}P \subseteq P$ for some $n \geq 1$. Hence $x^{-n} \in V$ and V is an AVD, as desired. Now, let x be a nonunit of $x \in V$. We show that $x^n \in R$ for some $n \geq 1$. Deny. Then $x \in E(R)$. The same argument as above leads to $x^{-n} \in V$ for some $n \geq 1$, a contradiction.

(3) $\Rightarrow$ (1) Let $x \in E(R)$. We show that $x^{-n}P \subseteq P$ for some $n \geq 1$; in which case P is pseudo-strongly prime in R, by [6, Lemma 2.1]. If $x \in E(V)$, then, as V is an AVD, $x^{-n} \in V$ for some $n \geq 1$. Therefore $x^{-n}P \subseteq P$, as desired. We may assume that $x^n \in V$ for some $n \geq 1$. Hence $x^n \in E(R)$. Thus x^n is a unit of V so $x^{-n} \in V$, that is, $x^{-n}P \subseteq P$, as desired. □

Corollary 1.6. *Let R be an integral domain. Then, R is a PAVD if and only if some maximal ideal of R is pseudo-powerful.*

Proof. This follows directly from [6, Theorem 2.5] and Proposition 1.5. □

Recall from [1] that a radical ideal I of R is called *strongly radical* if whenever $x \in K$ satisfies $x^n \in I$ for some $n \geq 1$, we have $x \in I$. Following [14], an integral domain R is called *rooty* if each radical ideal of R is strongly radical (equivalently, each prime ideal of R is strongly radical [4, Theorem 1.8]). Obviously, a powerful ideal of R is pseudo-powerful. We next show that the converse holds for proper ideals when R is a rooty domain.

Proposition 1.7. *Let R be a rooty domain and I a proper pseudo-powerful ideal of R. Then I is a powerful ideal of R.*

Proof. Let $x \in K \setminus R$. We claim that $x^{-n}I^n \subsetneq R$ for some $n \geq 1$. As R is rooty, two cases are then possible:

Case 1: "$x \in E(R)$".
Since I is pseudo-powerful in R, Lemma 1.1 yields $x^{-n}I \subseteq R$ for some $n \geq 1$. Then $x^{-n}I^n \subseteq R$. Moreover $x^{-n}I^n \neq R$; otherwise $x^n \in R$, a contradiction.

Case 2: "$x^n \in R$ is a unit for some $n \geq 1$".
Then $x^{-n} \in R$ so $x^{-n}I \subseteq I \subseteq R$. Therefore $x^{-n}I^n \subseteq R$. Moreover $x^{-n}I^n \neq R$; otherwise $x^n \in I^n \subseteq I$, contradicting the fact that $I \neq R$.

Hence $x^{-n}I^n \subsetneq R$ for some $n \geq 1$, as claimed. Thus $x^{-1}I \subseteq R$, again by the fact that R is a rooty domain. It follows from [7, Lemma 1.1] that I is powerful in R. The proof is complete. □

Remark 1.8. The assumption that "I is proper in R" is essential for Proposition 1.7. For example, take R any nonvaluation AVD which is a PVD (for example, $R = \mathbb{Z}_p + XF[X]$ where p is a positive prime integer and $F = \overline{\mathbb{Z}_p}$ is the algebraic closure of $\mathbb{Z}_p$ the integers mod p [3, Example 4.21 (b)]). As R is an AVD (resp., PVD), R is pseudo-powerful in R (resp., a rooty domain). However, as R is a nonvaluation domain, R is not powerful in R.

Recall from [15] that an integral domain R with quotient field K is called *root closed* if, whenever $x \in K$ and $x^n \in R$ for some $n \geq 1$, then $x \in R$. It was shown in [6, Theorem 2.13] that a root closed PAVD is a PVD. The following result generalizes this fact (cf. [10, Proposition 7 (c)]).

Corollary 1.9. *Let R be a rooty PAVD. Then R is a PVD.*

Proof. Let P be a prime ideal of R. As R is a PAVD, P is pseudo-strongly prime in R; in particular, P is pseudo-powerful in R. As R is a rooty domain, Proposition 1.7 yields P is powerful in R. It follows from [7, Proposition 1.3] that P is strongly prime in R. Hence R is a PVD, as desired. □

We next investigate the radical of a pseudo-powerful ideal (cf. [7, Propositions 1.9 and 1.12]).

Proposition 1.10. *Let R be an integral domain, K the quotient field of R and I be a proper pseudo-powerful ideal of R. Then whenever $xy \in \sqrt{I}$ with x, y elements of K, we have $x^n \in I$ or $y^n \in I$ for some positive integer $n = n(x, y)$. In particular $\sqrt{I}$ is prime in R.*

We have need of the following lemma.

Lemma 1.11. *Let R be an integral domain, K the quotient field of R, I be a proper pseudo-powerful ideal of R and x, y be elements of K. If $xy \in I$, then either $x^n \in I$ or $y^n \in I$ for some $n \geq 1$.*

Proof. Assume that $xy \in I$. Then $\frac{x^4}{xy} \cdot \frac{y^4}{x^2y^2} = xy \in I$. But I is pseudo-powerful in R so that either $\frac{x^{4n}}{x^n y^n} \in R$ or $\frac{y^{4n}}{x^{2n-1}y^{2n-1}} = \frac{y^{4n}}{x^{2n}y^{2n}} \cdot xy \in R$ for some $n \geq 1$. Therefore either $x^{4n} \in I$ or $y^{4n} \in I$, as desired. □

Proof of Proposition 1.10. The first assertion follows from Lemma 1.11. The “in particular” assertion is straightforward. This completes our proof. □

In spite of Proposition 1.10, the radical of a pseudo-powerful ideal needs not be pseudo-powerful, as the following example shows.

Example 1.12. Let $\mathbb{Q}$ be the field of rational numbers and $F = \mathbb{Q}(\sqrt{2})$. Set $S = \mathbb{Q} + \mathbb{Q}X + X^2F[[X]]$, $M = \mathbb{Q}X + X^2F[[X]]$ and $I = X^2F[[X]]$. Then I is a pseudo-powerful ideal of S. However, by [6, Example 4.8], $\sqrt{I} = M$ is not a pseudo-strongly prime ideal of S.

It is well known that the radical of a proper powerful ideal is strongly prime if and only if it is strongly radical [7, Proposition 1.12]. We next show that this fact remains valid if the hypothesis “powerful ideal” is weakened to “pseudo-powerful ideal”.

Proposition 1.13. *Let R be an integral domain, K the quotient field of R and I be a proper pseudo-powerful ideal of R. Then, $\sqrt{I}$ is strongly prime in R if and only if $\sqrt{I}$ is strongly radical in R. In particular, if R is rooty, then $\sqrt{I}$ is strongly prime in R.*

Proof. If $\sqrt{I}$ is strongly prime in R, then $\sqrt{I}$ is strongly radical in R. Conversely, assume that $\sqrt{I}$ is strongly radical in R. To show that $\sqrt{I}$ is strongly prime in R, let $xy \in \sqrt{I}$ with x, y elements of K. Then Proposition 1.10 yields $x^n \in I$ or $y^n \in I$ for some positive integer $n = n(x, y)$. But $\sqrt{I}$ is strongly radical in R, so that $x \in \sqrt{I}$ or $y \in \sqrt{I}$, as desired. The “in particular” assertion is straightforward. This completes our proof. □

It is known [7, Theorem 1.5 (2)] that a powerful ideal of R is comparable to any prime ideal. For pseudo-powerful ideals, we have the following:

Proposition 1.14. *Let R be an integral domain, I be a proper pseudo-powerful ideal of R and P a prime ideal of R. Then, either $I \subseteq P$ or $P \subseteq \sqrt{I}$.*

Proof. Deny. Then, by Proposition 1.10, P and $\sqrt{I}$ are two incomparable prime ideals of R. Therefore Proposition 1.2 yields $I \subseteq P$, a contradiction. □

Recall that [8] that an integral domain R is called *a divided domain* if each prime ideal P of R is *divided*, in the sense that P is comparable to each ideal of R. We have been unable to determine whether "a pseudo-powerful ideal of R is comparable to any prime ideal". Note that, this question was conjectured for PAVDs by Dobbs [9] as follows: whether "an PAVD is a divided domain" (since each proper ideal of a PAVD is pseudo-powerful by Proposition 1.4).

We next consider the stability of pseudo-powerful ideals under passage to homomorphic images (cf. [7, Proposition 1.2]).

Proposition 1.15. *Let R be an integral domain and $P \subseteq I$ be ideals of R with P prime. If I is pseudo-powerful in R, then I/P is pseudo-powerful in R/P.*

Proof. This is straightforward by Lemma 1.1 (3). □

Remark 1.16. The converse of Proposition 1.15 fails in general. For example, let F be a field, $H = F(X)$ be the quotient field of $F[X]$, $R = F + YH[[Y]]$ and $M = YH[[Y]]$. Now, R is not an AVD (since for every $n \geq 1$ neither X^n nor X^{-n} belongs to R). However, R/M is a field.

Let I be a pseudo-powerful ideal of an integral domain R. We next examine the extension of I in an overring of R (cf. [7, Propositions 1.13, 1.17 and 1.18]).

Proposition 1.17. *Let R be an integral domain, T be an overring of R and I be a pseudo-powerful ideal of R. Then, IT is a pseudo-powerful ideal of T. In particular, if $IT = T$, then T is an AVD.*

Proof. Let $x \in E(T)$. Then $x \in E(R)$. As I is pseudo-powerful in R, Lemma 1.1 (2) yields $x^{-n}I \subseteq R$ for some $n \geq 1$. Therefore $x^{-n}IT \subseteq T$ so that IT is pseudo-powerful in T, again by Lemma 1.1 (2), as desired. The "in particular" assertion is straightforward. This completes our proof. □

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