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SOME FIXED POINT RESULTS FOR NONLINEAR CONTRACTIVE TYPE MAPPING ON b-METRIC SPACES WITH ALTERNATING DISTANCE FUNCTIONS

Radia bouhafs, Wasfi Shatanawi and Abdalla Tallafha

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Abstract The purpose of this paper is to prove some fixed point results for a mapping satisfying generalized (ϕ, ψ)-contractive condition in a complete partially ordered b-metric space. Also, we prove some coincidence fixed point results for two self mappings S and f on a set X under a set of conditions, where S satisfies a generalized (ϕ, ψ)-contractive condition with respect to a function f in a complete partially ordered b-metric space. Our results generalize, extend and unify most of the fundamental metrical fixed point theorems in the existing literature. A few examples are illustrated to support our results.

1 Introduction

The Banach contraction principle [1] is one of the most important results in the fixed point theory which asserts that every contraction function in a complete metric space has a unique fixed point. After Banach [1] proposed this important theorem, many researchers extended it to many directions, for example see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. The usual metric space has been generalized and enhanced in many different directions, one of such generalization is a b-metric space, which was initiated by Bakhtin [21]. Czerwik in [22] extended the Banach contraction principle to the frame work of complete b-metric spaces. Then after, many researchers obtained many important results in $b-$metric spaces, for example see [23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36]. In 2015 Abdeljawad et al. [6] extended the Banach contraction principle to the frame of partial b metric spaces. In the present paper, we will studied some fixed and common fixed point theorems in the frame of ordered $b-$ metric spaces.

2 Mathematical Preliminaries

The following definitions and results will be needed in what follows.

Definition 2.1. [21, 22] A map $d : X \times X \longrightarrow [0, \infty)$, where X is a non-empty set is said to be a b-metric, if the following conditions are satisfied for all $x, y, z \in X$ and $s \geq 1$:
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
Then (X, d, s) is known as a b-metric space.
If $(X, \preceq)$ is still a partially ordered set, then $(X, d, s, \preceq)$ is called a partially ordered b-metric space.

Definition 2.2. [22] Let (X, d, s) be a b-metric space. Then
(1) A sequence $\{x_n\}$ is said to converges to x if $\lim_{n\longrightarrow+\infty} d(x_n, x) = 0$ and written as $\lim_{n\longrightarrow+\infty} x_n = x$.
(2) $\{x_n\}$ is said to be a Cauchy sequence in X if $\lim_{n,m\longrightarrow+\infty} d(x_n, x_m) = 0$.
(3) (X, d, s) is said to be complete if every Cauchy sequence in it is convergent.

Remark 2.3. If the metric d is complete, then $(X, d, s, \preceq)$ is called a complete partially ordered b-metric space.

3 Previous results

Definition 3.1. [35] Let $(X, \preceq)$ be a partially ordered set, let $f, S : X \longrightarrow X$ be two mappings. Then
(1) S is called a monotone nondecreasing, if $S(x) \preceq S(y)$ for all $x, y \in X$ with $x \preceq y$.
(2) An element $x \in X$ is called a coincidence (common fixed) point of $f$and S if $fx = Sx(fx = Sx = x)$.
(3) $f$and S are called commuting if $fSx = Sfx$, for all $x \in X$.
(4) f and S are called compatible if any sequence $\{x_n\}$ with $\lim_{n \longrightarrow +\infty} fx_n = \lim_{n \longrightarrow +\infty} Sx_n = \mu$, for $\mu \in X$, then $\lim_{n \longrightarrow +\infty} d(Sfx_n, fSx_n) = 0$.
(5) A pair of self maps (f, S) is called weakly compatible if $fSx = Sfx$, when $Sx = fx$ for some $x \in X$.
(6) S is called monotone f-nondecreasing if

$$fx \preceq fy \Longrightarrow Sx \preceq Sy,$$

for any $x, y \in X$.
(7) A non empty set X is called well ordered set if very two elements of it are comparable, i.e., $x \preceq y$ or $y \preceq x$ for $x, y \in X$.

Lemma 3.2. *[34] Let $(X, d, s, \preceq)$ be a b-metric space with $s > 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leq \lim_{n \longrightarrow +\infty} \inf d(x_n, y_n) \leq \lim_{n \longrightarrow +\infty} \sup d(x_n, y_n) \leq s^2 d(x, y).$$

In particular if $x = y$, then $\lim_{n \longrightarrow +\infty} d(x_n, y_n) = 0$. Moreover, for each $\tau \in X$, we have

$$\frac{1}{s} d(x, \tau) \leq \lim_{n \longrightarrow +\infty} \inf d(x_n, \tau) \leq \lim_{n \longrightarrow +\infty} \sup d(x_n, \tau) \leq s d(x, \tau).$$

4 Main Results

Definition 4.1. The notion of distance functions, given by Khan [37], play major role in our results. A self mapping ϕ defined on $[0, +\infty)$ is said to be an altering distance function, if it satisfies the following conditions:
(i) ϕ is continuous,
(ii) ϕ is non-decreasing,
(iii) $\phi(t) = 0$ if and only if $t = 0$.

Let $(X, d, s, \preceq)$ be a partially ordered b-metric space with parameter $s \geq 1$ and $S : X \longrightarrow X$ be a mapping. Set

$$M(x, y) = \max \left\{ \frac{d(y, Sy)[1 + d(x, Sx)]}{1 + d(x, y)}, \frac{d(x, Sy) + d(y, Sx)}{2s}, \frac{d(x, Sx) d(x, Sy)}{1 + d(x, Sy) + d(y, Sx)}, d(x, y) \right\}.$$

Definition 4.2. Let $\phi, \psi \in \Phi$. We call the mapping S a generalized (ϕ, ψ)-contraction mapping if it satisfies the following condition:

$$\phi(sd(Sx, Sy)) \leq \phi(M(x, y)) - \psi(M(x, y)), \tag{4.1}$$

for any $x, y \in X$ with $x \preceq y$.

Theorem 4.3. *Suppose that $(X, d, s, \preceq)$ is a complete partially ordered b-metric space with parameter $s \geq 1$. Let $S : X \longrightarrow X$ be a generalized (ϕ, ψ)-contractive mapping. Suppose S is a continuous, nondecreasing mapping with respect to $\preceq$. If there exists $x_0 \in X$ with $x_0 \preceq Sx_0$, then S has a fixed point in X.*

Proof. If there exists $x_0 \in X$ such that $Sx_0 = x_0$, then we have the result. Assume that $x_0 \prec Sx_0$. Then construct a sequence $\{x_n\} \subset X$ by $x_{n+1} = Sx_n$ for $n \geq 0$. Since S is nondecreasing, then by induction we obtain that

$$x_0 \prec Sx_0 = x_1 \preceq ... \preceq x_n \preceq Sx_n = x_{n+1} \preceq .... \tag{4.2}$$

If for some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of S and we have nothing to prove. Suppose that $x_n \neq x_{n+1}$ for all $n \geq 1$. Since $x_{n-1} \preceq x_n$ for all $n \geq 1$, then Condition (3) implies that

$$\begin{aligned} \phi(d(x_n, x_{n+1})) &= \phi(d(Sx_{n-1}, Sx_n)) \leq \phi(sd(Sx_{n-1}, Sx_n)) && (4.3)\\ &\leq \phi(M(x_{n-1}, x_n)) - \psi(M(x_{n-1}, x_n)), && (4.4)\\ & && (4.5) \end{aligned}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{\tfrac{d(x_n,Sx_n)[1+d(x_{n-1},Sx_{n-1})]}{1+d(x_{n-1},x_n)}, \tfrac{d(x_{n-1},Sx_n)+d(x_n,Sx_{n-1})}{2s},\right.\\ &\quad \left.\tfrac{d(x_{n-1},Sx_{n-1})d(x_{n-1},Sx_n)}{1+d(x_{n-1},Sx_n)+d(x_n,Sx_{n-1})}, d(x_{n-1}, x_n)\right\}\\ &\leq \max\left\{d(x_n, x_{n+1}), \tfrac{d(x_{n+1},x_n)+d(x_n,x_{n-1})}{2}, \tfrac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{1+)d(x_{n-1},x_{n+1})}, d(x_{n-1}, x_n)\right\}\\ &\leq \max\left\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right\}. \end{aligned}$$

If $\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_n, x_{n+1})$ for some $n \geq 1$, then (4.3) implies that

$$\phi(d(x_n, x_{n+1})) \leq \phi(d(x_n, x_{n+1})) - \psi(d(x_n, x_{n+1})) < \phi(d(x_n, x_{n+1})), \tag{4.6}$$

which is a contradiction. This means that $\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n)$ for $n \geq 1$. Hence, we obtain from (4.3) that

$$\phi(d(x_n, x_{n+1})) \leq \phi(d(x_n, x_{n-1})) - \psi(d(x_n, x_{n-1})) < \phi(d(x_n, x_{n-1})). \tag{4.7}$$

So $\{d(x_n, x_{n+1}) : n = 0, 1, 2, ...\}$ is a decreasing sequence. Thus there exists $t \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = t$. Letting $n \to +\infty$ in (4.6), we get that

$$\phi(t) \leq \phi(t) - \psi(t),$$

which is correct only if $\psi(t) = 0$ and hence $t = 0$. So

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{4.8}$$

Now, we will show that (x_n) is a Cauchy sequence. Suppose to the contrary. Then there exist $\epsilon > 0$ for which we can find two subsequences (x_{m_i}) and (x_{n_i}) such that n_i is the smallest index for which

$$d(x_{m_i}, x_{n_i}) \geq \epsilon, \quad n_i \geq m_i > i. \tag{4.9}$$

that means

$$d(x_{m_i}, x_{n_i-1}) < \epsilon. \tag{4.10}$$

Triangular inequality implies that

$$\begin{aligned} d(x_{m_i-1}, x_{n_i}) &\leq sd(x_{m_i-1}, x_{m_i}) + sd(x_{m_i}, x_{n_i})\\ &\leq sd(x_{m_i-1}, x_{m_i}) + s^2d(x_{m_i}, x_{n_i-1}) + s^2d(x_{n_i-1}, x_{n_i}), \end{aligned}$$

and

$$\begin{aligned} d(x_{m_i-1}, x_{n_i-1}) &\leq sd(x_{m_i-1}, x_{m_i}) + sd(x_{m_i}, x_{n_i-1})\\ &\leq sd(x_{m_i-1}, x_{m_i}) + s\epsilon. \end{aligned}$$

Taking lim sup in above inequalities and using (4.8) and (4.10), we arrive to

$$\limsup_{n\to+\infty} d(x_{m_i-1}, x_{n_i}) \leq s^2\epsilon, \tag{4.11}$$

and

$$\limsup_{n\to+\infty} d(x_{m_i-1}, x_{n_i-1}) \leq s\epsilon. \tag{4.12}$$

So, we have the following:

$$\limsup_{n\to+\infty} \frac{d(x_{n_i-1}, x_{n_i})(1 + d(x_{m_i-1}, x_{m_i}))}{1 + d(x_{m_i-1}, x_{n_i-1})} \leq \limsup_{n\to+\infty} d(x_{n_i-1}, x_{n_i})(1 + d(x_{m_i-1}, x_{m_i})) = 0, \tag{4.13}$$

$$\limsup_{n\to+\infty} \frac{d(x_{m_i-1}, x_{n_i}) + d(x_{n_i-1}, x_{m_i})}{2s} \leq \frac{s^2\epsilon + \epsilon}{2s} \leq s\epsilon, \tag{4.14}$$

and

$$\limsup_{n\to+\infty} \frac{d(x_{m_i-1}, x_{m_i} d(x_{m_i-1}, x_{n_i}))}{1 + d(x_{m_i-1}, x_{n_i}) + d(x_{n_i-1}, x_{m_i})} \leq \limsup_{n\to+\infty} d(x_{m_i-1}, x_{m_i} d(x_{m_i-1}, x_{n_i})) = 0. \tag{4.15}$$

Also, from triangular inequality, we get

$$\begin{aligned} \epsilon \leq d(x_{m_i}, x_{n_i}) &\leq sd(x_{m_i}, x_{m_i-1}) + sd(x_{m_i-1}, x_{n_i}) \\ &\leq sd(x_{m_i}, x_{m_i-1}) + s^2 d(x_{m_i-1}, x_{n_i-1}) + s^2 d(x_{n_i-1}, x_{n_i}). \end{aligned}$$

Taking lim sup in above inequalities and using (4.9), we get

$$\frac{\epsilon}{s^2} \leq \limsup_{n\to+\infty} d(x_{m_i-1}, x_{n_i-1}).$$

Using the properties of ψ,

$$-\psi(\liminf_{n\to+\infty} d(x_{m_i-1}, x_{n_i-1})) \leq -\psi\left(\frac{\epsilon}{s^2}\right). \tag{4.16}$$

Now, since x_{n_i} and x_{m_i} are comparable, we have

$$\phi(sd(x_{m_i}, x_{n_i})) = \phi(sd(Sx_{m_i-1}, Sx_{n_i-1})) \leq \phi(M(x_{m_i-1}, x_{n_i-1})) - \psi(M(x_{m_i-1}, x_{n_i-1})), \tag{4.17}$$

where

$$\begin{aligned} M(x_{m_i-1}, x_{n_i-1}) = \max\Big\{ &\tfrac{d(x_{n_i}, x_{n_i+1})[1+d(x_{m_i}, x_{m_i+1})]}{1+d(x_{m_i-1}, x_{n_i-1})}, \tfrac{d(x_{m_i-1}, x_{n_i}) + d(x_{n_i-1}, x_{m_i})}{2s}, \\ &\tfrac{d(x_{m_i-1}, x_{m_i}) d(x_{m_i-1}, x_{n_i})}{1+d(x_{m_i-1}, x_{n_i}) + d(x_{n_i-1}, x_{m_i})}, d(x_{m_i-1}, x_{n_i-1}) \Big\} \end{aligned}$$

So,

$$d(x_{m_i-1}, x_{n_i-1}) \leq M(x_{m_i-1}, x_{n_i-1}).$$

Using the properties of ψ, we get

$$-\psi(M(x_{m_i-1}, x_{n_i-1})) \leq -\psi(d(x_{m_i-1}, x_{n_i-1})).$$

From (4.17), we get

$$\phi(d(x_{m_i}, x_{n_i})) \leq \phi(M(x_{m_i-1}, x_{n_i-1})) - \psi(d(x_{m_i-1}, x_{n_i-1})). \tag{4.18}$$

Taking the lim sup in (4.18) and using the inequalities (4.9), (4.12) – (4.15), we get that

$$\phi(s\epsilon) \leq \phi(s\epsilon) - \psi\left(\frac{\epsilon}{s^2}\right).$$

the last inequality is true only if

$$\psi\left(\frac{\epsilon}{s^2}\right) = 0.$$

By properties of ψ, we conclude that $\epsilon = 0$, which is a contradiction. So (x_n) is cauchy. Since X is complete, then there exists $\mu \in X$ such that $x_n \longrightarrow \mu$.
Also, the continuity of S implies that

$$S\mu = S(\lim_{n\longrightarrow+\infty} x_n) = \lim_{n\longrightarrow+\infty} S(x_n) = \lim_{n\longrightarrow+\infty} x_{n+1} = \mu. \tag{4.19}$$

Therefore μ is a fixed point of S in X.

The continuity condition in Theorem 4.3 can be dropped if we assume that X satisfies the following conditions:
(i) if a nondecreasing sequence $\{x_n\} \longrightarrow \mu$ in X, then $x_n \preceq \mu$ for all $n \in \mathbb{N}$, i.e., $\mu = \sup x_n$.

Theorem 4.4. *In Theorem 4.3 all conditions are satisfied except the continuity of S. If X satisfies the condition (i), then a nondecreasing mapping S has a fixed point in X.*

Proof. From Theorem 4.3, we take the same sequence $\{x_n\}$ in X such that $x_0 \preceq x_1 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ...$, that is, $\{x_n\}$ is nondecreasing and converges to some $\mu \in X$. Thus from the hypotheses, we have $x_n \preceq \mu$, for any $n \in \mathbb{N}$, implies that $\mu = \sup x_n$.
Next, we prove that μ is a fixed point of S in X, that is $S\mu = \mu$. Suppose that $S\mu \neq \mu$.
Let

$$\begin{aligned} M(x_n, \mu) \quad &= \max\left\{\frac{d(\mu, S\mu)[1 + d(x_n, Sx_n)]}{1 + d(x_n, \mu)}, \frac{d(x_n, S\mu) + d(\mu, Sx_n)}{2s},\right. \\ &\left.\frac{d(x_n, Sx_n)d(x_n, S\mu)}{1 + d(x_n, S\mu) + d(\mu, Sx_n)}, d(x_n, \mu)\right\}. \end{aligned}$$

Letting $n \longrightarrow +\infty$ and from the fact that $\lim_{n\longrightarrow+\infty} x_n = \mu$, we get

$$\lim_{n\longrightarrow+\infty} M(x_n, \mu) = \max\left\{d(\mu, S\mu), \frac{d(\mu, S\mu)}{2s} 0, 0\right\} = d(\mu, S\mu). \tag{4.20}$$

We know that $x_n \preceq \mu$ for all n, then from contraction condition (3), we get

$$\phi(d(x_{n+1}, S\mu)) = \phi(d(Sx_n, S\mu)) \le \phi(sd(Sx_n, S\mu)) \le \phi(M(x_n, \mu)) - \psi(M(x_n, \mu)). \tag{4.21}$$

Letting $n \longrightarrow +\infty$. By (4.20), we get

$$\phi(d(\mu, S\mu)) \le \phi(d(\mu, S\mu)) - \psi(d(\mu, S\mu)) < \phi(d(\mu, S\mu)), \tag{4.22}$$

which is a contradiction. Thus, $S\mu = \mu$, that is S has a fixed point μ in X.

Theorem 4.5. *In addition to the hypotheses of Theorem 4.3 (or Theorem 4.4), and adding this condition: Every pair of elements has a lower bound or an upper bound. Then the fixed point of S is unique.*

Proof. From Theorem 4.3 (or Theorem 4.4), we conclude that S has a nonempty set of fixed points. Suppose that x^* and y^* be two fixed points of S. Then we claim that $x^* = y^*$.
Suppose that $x^* \neq y^*$. From the hypotheses we have

$$\phi(d(Sx^*, Sy^*)) \le \phi(sd(Sx^*, Sy^*)) \le \phi(M(x^*, y^*)) - \psi(M(x^*, y^*)). \tag{4.23}$$

where

$$\begin{aligned} M(x^*,y^*) &= \max\left\{\frac{d(y^*,Sy^*)[1+d(x^*,Sx^*)]}{1+d(x^*,y^*)}, \frac{d(x^*,Sy^*)+d(y^*,Sx^*)}{2s},\right. \\ &\quad \left.\frac{d(x^*,Sx^*)d(x^*,Sy^*)}{1+d(x^*,Sy^*)+d(y^*,Sx^*)}, d(x^*,y^*)\right\} \\ &= \max\left\{\frac{d(y^*,y^*)[1+d(x^*,x^*)]}{1+d(x^*,y^*)}, \frac{d(x^*,y^*)+d(y^*,x^*)}{2s}, \frac{d(x^*,x^*)d(x^*,y^*)}{1+d(x^*,y^*)+d(y^*,x^*)},\right. \\ &\quad \left. d(x^*,y^*)\right\}. \\ &= \max\left\{0, \frac{d(x^*,y^*)}{s}, 0, d(x^*,y^*)\right\} \\ &= d(x^*,y^*). \end{aligned}$$

From (4.23), we obtain that

$$\phi(d(x^*,y^*)) = \phi(d(Sx^*,Sy^*)) \le \phi(d(x^*,y^*)) - \psi(d(x^*,y^*)) < \phi(d(x^*,y^*)). \tag{4.24}$$

which is a contradiction. Hence, $x^* = y^*$. This completes the proof.
Let $(X,d,s,\preceq)$ be a partially ordered b-metric space with parameter $s \ge 1$, and let $S,f : X \longrightarrow X$ be two mappings. Set

$$\begin{aligned} M_f(x,y) \quad = \quad \max\left\{\frac{d(fy,Sy)[1+d(fx,Sx)]}{1+d(fx,fy)}, \frac{d(fx,Sy)+d(fy,Sx)}{2s},\right. \\ \left.\frac{d(fx,Sx)d(fx,Sy)}{1+d(fx,Sy)+d(fy,Sx)}, d(fx,fy)\right\}. \end{aligned} \tag{4.25}$$

Now, we introduce the following definition.

Definition 4.6. Let $(X,d,s,\preceq)$ be a partially ordered b-metric space with $s \ge 1$. The mapping $S : X \longrightarrow X$ is called a generalized (ϕ,ψ)-contraction mapping with respect to $f : X \longrightarrow X$ for some $\phi \in \Phi$ and $\psi \in \Psi$ if

$$\phi(sd(Sx,Sy)) \le \phi(M_f(x,y)) - \psi(M_f(x,y)), \tag{4.26}$$

for any $x,y \in X$ with $fx \preceq fy$, where $M_f(x,y)$ as given by (4.25).

Theorem 4.7. *Suppose that $(X,d,s,\preceq)$ be a complete partially ordered b-metric space with $s \ge 1$. Let $S : X \longrightarrow X$be a generalized (ϕ,ψ)-contractive mapping with respect to $f : X \longrightarrow X$. Assume the following hypotheses:*
(1) f and S are continuous.
(2) S is a compatible with f.
(3) $SX \subseteq fX$.
(4) S is a monotone f-non decreasing mapping.
If for some $x_0 \in X$ such that $fx_0 \preceq Sx_0$, then S and f have a coincidence point in X.

Proof. From (3) and (4), we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X with

$$y_n = Sx_n = fx_{n+1}, \quad \forall n \ge 0, \tag{4.27}$$

such that

$$fx_0 \preceq fx_0 \preceq ... \preceq fx_n \preceq fx_{n+1} \preceq .... \tag{4.28}$$

If $y_n = y_{n+1}$ for some $n = 0,1,2,3,\ldots$. Then $y_n = fy_n = Sy_n$, that is y_n is a coincidence point of f and S. So, we may assume that $y_n \ne y_{n+1}$ for all $n = 0,1,2,3,\ldots$.
Given $n \in \{0,1,2,3,\ldots\}$. Since $y_n = fx_{n+1}$ and $y_{n+1} = fx_{n+2}$ are comparable, we have

$$\phi(sd(y_n,y_{n+1})) = \phi(sd(Sx_n,Sy_{n+1})) \le \phi(M_f(x_n,x_{n+1})) - \psi(M_f(x_n,x_{n+1})), \tag{4.29}$$

where

$$\begin{aligned} M_f(x_n, x_{n+1}) &= \max\Big\{ \tfrac{d(fx_{n+1}, Sx_{n+1})[1+d(fx_n, Sx_n)]}{1+d(fx_n, fx_{n+1})}, \\ &\quad \tfrac{d(fx_n, Sx_{n+1})+d(fx_{n+1}, Sx_n)}{2s}, \tfrac{d(fx_n, Sx_n)d(fx_n, Sx_{n+1})}{1+d(fx_n, Sx_{n=1})+d(fx_{n+1}, Sx_n)}, d(fx_n, fx_{n+1})\Big\} \\ &= \max\Big\{ \tfrac{d(y_n, y_n+1)[1+d(y_{n-1}, y_n)]}{1+d(y_{n-1}, y_n)}, \tfrac{d(y_{n-1}, y_{n+1})+d(y_n, y_n)}{2s}, \\ &\quad \tfrac{d(y_{n-1}, y_n)d(y_{n-1}, y_{n+1})}{1+d(y_{n-1}, y_{n+1})+d(y_n, y_n)}, d(y_{n-1}, y_n)\Big\} \\ &\leq \max\Big\{ \tfrac{d(y_n, y_n+1)[1+d(y_{n-1}, y_n)]}{1+d(y_{n-1}, y_n)}, \tfrac{d(y_{n-1}, y_n)+d(y_n, y_{n+1})}{2}, \\ &\quad \tfrac{d(y_{n-1}, y_n)d(y_{n-1}, y_{n+1})}{1+d(y_{n-1}, y_{n+1})+d(y_n, y_n)}, d(y_{n-1}, y_n)\Big\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} \end{aligned}$$

Therefore from equation (4.29), we get

$$\phi(sd(y_n, y_{n+1})) \leq \phi(\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}) - \psi(\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}). \quad (4.30)$$

If $0 < d(y_{n-1}, y_n) \leq d(y_n, y_{n+1})$ for some $n \in \mathbb{N}$, then (4.30) implies that

$$\phi(d(y_n, y_{n+1})) \leq \phi(sd(y_n, y_{n+1})) \leq \phi(d(y_n, y_{n+1})) - \psi(d(y_n, y_{n+1})) \leq \phi(d(y_n, y_{n+1})). \quad (4.31)$$

So, we conclude that

$$\psi(d(y_n, y_{n+1})) = 0. \quad (4.32)$$

Hence $y_n = y_{n+1}$, which is a contradiction. Hence from (4.30) we obtain that

$$\{d(y_n, y_{n+1}) : n = 0, 1, 2, 3, ...\} \quad (4.33)$$

is a decreasing sequence. So, there exist $t \in [0, +\infty)$ such that

$$\lim_{n \to +\infty} d(y_n, y_{n+1}) = t.$$

Letting $n \to +\infty$ in (4.31), we get

$$\phi(t) \leq \phi(t) - \psi(t),$$

this is true only if $\psi(t) = 0$ and hence $t = 0$. Similar arguments to those given in Theorem 4.3, we conclude that $y_n = \{Sx_n\} = \{fx_{n+1}\}$ is a Cauchy sequence in X. Since X is complete, then there exists $\mu \in X$ such that

$$\lim_{n \longrightarrow +\infty} Sx_n = \lim_{n \longrightarrow +\infty} fx_{n+1} = \mu.$$

Thus by the compatibility of S and f, we obtain that

$$\lim_{n \longrightarrow +\infty} d(f(Sx_n), S(fx_n)) = 0. \quad (4.34)$$

From the continuity of S and f, we have

$$\lim_{n \longrightarrow +\infty} f(Sx_n) = f\mu, \qquad \lim_{n \longrightarrow +\infty} S(fx_n) = S\mu. \quad (4.35)$$

Further by use of triangular inequality and from equations (4.34) and (4.35) , we get

$$\frac{1}{s} d(S\mu, f\mu) \leq d(S\mu, S(fx_n)) + sd(S(fx_n), f(Sx_n)) + sd(f(Sx_n), f\mu). \quad (4.36)$$

Finally, we arrive at $d(S\mu, f\mu) = 0$ as $n \longrightarrow +\infty$ in (4.36). Therefore, μ is a coincidence point of S and f in X.

The continuity of the functions f and S in Theorems 4.7 by adding another suitable conditions.

Theorem 4.8. *Assume all the hypotheses of Theorem 4.7 are satisfied, except the condition of the continuity of f and S. Also, suppose the following conditions:*
(1) $f(X)$ is a closed subspace of X.
(2) If $\lim_{n\longrightarrow+\infty} fx_n = fx$, then $fx_n \preceq f(x) \quad \forall n \in \mathbf{N}$ and $fx \preceq f(fx)$.
(3) S and f are weakly compatible.
If there exists $x_0 \in X$ such that $fx_0 \preceq Sx_0$, then f and S have a coincidence point in X.
Furthermore, if S and f commute at their coincidence point, then S and f have a common fixed point.

Proof. Following the proof of Theorem 4.7 line by line, we construct the Cauchy sequence $y_n = Sx_n = fx_{n+1}$. Since fX is closed, then there is some $\mu \in X$ such that

$$\lim_{n\longrightarrow+\infty} Sx_n = \lim_{n\longrightarrow+\infty} fx_n = f\mu.$$

Thus from the hypotheses, we have $fx_n \preceq f\mu$ for all $n \in \mathbb{N}$. Now, we have to prove that μ is a coincidence point of S and f. From equation (4.26), we have

$$\phi(sd(Sx_n, S\mu)) \leq \phi(M_f(x_n,\mu)) - \psi(M_f(x_n,\mu)), \tag{4.37}$$

where

$$\begin{aligned} M_f(x_n,\mu) \quad &= \max\left\{\frac{d(f\mu,S\mu)[1+d(fx_n,Sx_n)]}{1+d(fx_n,f\mu)}, \frac{d(fx_n,S\mu)+d(f\mu,Sx_n)}{2s},\right. \\ &\left.\frac{d(fx_n,Sx_n)d(fx_n,S\mu)}{1+d(fx_n,S\mu)+d(f\mu,Sx_n)}, d(fx_n,f\mu)\right\} \\ &\longrightarrow \max\left\{d(f\mu,S\mu), \frac{d(f\mu,S\mu)}{2s}, 0, 0\right\} \\ &= d(f\mu,S\mu),\ n\longrightarrow+\infty. \end{aligned}$$

By letting $n \to +\infty$ in (4.37) we get

$$\phi(sd(f\mu,S\mu)) \leq \phi(d(f\mu,S\mu)) - \psi(d(f\mu,S\mu)), \tag{4.38}$$

which true only if $\psi(d(f\mu,S\mu))$. Hence $f\mu = S\mu$. So μ is a coincidence of f and S. Now, let $f\mu = S\mu = \rho$. Since f and S are commute at ρ, then $S\rho = S(f\mu) = f(S\mu) = f\rho$. From Condition (2), we conclude that $f\mu \preceq f(f\mu) = f\rho$. So $f\mu$ and $f\rho$. Thus quation (4.37) implies that

$$\phi(sd(S\mu,S\rho)) \leq \phi(M_f(\mu,\rho)) - \psi(M_f(\mu,\rho)). \tag{4.39}$$

By simple calculation, we find $M_f(\mu,\rho) = d(S\mu,Sp)$. Thus (4.39) becomes:

$$\phi(sd(S\mu,S\rho)) \leq \phi(d(\mu,\rho)) - \psi(d(\mu,\rho)),$$

which is true only if $\psi(d(\mu,\rho)) = 0$. Properties of ψ implies that $S\mu = S\rho$; that is, ρ is a common fixed point of S and f.

Example 4.9. Define a metric $d : X \longrightarrow X$, where $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\frac{1}{n}, ...\}$ with usual order $\leq$ is as follows

$$d(x,y) = \begin{cases} 0 & ,\ \text{if } x = y, \\ 1 & ,\ \text{if } x \neq y \in \{0,1\}. \\ |x-y| & ,\ \text{if } x,y \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\}, \\ 5 & ,\ \text{if } otherwise. \end{cases}$$

A map $S : X \longrightarrow X$ be such that $S0 = 0$, $S\frac{1}{n} = \frac{1}{10n}$ for all $n \geq 1$ and let $\phi(t) = t, \psi(t) = \frac{3t}{4}$ for $t \in [0, +\infty)$. Then, S has a fixed point in X.

Proof. It is obvious that for $s = \frac{10}{4}$, $(X, d, s, \preceq)$ is a complete partially ordered b-metric space and also by definition, d is discontinuous b-metric space. Now for $x, y \in X$ with $x < y$, we have the following cases:

Case I: Let $x = 0$ and $y = \frac{1}{n}, n \geq 1$, we have two subcases:
Subcase 1: If $n = 2t$ for some $t \geq 1$, then $d(Sx, Sy) = \frac{1}{10n}$ and $M(x, y) = \frac{1}{n}$. Therefore, we have

$$\phi\left(\frac{10}{4}d(Sx, Sy)\right) = \frac{1}{4n} \leq \frac{M(x, y)}{4} = \phi(M(x, y)) - \psi(M(x, y)).$$

Subcase 2: if $n = 2t - 1$ for some $t \geq 1$, then $d(Sx, Sy) = \frac{1}{10n}$ and $M(x, y) = 5$. Therefore, we have

$$\phi\left(\frac{10}{4}d(Sx, Sy)\right) \leq \frac{M(x, y)}{4} = \phi(M(x, y)) - \psi(M(x, y)).$$

Case II: Let $x = \frac{1}{m}$ and $y = \frac{1}{n}$ with $m > n \geq 1$, we have three cases:
Subcase 1: If $n = 2t$ and $m = 2k$ for $t, k \geq 1$, then $d(Sx, Sy) = \frac{1}{10n} - \frac{1}{10m}$ and $d(x, y) = \frac{1}{n} - \frac{1}{m} \leq M(x, y)$. Therefore, we have

$$\phi\left(\frac{10}{4}d(Sx, Sy)\right) = \frac{1}{4n} - \frac{1}{4m} \leq \frac{M(x, y)}{4} = \phi(M(x, y)) - \psi(M(x, y)).$$

Subcase 2: if $n = 2t - 1$ and $m = 2k - 1$ for $t, k \geq 1$, then $d(Sx, Sy) = \frac{1}{10n} - \frac{1}{10m}$ and $M(x, y) = 5$. Therefore, we have

$$\phi\left(\frac{10}{4}d(Sx, Sy)\right) = \frac{1}{4n} - \frac{1}{4m} \leq \frac{5}{4} = \frac{M(x, y)}{4} = \phi(M(x, y)) - \psi(M(x, y)).$$

Subcase 3: Similar to those arguments given in Subcase 2.
Hence, condition (3) of Theorem 4.3 and remaining assumptions are satisfied. Thus, S has a fixed point in X.

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References

[1] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales. *Fundam. Math.* **3**, 133-181 (1922)

[2] T. Abdeljawad , Meir-Keeler α-contractive fixed and common fixed point theorems, *Fixed Point Theory and Applications* **2013**, 2013:19, 10 pages.

[3] T. Abdeljawad, K. Ullah, J. Ahmad and N. Mlaiki, Iterative Approximation of Endpoints for Multivalued Mappings in Banach Spaces, *J. Funct. Spaces 2020,* Article ID 2179059 **2020**. https://doi.org/10.1155/2020/2179059.

[4] T. Abdeljawad, N. Mlaik, H. Aydi and N. Souayah, Double controlled metric type spaces and some fixed foint results, *Mathematics***6(12),** Article ID 320 (2018), https://doi.org/10.3390/math6120320

[5] T. Abdeljawad, K. Abodayeh and N. Mlaiki, On fixed point generalizations to partial b-metric spaces, *J.Comput.Anal. Appl.* **19(5)**, 883–891 (2015)

[6] A. Khan, T. Abdeljawad, W. Shatanawi, H. Khan, Fixed Point Theorems for Quadruple Self-Mappings Satisfying Integral Type *Inequalities, Filomat* 34:3 **2020**, 905-917 https://doi.org/10.2298/FIL2003905K

[7] W. Shatanawi and A. Pitea, Fixed and coupled fixed point theorems of omega-distance for nonlinear contraction, *Fixed Point Theory and Applications,* **11** 2013,DOI: 10.1186/1687-1812-2013-275.

[8] W. Shatanawi, Z. Mustafa, N. Tahat, Some coincidence point theorems for nonlinear contraction in ordered metric spaces, *Fixed Point Theory and Applications*, **Volume 2011**, October 2011, Article number 68

[9] W. Shatanawi, A. Bataihah and A. Tallafha, Four-Step Iteration Scheme to Approximate Fixed Point for Weak Contractions, Computers, *Materials & Continua CMC*, **vol.64**, no.3, pp.1491-1504, 2020

[10] W. Shatanawi, On w-compatible mappings and common coupled coincidence point in cone metric spaces, *Applied Mathematics Letters*, **25**, Issue 6, June 2012, Pages 925–931

[11] W. Shatanawi, Some fixed point results for a generalized $\psi-$weak contraction mappings in orbitally metric spaces, *Chaos, Solitons and Fractals*, **45**, Issue 4, April 2012, Pages 520–526

[12] W. Shatanawi, E. Karapnar, H. Aydi, Coupled coincidence points in partially ordered cone metric spaces with a c-distance, *Journal of Applied Mathematics*, **Volume 2012**, 2012, Article number 312078

[13] W. Shatanawi, M. Postolache, Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces, *Fixed Point Theory and Applications*, **Volume 2013**, November 2013, Article number 271

[14] H. Aydi, W. Shatanawi, M, Postolache, Z. Mustafa, N. Tahat, Theorems for Boyd-Wong-type contractions in ordered metric spaces, *Abstract and Applied Analysis*, **Volume 2012**, 2012, Article number 359054

[15] W. Shatanawi, V.C. Rajic, S.C. Radenovic, A. Al-Rawashdeh, Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces,*Fixed Point Theory and Applications*, **Volume 2012**, 2012, Article number 106

[16] W. Shatanawi, A. Al-Rawashdeh: Common fixed point of almost generalized (ψ, ϕ)-cotractive mappings in ordered metric space. *Fixed point theory Appl*. **2012**, Article ID80(2012.

[17] W. Shatanawi, S. Manro: Fixed point results for cyclic (ψ, ϕ, A, B) -contraction in partial metric space. *Fixed point Theory Appl*. **2012**. Article ID. 165(2012)

[18] A. Bataihah, W. Shatanawi, T. Qawasmeh, R. Hatamleh, On H-Simulation functions and fixed point results in the setting of wt-distance mappings with application on matrix equations, *Mathematics*, **8** (2020), Article number 837.

[19] A. Bataihah, W. Shatanawi, A. Tallafha, Fixed point results with simulation functions, *Nonlinear Functional Analysis and Applications* **25** (2020), 13–23.

[20] A. Bataihah, A. Tallafha, W. Shatanawi, Fixed point results with $\Omega-$distance by utilizing simulation functions, *Italian Journal of Pure and Applied Mathematics* **43** (2020), 185–196.

[21] A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal, Gos. Ped. Inst. Unianowsk,* **30**, (1989), 26–37.

[22] S. Czerwik, Contraction mappings in b-metric spaces,*Acta Math Inf Univ Ostraviensis,***1** (1993) 5-11.

[23] A. Mukheimer, N. Mlaiki, K. Abodayeh and W.Shatanawi, New theorems on extended b-metric spaces under new contractions, *Nonlinear Anal,* Model. Control **24**(6), 870–883 (2019).

[24] K. Kukiĉ, W. Shatanawi, M.G. Filipoviĉ, Khan and Ĉiriĉ contraction princples in almost $b-$metric space, *U.P.B. Sci. Bull.*, Series A, **Vol. 82**, Iss. 1, 2020

[25] W. Shatanawi, Z.D. Mitrovic, N. Hussain and S. Radenovic, On Generalized Hardy Rogers Type $\alpha-$Admissible Mappings in Cone b-Metric Spaces over Banach Algebras, **Symmetry** 2020, 12, 81; doi:10.3390/sym12010081

[26] W. Shatanawi, K. Abodayeh, A. Mukheimer, Some Fixed Point Theorems in Extended $b-$metric spaces, *U.P.B. Sci. Bull.*, Series A, **Vol. 80**, Iss. 4, 2018.

[27] K. P. R. Rao, W. Shatanawi, G.N.V. Kishore, K. Abodayeh and D.R. Prasad, Existeness and Uniqeness of Suzuki Type Results in S_b- Metric Spaces with Applications to Integral Equations, *Nonlinear Functional Analysis and Applications* **Vol. 23**, No. 2 (2018), pp. 225-245

[28] S. Sedghi, N. Shobkolaei, J. Rezaei Roshan, and W. Shatanawi, Coupled fixed point theorems in G_b-metric spaces, *Matematicki Vesnik*, **Volume 66**, Issue 2, 2014, Pages 190-201

[29] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, W. Shatanawi, Common Fixed Points of Almost Generalized $(\psi, \phi)_s-$Contractive Mappings in Ordered $b-$Metric Spaces, *Fixed Point Theory and Applications* **2013**, 2013:159, 1–23

[30] H. Qawaqneh, M.S.M. Noorani, S. Shatanawi, H. Aydi, H. Alsamir, Fixed point results for multi-valued contractions in b-metric spaces and an application, *Mathematics*, **Volume 7**, Issue 2, 1 February 2019, Article number 132.

[31] T. Kamran, M. Postolache, M. U. Ali and Q. Kiran, Feng and Liu type F-contraction in b-metric spaces with application to integral equations, *J. Math. Anal.,* **7** (2016), 18-27.

[32] Huaping Huang, Radenović , S., Jelena Vujaković: On some recent coincidence and immediate consequences in partially ordered b-metric spaces. *Fixed Point Theory and Appl*. 2015:63 **2015**. doi: 10.1186/s13663-015-0308-3

[33] H. Qawaqneh, M. S. M. Noorani and W. Shatanawi, Fixed point theorems for (α, k, θ)-contractive multi-valued mapping in b-metric space and applications, *Int. J. Math. Comput. Sci.* **14**(1), 263–283 (2019).

[34] A. Aghajani, R. Arab : Fixed points of (ψ, ϕ, θ)-contractive mappings in partially ordered b-metric spaces and applications to quadratic integral equations. *Fixed Point Theory Appl*. **2013**, Article ID 245 (2013).

[35] J. Harjani, B. López, K. Sadarangani: Fixed point theorems for mixed monotone operators and applications to integral equations. *Nonlinear Anal*. **74** (2011), 1749–1760.

[36] H. Kerim, W. Shatanawi and A. Tallafha: Common fixed point theorem for set-valued maps on modular b-gauge spaces. *Palestine Journal of Mathematics* **Vol. 9**(2)(2020), 1-10.

[37] Khan, MS, Swaleh, M, Sessa, S: Fixed point theorems by altering distances between the points. *Bull. Aust. Math. Soc.* **30**, 1-9 (1984)

Author information

Radia bouhafs, Wasfi Shatanawi and Abdalla Tallafha,
Radia Bouhafs, Abdalla Tallafha, Department of Mathematics, The University of Jordan, Amman, Jordan
Wasfi Shatanawi, Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia,.
E-mail: `rad9170450@ju.edu.jo, radia.93.bou@gmail.com`

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