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LINEAR DIFFERENTIAL POLYNOMIALS SHARING A SET IM

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Abstract: In the paper, we study the uniqueness of meromorphic functions when linear differential polynomials of some power of two meromorphic functions share a set of roots of unity ignoring multiplicities. Our results are inspired by a recent work of Lahiri and Sinha [Commun. Korean Math. Soc. 35(2020), no. 3. 773-787].

1 Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [6, 13]. Let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $(r \to \infty, r \notin E)$.

A recent increment to uniqueness theory has to considering weighted sharing instead of sharing IM or CM, this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2000, which measure how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

Definition 1.1. [8, 9] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [8] For $S \subset \mathbb{C} \cup \{\infty\}$ we define $E_k(S; f) = \cup_{a \in S} E_k(a; f)$, where k is a non negative integer or infinity. If $E_k(S; f) = E_k(S; g)$, then we say that f and g share the set S with weight k and we write f and g share (S, k).

In 1996-1997, Fang and Hua [5], Yang and Hua [12] proved the following uniqueness theorem for nonconstant entire functions when the first derivative of some power of those share a nonzero value.

Theorem A. Let f and g be two nonconstant entire functions and $n(\geq 6)$ be a positive integer. If $\left(\frac{f^{n+1}}{n+1}\right)'$ and $\left(\frac{g^{n+1}}{n+1}\right)'$ share 1 CM, then either $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f = \omega g$ for a constant ω such that $\omega^{n+1} = 1$.

In 2002, Fang [4] improved Theorem A by generalizing the order of the derivative. Here is the result.

Theorem B. Let f and g be two nonconstant entire functions and let $n,\ k$ be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = \omega g$ for a constant ω such that $\omega^n = 1$.

Yang and Hua [12] also proved that Theorem A is valid for nonconstant meromorphic functions f and g, provided $n \geq 11$. In 2018, An and Khoai [1] improved that result of Yang and Hua [12] and above theorem by considering some set sharing instead of value sharing. Their result is as follows.

Theorem C. Let f and g be two nonconstant meromorphic functions. Suppose that $n,\ d,\ k$ be positive integers with $n > 2k + \frac{2k+8}{d}$ and $d \geq 2$. If $E(S,(f^n)^{(k)}) = E(S,(g^n)^{(k)})$, where $S = \{z \in \mathbb{C} : z^d = 1\}$, then one of the following holds:

(i) $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three constants satisfying $(-1)^{dk}(c_1c_2)^{nd}(nc)^{2dk} = 1$;

(ii) $f = \omega g$ for a constant ω such that $\omega^{nd} = 1$.

Recently, Lahiri and Sinha [11] found some gap in the proof of Theorem C (Theorem 1 of [1]) and also in the proof of Theorem 2 of [2]. They also suggested some corrections for the same. In [11] Lahiri and Sinha improved Theorem C by relaxing the nature of sharing and using a particular type of linear differential polynomial defined as follows.

Definition 1.3. [11] Let f be a nonconstant meromorphic function. Then $L(f)$ a differential polynomial of the following form: $L(f) = f^{(l)}$ for $l = 1,\ 2,\ 3$ and $L(f) = \sum_{j=1}^{l-3} a_j f^{(j)} + f^{(l)}$ for $l \geq 4$, where $a_1,\ a_2,\ \cdots, a_{l-3}$ are constants.

The following are the results of Lahiri and Sinha [11].

Theorem D. Let f and g be two nonconstant meromorphic functions sharing $(\infty, 0)$ and $n,\ d,\ l$ be positive integers with $n > 2l + \frac{2l+8}{d}$ and $d \geq 2$. Let $S = \{a \in \mathbb{C} : a^d = 1\}$. If $L(f^n)$ and $L(g^n)$ share $(S, 2)$, then one of the following holds:

(i) $L(f^n) = \omega L(g^n)$, where $\omega^d = 1$;

(ii) $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three constants satisfying $(c_1c_2)^n\{A\sum_{j=1}^{l-3} a_j(nc)^j + (nc)^l\}\{A\sum_{j=1}^{l-3} a_j(-nc)^j + (-nc)^l\} = \omega$ and $\omega^d = 1$ and $A = 0$ if $l = 1,\ 2,\ 3$ and $A = 1$ if $l \geq 4$.

Theorem E. Let f and g be two nonconstant meromorphic functions sharing $(\infty, 0)$ and $n,\ d,\ l$ be positive integers with $n > \max\{3, 2l + \frac{2l+8}{d}\}$ and $d \geq 2$. Let $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(f^n)^{(l)}$ and $(g^n)^{(l)}$ share $(S, 2)$, then one of the following holds:

(i) $f = \omega g$ for a constant ω such that $\omega^{nd} = 1$.

(ii) $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three constants satisfying $(-1)^l(c_1c_2)^n(nc)^{2l} = \omega$ and $\omega^d = 1$.

Regarding Theorems D and E it is natural to ask the following question.

Question 1.1. What will happen if we relax the nature of sharing of the set in Theorems D and E?

We concentrate our attention to the above question and provide possible answers in this direction. We now state our main results of the paper.

Theorem 1.4. *Let f and g be two nonconstant meromorphic functions sharing $(\infty, 0)$ and $n,\ d,\ l$ be positive integers with $n > 2l + \frac{8l+14}{d}$ and $d \geq 2$. Let $S = \{z \in \mathbb{C} : z^d = 1\}$. If $L(f^n)$ and $L(g^n)$ share $(S, 0)$, then one of the following holds :*

(i) $L(f^n) = \omega L(g^n)$ *where* $\omega^d = 1$,

(ii) $f(z) = c_1 exp(cz)$ *and* $g(z) = c_2 exp(-cz)$, *where* $c_1,\ c_2$ *and* c *are three constants satisfying* $(c_1c_2)^n\{A\sum_{j=1}^{l-3} a_j(nc)^j + (nc)^l\}\{A\sum_{j=1}^{l-3} a_j(-nc)^j + (-nc)^l\} = \omega$ *and* $\omega^d = 1$ *and* $A = 0$ *if* $l = 1,\ 2,\ 3$ *and* $A = 1$ *if* $l \geq 4$.

Theorem 1.5. *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(\infty, 0)$ *and* $n,\ d,\ l$ *be positive integers with* $n > \max\{3, 2l + \frac{8l+14}{d}\}$ *and* $d \geq 2$. *Let* $S = \{z \in \mathbb{C} : z^d = 1\}$. *If* $(f^n)^{(l)}$ *and* $(g^n)^{(l)}$ *share* $(S, 0)$, *then one of the following holds :*

(i) $f = \omega g$ *for a constant* ω *such that* $\omega^{nd} = 1$,

(ii) $f(z) = c_1 exp(cz)$ *and* $g(z) = c_2 exp(-cz)$, *where* $c_1,\ c_2$ *and* c *are three constants satisfying* $(-1)^l(c_1c_2)^n(nc)^{2l} = \omega$ *and* $\omega^d = 1$.

Though the standard definitions and notations of the value distribution theory are available in [6, 13], we give following definitions and notations used in this paper.

Definition 1.6. [7] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer p, we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function.

Analogously we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 1.7. [10] Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.8. [15] Let f and g be two nonconstant meromorphic functions sharing a value $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f of order p and an a-point of g of order q. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the counting function of those common a-points of f and g where $p > q$ ($q > p$), each a-point being counted once only.

Definition 1.9. [6, 8] Let f and g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those common a-points of f and g whose multiplicities are not the same. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

Now we give some lemmas which will be needed to prove the results of this paper.

Lemma 2.1. *[6, p. 43] Let* f *be a transcendental meromorphic function and let* $a_1,\ a_2, \cdots, a_q$ *be* q *distinct points in* $\mathbb{C} \cup \{\infty\}$. *Then*

$$(q-2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, a_j; f) + S(r, f).$$

Lemma 2.2. *[3] Let* f *and* g *be two nonconstant meromorphic functions sharing* $(1, 0)$. *Then one of the following cases holds.*

(i) $$T(r, f) + T(r, g) \leq 2\{N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g)\} + 3\overline{N}_*(r, 1; f, g) + S(r, f) + S(r, g),$$

(ii) $f = \frac{(B+1)g + (A-B-1)}{Bg + (A-B)}$, *for some constants* $A,\ B$.

Lemma 2.3. *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(1, 0)$. *Then*

$$\overline{N}_*(r, 1; f, g) \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g).$$

Proof. The results follows from Lemma 8 of [14] and Definition 1.9. □

Lemma 2.4. *[6, p. 55] Let h be a nonconstant meromorphic function. Then*

$$T\left(r, L(h)\right) = (l+1)T(r,h) + S(r,h).$$

Lemma 2.5. *[11] Let f be a nonconstant meromorphic function and $n,\ l$ be two positive integers with $n > 2l$. Then*

(i) $(n-2l)T(r,f) + lN(r,\infty;f) + N\left(r, \frac{f^{n-l}}{L(f^n)}\right) \leq T(r, L(f^n)) + S(r,f)$,

(ii) $N\left(r, \frac{f^{n-l}}{L(f^n)}\right) \leq lN(r,0;f) + l\overline{N}(r,\infty;f) + S(r,f)$.

Lemma 2.6. *[11] Let f be a nonconstant meromorphic function and n, l be two positive integers with $n \geq l+1$. Then*

$$T(r,f) \leq T(r, L(f^n)) + S(r,f).$$

Lemma 2.7. *[11] Let f and g be two nonconstant meromorphgic functions sharing $(\infty, 0)$ and such that $L(f^n)L(g^n) = \omega$, where ω is a nonzero constant and $n \geq l+1$. Then*

$$f(z) = c_1 \exp(cz) \text{ and } g(z) = c_2 \exp(-cz),$$

where $c_1,\ c_2$ and c are three constants satisfying

$$(c_1c_2)^n\{A\sum_{j=1}^{l-3} a_j(nc)^j + (nc)^l\}\{A\sum_{j=1}^{l-3} a_j(-nc)^j + (-nc)^l\} = \omega,$$

with $A = 0$ if $l = 1,\ 2,\ 3$ and $A = 1$ if $l \geq 4$.

Lemma 2.8. *[11] Let f be a nonconstant meromorphic function and $n,\ l$ be two positive integers with $n \geq l+2$. If $a \in \mathbb{C} - \{0\}$, then*

$$\frac{n-l-2}{n+l}T(r,f) \leq \overline{N}\left(r, \frac{1}{L(f^n)-a}\right) + S(r,f).$$

3 PROOF OF THE THEOREMS

Proof of Theorem 1.4. Let $F = \{L(f^n)\}^d$ and $G = \{L(g^n)\}^d$. Also let S be the set given by $\{z \in \mathbb{C} : z^d = 1\}$ and $\mu_i,\ i = 1,\ 2, \cdots, d$ are roots of $z^d = 1$. We define

$$E_0(\{L(f^n)\}^d, 1) = \cup_{i=1}^d E_0(L(f^n), \mu_i) \text{ and } E_0(\{L(g^n)\}^d, 1) = \cup_{i=1}^d E_0(L(g^n), \mu_i).$$

From Lemma 2.8 we can see that $E_0(\{L(f^n)\}^d, 1)$ and $E_0(\{L(g^n)\}^d, 1)$ are non empty. Since $L(f^n)$ and $L(g^n)$ share $(S, 0)$, we get F and G share $(1, 0)$. From Lemma 2.2 we now consider the following two cases:

Case 1. Let

$$\begin{aligned} T(r,F) + T(r,G) &\leq 2\{N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G)\} \\ &+ 3\overline{N}_*(r,1;F,G) + S(r,F) + S(r,G) \end{aligned} \tag{3.1}$$

Since $d \geq 2$, we get

$$N_2(r,\infty;F) = 2\overline{N}(r,\infty;F) = 2\overline{N}(r,\infty;f)$$

$$\text{and } N_2(r,\infty;G) = 2\overline{N}(r,\infty;G) = 2\overline{N}(r,\infty;g); \tag{3.2}$$

$$N_2(r,0;F) = 2\overline{N}(r,0;F) \text{ and } N_2(r,0;G) = 2\overline{N}(r,0;G). \tag{3.3}$$

From Lemma 2.3 we get,

$$\begin{aligned}\overline{N}_*(r,1;F,G) &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &+ S(r,F)+S(r,G). \end{aligned} \tag{3.4}$$

Now

$$\begin{aligned}\overline{N}(r,0;F)=\overline{N}(r,0;L(f^n)) &\leq \overline{N}\left(r,\frac{f^{n-l}}{L(f^n)}\right)+\overline{N}(r,0;f^{n-l})\\ &= \overline{N}\left(r,\frac{f^{n-l}}{L(f^n)}\right)+\overline{N}(r,0;f) \end{aligned} \tag{3.5}$$

Similarly,

$$\overline{N}(r,0;G)\leq \overline{N}\left(r,\frac{g^{n-l}}{L(g^n)}\right)+\overline{N}(r,0;g). \tag{3.6}$$

From Lemmas 2.4 and 2.6 we get,

$$S(r,F)=S(r,L(f^n))=S(r,f) \text{ and } S(r,G)=S(r,L(g^n))=S(r,g). \tag{3.7}$$

Combining (3.1) - (3.7) we get,

$$\begin{aligned}T(r,F)+T(r,G) &\leq 7\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &+ 7\left\{\overline{N}\left(r,\frac{f^{n-l}}{L(f^n)}\right)+\overline{N}\left(r,\frac{g^{n-l}}{L(g^n)}\right)\right\}+S(r,f)+S(r,g). \end{aligned} \tag{3.8}$$

For $n>2l$ and $d\geq 2$, using (1) of Lemma 2.5, we get from (3.8)

$$\begin{aligned}d(n-2l)\{T(r,f)+T(r,g)\} &\leq 7\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &+ 7\left\{\overline{N}\left(r,\frac{f^{n-l}}{L(f^n)}\right)+\overline{N}\left(r,\frac{g^{n-l}}{L(g^n)}\right)\right\}\\ &- d\left\{N\left(r,\frac{f^{n-l}}{L(f^n)}\right)+N\left(r,\frac{g^{n-l}}{L(g^n)}\right)\right\}\\ &- dl\{N(r,\infty;f)+N(r,\infty;g)\}+S(r,f)+S(r.g)\\ &\leq 7\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &+ 5\left\{\overline{N}\left(r,\frac{f^{n-l}}{L(f^n)}\right)+\overline{N}\left(r,\frac{g^{n-l}}{L(g^n)}\right)\right\}\\ &- dl\{N(r,\infty;f)+N(r,\infty;g)\}+S(r,f)+S(r.g) \end{aligned} \tag{3.9}$$

Now using (2) of Lemma 2.5, from (3.9) we get,

$$\begin{aligned}d(n-2l)\{T(r,f)+T(r,g)\} &\leq 7\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &+ 5l\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+N(r,0;f)+N(r,0;g)\}\\ &- dl\{N(r,\infty;f)+N(r,\infty;g)\}+S(r,f)+S(r.g)\\ &\leq (7+3l)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\ &+ (7+5l)\{N(r,0;f)+N(r,0;g)\}+S(r,f)+S(r.g)\\ &\leq (14+8l)\{T(r,f)+T(r,g)\}+S(r,f)+S(r.g). \end{aligned}$$

Therefore $n\leq 2l+\frac{8l+14}{d}$. This is a contradiction with the assumption that $n>2l+\frac{8l+14}{d}$.

Case 2. Let

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)}. \tag{3.10}$$

Now we consider the following two subcases.

Subcase 2.1. Let $A = B\ (\neq 0)$. Then from (3.10) we get,

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.11}$$

If $B = -1$, then from (3.11) we get $F.G = 1$, i.e., $L(f^n)L(g^n) = \alpha$, where α is a constant such that $\alpha^d = 1$. Then for $n \geq l+1$ from Lemma 2.7 we get $f(z) = c_1 exp(cz)$ and $g(z) = c_2 exp(-cz)$, where $c_1,\ c_2$ and c are three constants satisfying

$$(c_1c_2)^n\{A\sum_{j=1}^{l-3} a_j(nc)^j + (nc)^l\}\{A\sum_{j=1}^{l-3} a_j(-nc)^j + (-nc)^l\} = \omega$$

where $\omega^d = 1$, $A = 0$ if $l = 1,\ 2,\ 3$ and $A = 1$ if $l \geq 4$.

If $B \neq -1$, then from (3.11) we get

$$\frac{1}{F} = \frac{BG}{(B+1)G-1} \quad \text{and} \quad G = \frac{-1}{BF-(B+1)}.$$

Therefore,

$$\overline{N}(r,0;F) = \overline{N}\left(r, \frac{1}{B+1}; G\right) \quad \text{and} \quad \overline{N}(r,\infty;G) = \overline{N}\left(r, \frac{B+1}{B}; F\right). \tag{3.12}$$

Using (3.12) and from the second fundamental theorem of Nevanlinna we get,

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}\left(r, \frac{B+1}{B}; F\right) + \overline{N}(r,\infty;F) + S(r,F) \\ &= \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,\infty;F) + S(r,F) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{B+1}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &= \overline{N}(r,0;G) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + S(r,G). \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\begin{aligned} T(r,F) + T(r,G) &\leq 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) \\ &+ S(r,F) + S(r,G). \end{aligned} \tag{3.15}$$

Now since $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;f)$ and $\overline{N}(r,\infty;G) = \overline{N}(r,\infty;g)$, using (3.5), (3.6), (3.7) and (3.15) we get,

$$\begin{aligned} T(r,F) + T(r,G) &\leq \left\{2\overline{N}\left(r, \frac{f^{n-l}}{L(f^n)}\right) + \overline{N}\left(r, \frac{g^{n-l}}{L(g^n)}\right)\right\} + \{2\overline{N}(r,0;f) + \overline{N}(r,0;g)\} \\ &+ \{\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g)\} + S(r,f) + S(r,g). \end{aligned} \tag{3.16}$$

For $n > 2l$ and $d \geq 2$, using (1) of Lemma 2.5 we get from (3.16)

$$\begin{aligned} d(n-2l)\{T(r,f) + T(r,g)\} &\leq \{\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g)\} - dl\{N(r,\infty;f) + N(r,\infty;g)\} \\ &+ \left\{2\overline{N}\left(r, \frac{f^{n-l}}{L(f^n)}\right) - dN\left(r, \frac{f^{n-l}}{L(f^n)}\right)\right\} \\ &+ \left\{\overline{N}\left(r, \frac{g^{n-l}}{L(g^n)}\right) - dN\left(r, \frac{g^{n-l}}{L(g^n)}\right)\right\} \\ &+ \{2\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + S(r,f) + S(r,g) \\ &\leq 2\overline{N}(r,0;f) + \overline{N}(r,0;g) + S(r,f) + S(r,g) \\ &\leq 2T(r,f) + T(r,g) + S(r,f) + S(r,g). \end{aligned}$$

Therefore $n \leq 2l + \frac{2}{d}$. This is a contradiction with the assumption that $n > 2l + \frac{8l+14}{d}$.

Subcase 2.2. Let $A \neq B$.

If $B \neq 0$, then from (3.10) we get $\overline{N}(r,\infty;F) = \overline{N}(r, \frac{B-A}{B}; G)$. Also from (3.10) we can write

$$G = \frac{(B-A)F + (A-B-1)}{BF-(B+1)}.$$

From this it follows that $\overline{N}(r,\infty;G) = \overline{N}(r, \frac{B+1}{B}; F)$.
Therefore from the second fundamental theorem of Nevanlinna we get,

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0,F) + \overline{N}\left(r, \frac{B+1}{B}; F\right) + \overline{N}(r,\infty;F) + S(r,F) \\ &= \overline{N}(r,0,F) + \overline{N}(r,\infty;G) + \overline{N}(r,\infty;F) + S(r,F) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0,G) + \overline{N}\left(r, \frac{B-A}{B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &= \overline{N}(r,0,G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,G). \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18) we get,

$$\begin{aligned} T(r,F)+T(r,G) &\leq \{\overline{N}(r,0,F)+\overline{N}(r,0,G)\} + 2\{\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\} \\ &+ S(r,F)+S(r,G). \end{aligned} \tag{3.19}$$

Since $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;f)$ and $\overline{N}(r,\infty;G) = \overline{N}(r,\infty;g)$, using (3.5), (3.6), (3.7) and (3.19) we get,

$$\begin{aligned} T(r,F)+T(r,G) &\leq \left\{\overline{N}\left(r, \frac{f^{n-l}}{L(f^n)}\right) + \overline{N}\left(r, \frac{g^{n-l}}{L(g^n)}\right)\right\} + \{\overline{N}(r,0;f)+\overline{N}(r,0;g)\} \\ &+ 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} + S(r,f)+S(r,g). \end{aligned} \tag{3.20}$$

For $n > 2l$ and $d \geq 2$, using (1) of Lemma 2.5, we get from (3.20)

$$\begin{aligned} d(n-2l)\{T(r,f)+T(r,g)\} &\leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} - dl\{N(r,\infty;f)+N(r,\infty;g)\} \\ &+ \left\{\overline{N}\left(r, \frac{f^{n-l}}{L(f^n)}\right) + \overline{N}\left(r, \frac{g^{n-l}}{L(g^n)}\right)\right\} \\ &- d\left\{N\left(r, \frac{f^{n-l}}{L(f^n)}\right) + N\left(r, \frac{g^{n-l}}{L(g^n)}\right)\right\} \\ &+ \{\overline{N}(r,0;f)+\overline{N}(r,0;g)\} + S(r,f)+S(r,g) \\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+S(r,f)+S(r,g) \\ &\leq T(r,f)+T(r,g)+S(r,f)+S(r,g). \end{aligned}$$

Therefore $n \leq 2l + \frac{1}{d}$. This is a contradiction with the assumption $n > 2l + \frac{8l+14}{d}$.

If $B = 0$, then from (3.10) we get,

$$AF = G + (A-1) \;\text{ and }\; G = AF - (A-1). \tag{3.21}$$

If $A \neq 1$, then $\overline{N}(r,0;F) = \overline{N}(r, 1-A; G)$ and $\overline{N}(r,0;G) = \overline{N}\left(r, \frac{A-1}{A}; F\right)$. Proceeding as above and using Nevanlinna second fundamental theorem we get,

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\{\overline{N}(r,0,F)+\overline{N}(r,0,G)\} + \{\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\} \\ &+ S(r,F)+S(r,G). \end{aligned} \tag{3.22}$$

Since $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;f)$ and $\overline{N}(r,\infty;G) = \overline{N}(r,\infty;g)$, using (3.5), (3.6), (3.7), (3.22) and (1) of Lemma 2.5 we get,

$$\begin{aligned} d(n-2l)\{T(r,f)+T(r,g)\} &\leq 2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g) \\ &\leq 2\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \end{aligned}$$

Therefore $n \leq 2l + \frac{2}{d}$, which is again a contradiction.

If $A = 1$, then from (3.21) we get $F = G$, i.e., $L(f^n) = \omega L(g^n)$, where ω is a constant such that $\omega^d = 1$. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. The proof of this theorem is similar to the proof of Theorem 1.2 of [11]. Hence we omit the proof. □

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