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NON-PLANARITY USING CYCLES

Jasmine Mathew and K.Reji Kumar

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Abstract This article proposes a method to examine whether a given undirected simple graph is non-planar using Cycles. The cyclic rotation of the available cycle produces a new cycle. The non-planarity can be fixed depending on whether both cycles are subgraphs of the given graph. Pseudocode for setting up an algorithm for detecting non-planarity for any graph containing at-least one cycle is the highlight of the article.

1 Introduction

Some graphs are drawn in a plane such that any intersection of edges occurs only at the vertices. Such graphs are called planar graphs. But it is not possible to depict every graph without any edge crossing. The problem of determining whether a graph is planar or not has many practical applications, such as in VLSI design [1]. Danny Dolev et al. [2] provides details of some applications.

Planar drawings represent Integrated circuit layouts to the patterns of semiconductor or metal oxide; which are the components of an integrated circuit. The models are usually large graphs. For large n, finding non-planar subgraphs will help to decompose graphs into planar subgraphs. The union of C and C^r is a minimal non-planar graph. Detection of the minimal non-planar sub-graph will simplify the process of planar graph decomposition. Planar Decomposition is the partitioning of the edge set so that each subgraph induced by the corresponding edge set is a planar graph. Isomorphic decomposition is the partitioning of the edge set such that each induced subgraphs are isomorphic to the other. If each subgraph is a path the decomposition is path decomposition. Analogously if each subgraph is a star the decomposition is a star decomposition. The path and star decomposition of Fibonacci graphs are given in [3]. The crossing number of a graph is closely associated with the nonplanarity of a graph. It is the minimum number of crossings in an optimal drawing of a graph. The pseudocode will help to improve the available results in the literature regarding crossing numbers. The determination of crossing numbers in Complete Graphs is of great interest [4]. Several graph parameters exist in the literature to analyze graph properties. A survey on various graph parameters can be found in [5]

Many studies have been done on the verification of planarity. In some problems, it is sufficient to check whether the given graph is non-planar. For testing the non-planarity, Euler's theorem [6] can be used. According to the theorem, $m \leq 3n-6$ for every planar graph and for bipartite planar graph $m \leq 2n-4$, where n is the number of vertices and m is the number of edges in G. This condition is necessary but not sufficient. So in the case where the above conditions are satisfied, the determination of non-planarity can be done with the well-known Kuratowski's Theorem[6]. The theorem states that a graph is planar if and only if it does not have K_5 and $K_{3,3}$ as topological minors. But, to find a subgraph homeomorphic to K_5 or $K_{3,3}$ is not easy. This article proposes an easy method finding a subgraph homeomorphic to K_5 or $K_{3,3}$ by construction. A graph is tested if it contains a cycle of size greater than or equal to five. The permutation multiplication of cycles proposed by Walecki [7] is used.

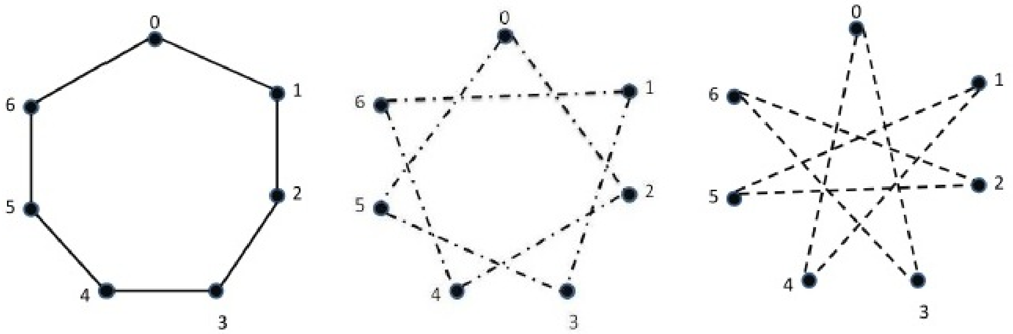

Figure 1. C_7, C_7^2 and C_7^3

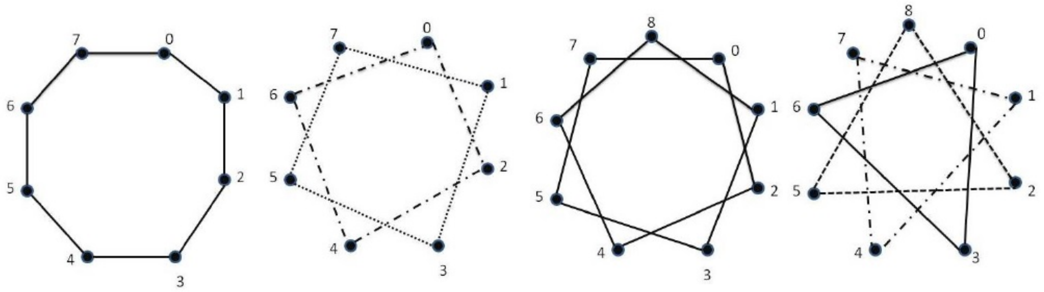

Figure 2. C_8, C_8^2, C_9^2 and C_9^3

2 Definition and Terminology

Since the planarity is not affected by loops and parallel edges, this paper discusses only the undirected simple graphs. The readers are expected to be familiar with the elementary graph-theoretical terms, such as subgraphs, paths, cycles, the union of graphs, etc. For basic definitions refer [6], [1]. C represents a cycle according to the context.

Permutation multiplication of vertex labels of a cycle produces new cycles. Walecki used this method for decomposing complete graphs into Hamilton cycles.

Definition 2.1 (Permutation Multiplication). :
A cycle of size p in a graph G is expressed using a permutation of vertex labels $0, 1, 2 \ldots p-1$. That is $C_p = (0\ 1\ 2\ \ldots\ p-1)$. Then permutation multiplication of two cycles C_p and C_p represented by, $C_p \circ C_p = C_p^2$ is given by, $C_p \circ C_p = (0\ 1\ 2\ \ldots\ p-1) \circ (0\ 1\ 2\ \ldots\ p-1) = (0\ 2\ 4\ \ldots\ p-2\ 1\ 3\ \ldots\ p-1)$ Similarly C_p^r is defined as $C_p^r = C_p \circ C_p \circ \ldots C_p$ (r- times).

The new graph produced has the same vertex set as C_p; hence for $C_p \bigcup C_p^2$.

Remark 2.2. From Figure 1 we observe that C_7^2 is a 7-cycle. From Figure 2, it is clear that C_8^2 is not a cycle, but the union of two, 4-cycles and C_9^2 is a 9-cycle, but C_9^3 is the union of three, 3-cycles.

3 Main Results

This section shows that the union of an odd cycle and its square is non-planar. Hence if an odd cycle and its square are in G, then G is non-planar.

Theorem 3.1. *Let G be a graph containing C_{2n+1}, where C_{2n+1} is an odd cycle for $n \geq 2$. If $C_{2n+1} \bigcup C_{2n+1}^2 \subseteq G$, then G is non-planar.*

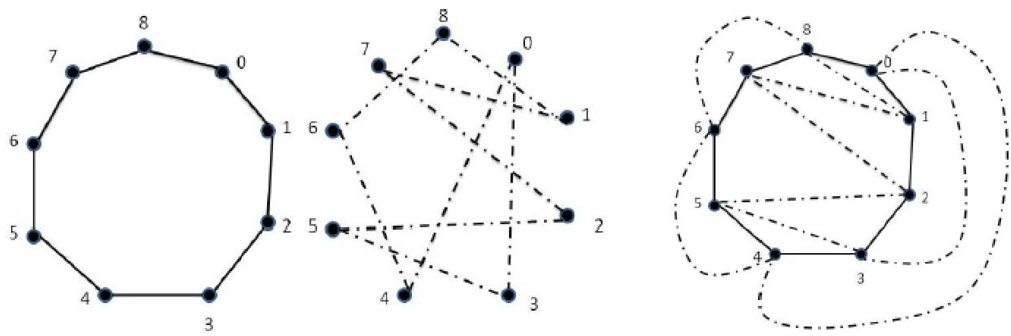

Figure 3. C_9,$C_9^{'}$ and $C_9 \bigcup C_9^{'}$

Proof. Let G be a graph and C_{2n+1} be an odd cycle[7], $C_{2n+1} = (\ 0\ 1\ 2\ \ldots\ 2n)$. Then, $C_{2n+1}^2 = C_{2n+1}\, o\, C_{2n+1} = (0\,1\,2\ \ldots\ 2n) \circ (0\,1\,2\ \ldots\ 2n) = (0\,2\,4\ \ldots\ 2n\ \ 1\,3\ \ldots\ 2n-1\,0)$ which is a cycle in G. Consider the graph $C_{2n+1} \bigcup C_{2n+1}^2$, We draw $C_{2n+1} \bigcup C_{2n+1}^2$ in a plane as follows. Since C_{2n+1} is a cycle with a length of $2n+1$ it can be embedded on a plane without crossing any edges. Draw C_{2n+1}^2 on C_{2n+1} to produce $C_{2n+1} \bigcup C_{2n+1}^2$ according to the following rule. Without loss of generality, draw the n edges $(0,2),(2,4),\ldots,(2n-2,2n)$ inside C_{2n+1}, which will not produce any edge crossing, so that the resulting graph is planar. Then draw n edges $(2n,1),(1,3),(3,5),\ldots,(2n-3,2n-1)$ outside C, which is also possible without any edge crossing, hence the resulting graph is planar. One more edge $(2n-1,0)$ is remaining to produce $C_{2n+1} \bigcup C_{2n+1}^2$, If $(2n-1,0)$ is drawn inside C_{2n+1}, it will meet $(2n-3,2n-1)$, otherwise, it will meet $(2n,1)$. Hence $C_{2n+1} \bigcup C_{2n+1}^2$ is non-planar. If $C_{2n+1} \bigcup C_{2n+1}^2 \subseteq G$, then G is also non-planar. □

Arbitrary selection of two cycles with the same vertex set need not produce a non-planar subgraph.

Example 3.2. The union of two distinct odd cycles of G with the same vertex set can be planar.

Proof. Consider the odd cycle C_9, labeled as $C_9 = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$. Another odd cycle C_9^1 with same vertices, labeled as $C_9^{'} = (0\,3\,5\,2\,7\,1\,8\,6\,4)$. The planar embedding of $C_9 \cup C_9^{'}$ shows that the arbitrary selection of two cycles need not produce a non-planar sub graph. □

An immediate consequence of theorem 3.1 is that the complement of an odd cycle is always non-planar when the size is greater than or equal to 7.

Corollary 3.3. C_{2n+1}^c *is non-planar for* $2n+1 \geq 7$.

Proof. K_{2n+1} can be decomposed into n cycles [7], also these cycles can be expressed as $C_{2n+1}, C_{2n+1}^2, C_{2n+1}^3, \ldots, C_{2n+1}^n$ by Walecki's construction method. Thus C_{2n+1}^c, the complement of C_{2n+1} can be decomposed into $n-1$ cycles. If $2n+1 \geq 7$, C_{2n+1}^c can be decomposed into two or more cycles of the form $C_{2n+1}, C_{2n+1}^2, C_{2n+1}^3, \ldots, C_{2n+1}^{n-1}$. Thus, C_{2n+1}^c is non-planar. □

C_{2n+1}^2 does not need to be always a subgraph of G. But the following Theorem says that if at least one power $C_{2n+1}^r \subseteq G$, for $2 \leq r \leq n$, then also G is non-planar. Here we have an upper bound n for r, since C_{2n+1}^r is produced by permutation multiplication of C_{2n+1}, r-times, we have only n distinct graphs.

Theorem 3.4. *If G be a graph having an odd cycle* C_{2n+1} *with* $n \geq 2$, *if* $C_{2n+1} \bigcup C_{2n+1}^r \subseteq G$ *for some* $2 \leq r \leq n$, *then G is non-planar.*

Proof. Let C_{2n+1} be the cycle, $C_{2n+1} = (0\,1\,2\ \ldots\ 2n)$, $C_{2n+1}^r = C_{2n+1}\, o\, C_{2n+1}\, o\ \ \ldots\ o\, C_{2n+1}$ (r times)

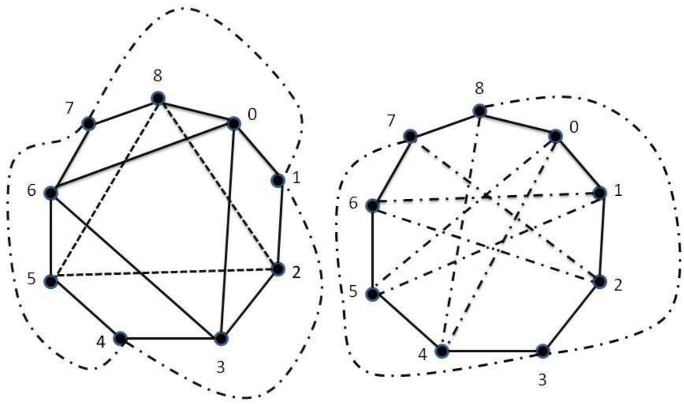

Figure 4. $C_9 \bigcup C_9^3$ and $C_9 \bigcup C_9^4$

Case 1: If $2n+1 = mr$ for some integer m. $C_{2n+1}^r = (0\ r\ 2r\ \dots\ (m-1)r) \bigcup (1\ r+1\ 2r+1 \dots ((m-1)r+1)) \bigcup \dots ((r-1)\ (2r-1)\ \dots\ ((m-1)r+(r-1))) = C_1 \bigcup C_2 \bigcup \dots \bigcup C_r$. Hence C_{2n+1}^r is the union of r, m-cycles. Draw C_{2n+1} in a plane. Draw C_{2n+1}^r on C_{2n+1} according to the following rule. Without loss of generality draw C_1 inside and C_2 outside C_{2n+1}, the planarity of C_{2n+1} will not be affected. Draw C_3 inside C_{2n+1}, the edge $(2, r+2)$ will cut $(0, r)$ and $(r, 2r)$. If C_3 is drawn outside C_{2n+1}, the edge $(2, r+2)$ will cut $(1, r+1)$ and $(r+1, 2r+1)$. Hence $C_{2n+1} \bigcup C_1 \bigcup C_2 \bigcup C_3$ is non-planar and$C_1 \bigcup C_2 \bigcup C_3 \subseteq C_{2n+1}^r$. Then, $C_{2n+1} \bigcup C_{2n+1}^r$ is non-planar. So if $C_{2n+1} \bigcup C_{2n+1}^r \subseteq G$, then G is non-planar.

Case II:If r does not divide $2n+1$, i.e, $r = mr + q$, where m, q are integers such that $0 < q < r$. Then $C_{2n+1}^r = (0\ \ r\ \ 2r\ \ \dots\ \ (m-1)r\ \ mr\ \ (m+1)r\ \ (m+2)r \dots\ (2m+1)r\ \ (2m+2)r\ \ \dots\ \ (3m+1)r\ \ (3m+2)r\ \ \dots\ \ (pm+l)r\ \ \dots\ \ ((m-1)r+q)$ which is a cycle; where $(m+1)r = r-q$, $(m+2)r = 2r-q, \dots\ (2m+1)r = r-2q,\ \ (2m+2)r = 2r-2q,\ \ \dots\ \ (3m+1)r = r-3q,\ \ (3m+2)r = 2r-3q\ \ , \dots\ \ (pm+l)r = lr-pq,\ \dots\ ((m-1)r+q)$. Without loss of generality draw m edges $(0, r), (r, 2r), \dots, ((m-1)r, mr)$ inside, (call the graph produced by these m edges D_1) and the m edges $(mr,\ \ (m+1)r), ((m+1)r,\ \ (m+2)r),\ \ \dots,\ \ ((m+m-1)r,\ \ 2mr)$, outside (call the graph produced by these m edges D_2) C_{2n+1} in a plane. Now if the edge $(2mr,\ \ (2m+1)r)$ is drawn inside C_{2n+1}, it will cut at least $((m-1)r,\ \ mr)$ and if drawn outside C_{2n+1}, it will meet at least $(mr,\ (m+1)r)$. Then $C_{2n+1} \bigcup D_1 \bigcup D_2 \bigcup (2mr,\ \ (2m+1)r)$ is non-planar. Since $D_1 \bigcup D_2 \bigcup (2mr,\ \ (2m+1)r)\ \subseteq C_{2n+1}^r$, $C_{2n+1} \bigcup C_{2n+1}^r$ is non-planar. Hence if $C_{2n+1} \bigcup C_{2n+1}^r \subseteq G$, then G is also non planar. □

For even cycles $C_{2n} \bigcup C_{2n}^2$ is planar. But theorem 3.5 proves that $C_{2n} \bigcup C_{2n}^r$ is non-planar for all $r \geq 3$.

Theorem 3.5. *Let C_{2n} be an even cycle in G, with $n \geq 3$. If $C_{2n} \bigcup C_{2n}^r \subseteq G$ for at least one r such that $3 \leq r \leq n$, then G is non-planar.*

Proof. Let $C_{2n} = (0\ 1\ 2\ \dots\ 2n-1)$. Then $C_{2n}^2 = C_{2n}\ \ o\ \ C_{2n} = (0\ \ 1\ \ 2\ \ \dots\ \ 2n-1) \circ (0\ \ 1\ \ 2\ \ \dots\ \ 2n-1) = (0\ \ 2\ \ 4\ \ \dots\ \ 2n-4\ \ 2n-2) \cup (1\ \ 3\ \ 5\ \ \dots\ \ 2n-1)$. $C \bigcup C_{2n}^2$ is planar. Consider the following cases. $C_{2n}^r = C_{2n}\ \ o\ \ C_{2n}\ \ o\ \ \dots\ \ o\ \ C_{2n}$ (r times)

Case I: If $2n = mr$, then, $C_{2n}^r = (0\ \ r\ \ 2r\ \ \dots\ \ (m-1)r) \bigcup (1\ \ r+1\ \ 2r+1 \dots\ ((m-1)r+1)) \bigcup\ \dots \bigcup ((r-1)\ \ (2r-1)\ \ \dots\ \ ((m-1)r+(r-1))) = C_1 \bigcup C_2 \bigcup \dots \bigcup C_r$. C_{2n}^r is not a cycle, but it is the union of r different m-cycles.

Draw C_{2n} in a plane. Draw C_{2n}^r on C_{2n} according to the following rule. With out loss of generality draw C_1 inside and C_2 outside C_{2n}. Draw C_3 inside C_{2n}, the edge $(2, r+2)$ will

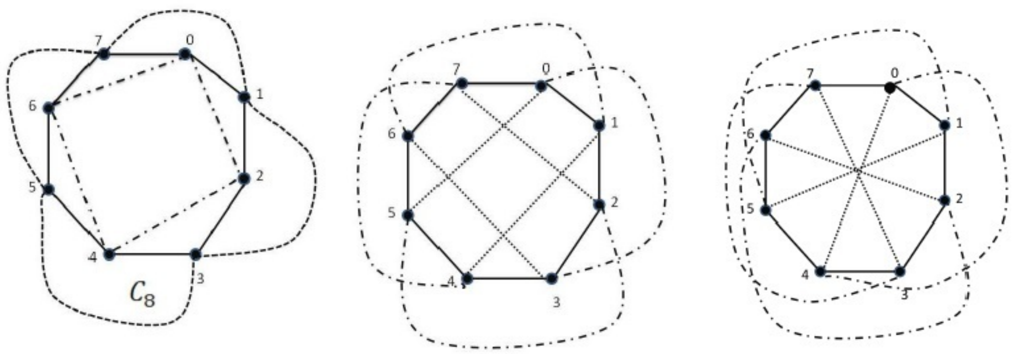

Figure 5. $C_8 \bigcup C_8^2$, $C_8 \bigcup C_8^3$ and $C_8 \bigcup C_8^4$

cut $(0, r)$ and $(r, 2r)$. If C_3 is drawn outside C_{2n}, the edge $(2, r+2)$ will cut $(1, r+1)$ and $(r+1, 2r+1)$. Hence $C_{2n} \bigcup C_1 \bigcup C_2 \bigcup C_3$ is non-planar.
Then, since $C_1 \bigcup C_2 \bigcup C_3 \subseteq C_{2n}^r$, $C_{2n} \bigcup C_{2n}^r$ is non-planar. So if $C_{2n} \bigcup C_{2n}^r \subseteq G$, then G is non-planar.

Case II: If $2n = mr + q$, where $0 < q < r$, then $C_{2n}^r = (0\ \ r\ \ 2r\ \ \ldots\ \ (m-1)r\ \ mr\ \ (m+1)r\ \ (m+2)r\ \ \ldots\ \ (2m+1)r\ \ (2m+2)r\ \ \ldots\ \ (3m+1)r(3m+2)r\ \ \ldots\ \ (pm+l)r\ \ \ldots\ \ ((m-1)r+q))$ which is a cycle; where $(m+1)r = r - q$, $(m+2)r = 2r - q$, $\ldots$ $(2m+1)r = r - 2q$,$(2m+2)r = 2r - 2q$,$\ldots$ $(3m+1)r = r - 3q$, $(3m+2)r = 2r - 3q$, $\ldots$ $(pm+l)r = lr - pq$,$\ldots$ $((m-1)r+q))$. Without loss of generality draw m edges $(0, r), (r, 2r), \ldots, ((m-1)r, mr)$ inside (call the graph produced by these m edges C_1) and the m edges $(mr, (m+1)r), ((m+1)r, (m+2)r), \ldots, ((2m-1)r, 2mr)$, outside (call the graph produced by these m edges C_2) C_{2n} in a plane. If the edge $(2mr, (2m+1)r)$ is drawn inside C_{2n}, it will cut at least $((m-1)r, mr)$ and if it is drawn outside C_{2n}, it will meet at least $(mr, (m+1)r)$.
Then $C_{2n} \bigcup C_1 \bigcup C_2 \bigcup (2mr, (2m+1)r)$ is non-planar. Since $C_1 \bigcup C_2 \bigcup (2mr, (2m+1)r) \subseteq C_{2n}^r$, $C_{2n} \bigcup C_{2n}^r$ is non-planar. Hence if $C_{2n} \bigcup C_{2n}^r \subseteq G$, then G is also non-planar. □

It is nice to have an algorithm for checking non-planarity since real-life problems handle large graphs in VLSI design, networks, etc. This article provides pseudo-code for the algorithm to check whether a given graph is non-planar. The concept of fundamental cycles[8] and their union produce all possible cycles in G. By checking whether the powers of each cycle belong to the graph, we confirm the non-planarity.
We select an arbitrary spanning tree T and collect all edges which are not present in T. Each edge will contribute to one fundamental cycle. The collection of edges that produce one fundamental cycle each is denoted as E_1. Several algorithms exist in literature to generate fundamental cycles in a graph. Any cycle in G can be expressed as the union of fundamental cycles. Hence we search for the existence of all possible unions of cycles in G. Then search for the existence of the powers too. The existence of a cycle and its power will confirm the non-planarity of G. The fundamental cycles are the input of our algorithm. The steps of the algorithm are given below.

(i) Set $j = 0$. Input the fundamental cycles $S = \{C_{0i}$, produced by each edge $e_i \in E_1$; where $1 \leq i \leq m - n + 1$. Go to Step 2.

(ii) Set $i = 1$ and go to step 3.

(iii) $n_i = |C_{ji}|$. If n_i is odd, set $r = 2$, else set $r = 3$. Go to step 4.

(iv) Check $C_{ji} \bigcup C_{ji}^r \subseteq G$. If yes conclude that G is non-planar and exit, else go to step 5.

(v) Set $r = r + 1$. If $r \leq \lfloor \frac{n_i}{2} \rfloor$, go to step 4, else set $i = i + 1$ and go to step 6.

(vi) Check $i \leq m - n + 1-$. If yes, go to step 3, else set $i = 1$ and go to step 7.

(vii) set $j = j + 1$, check $j \leq m - n$. If yes go to Step 8, else exit the algorithm without a conclusion regarding non-planarity.

(viii) Form a new cycle $C_{j\ i} = C_{j-1\ i} \bigcup C_{j-1\ i+1} - (C_{j-1\ 1} \bigcap C_{j-1\ i+1})$.
If $E(C_{j-1\ 1} \bigcap C_{j-1\ i+1}) \neq \phi$, $C_{j\ i}$ is a cycle. go to step 9, else go to Step 10.

(ix) Add $C_{j\ i}$ to S and go to Step 3.

(x) $C_{j\ i}$ is not a cycle. Update $i = i + 1$ and go to step 11.

(xi) Check $C_{j-1\ i+1} \in S$. If yes go to Step 8, else go to step 10.

There are $m - n + 1$ fundamental cycles in G and their unions will produce all cycles in G. A cycle has maximum $\lfloor \frac{n_i}{2} \rfloor$ distinct powers and since $n_i \leq n$, we have maximum $\lfloor \frac{n}{2} \rfloor$ powers for each cycle. Hence the complexity of the algorithm is $O(nm^2)$.

4 Conclusion

In many practical problems determination of non-planarity is a tedious job. There does not exist a fast recognition algorithm for detecting subgraph homeomorphic to K_5 or $K_{3,3}$. This paper opens an easy method by detecting the presence of the union of some cycles and its r^{th} power, for some r. More clearly, any graph G which contains an odd cycle and its square or even cycle and its cube is non planar. In general, every graph G with an odd cycle and its r^{th} power for $r \geq 2$ or even cycle and its r^{th} power for $r > 2$ is non planar. We can observe that $K_5 = C_5 \bigcup C_5^2$ and $K_{3,3} = C_6 \bigcup C_6^3$. $C_{2n+1} \bigcup C_{2n+1}^r$ is homeomorphic to K_5; if $2n+1 = mr+q$ with $0 < q < r$ and $C_{2n+1} \bigcup C_{2n+1}^r$ is homeomorphic to $K_{3,3}$; if $2n+1 = mr$. $C_{2n} \bigcup C_{2n}^r$ is homeomorphic to K_5; if $2n = mr + q$ with $0 < q < r$ and $C_{2n} \bigcup C_{2n}^r$ is homeomorphic to $K_{3,3}$; if $2n = mr$. Thus the theorem plays the role of sufficiency part of the Kurtowski's theorem, but reduces its complexity. Unfortunately it is not necessary that a non planar graph must contain either $C_{2n+1} \bigcup C_{2n+1}^r$ or $C_{2n} \bigcup C_{2n}^r$; Peterson graph is the counter example. The algorithm is needed only when $m \leq 3n - 6$.

References

[1] G. Agnarsson, R. Greenlaw, *Graph Theory, modelling Applications and Algorithms*, Dorling Kindersley (India) Pvt Ltd, (2007).

[2] D. Dolev, T. Leighton, H. Trickey, *Planar Embedding of Planar Graphs*, Advances in Computing Research, vol. 2 pages 147-161, (1984).

[3] K. Reji Kumar, Jasmine Mathew, *On some decomposition problems of Fibonacci graphs, Complete Bipartite Graphs and Complete Graphs*, Mathematical Sciences International Research Journal, Volume 6, Special Issue, pp 39-43 (2017).

[4] F. Harary, A. Hill, *On the number of crossings in a complete graph, Open Graph Problems*, Harary Hills (1963).

[5] Jasmine Mathew, K. Reji Kumar, *Communication Structures, its graph representations and decomposition possibilities*, Journal of Physics: Conference Series, 1706, 012050, (2020), doi:10.1088/1742-6596/1706/1/012050

[6] D.B. West, *Introduction to Graph Theory*, Prentince Hall of India Pvt Ltd, New Delhi, (1999).

[7] B. Alspach, J.C. Bermond, D. Sotteau, *Decomposition in to cycles I: Hamilton Decomposition*, Kluwer Academic Publishers, (1990).

[8] Gross, Jonathan L., Yellen, Jay, *Graphs and Vector Spaces: Section 6 of the fourth chapter of the the book Graph Theory and Its Applications (2nd ed.)*, CRC Press, pp. 197–217, (2005).

Author information

Jasmine Mathew, Department of Basic Sciences, Amal Jyothi College of Engineering, Kanjirappally, Kottayam. Department of Mathematics, Mar Thoma College, Tiruvalla, India.
E-mail: jasminebinu@gmail.com

K.Reji Kumar, Department of Mathematics, N.S.S College
Cherthala, Alleppey, India.
E-mail: rkkresearch@yahoo.in

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