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UNIQUENESS OF A POLYNOMIAL AND DIFFERENTIAL POLYNOMIAL SHARING A SMALL FUNCTION

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Abstract In this paper taking a question in [5] into background we investigate the uniqueness of a non-constant polynomial with the differential polynomial generated by a non-constant meromorphic function f. Our result will also extend a result of Banerjee-Chakraborty [4] given earlier. We provide some examples to show that certain conditions used in the paper can not be removed.

1 Introduction and and main results

In this paper, meromorphic functions mean meromorphic in the complex plane. We use the standard notations of Nevanlinna theory, which can be found in [23]. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$, i.e., $T(r,a) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite linear measure. We say that two meromorphic functions f and g share a small function a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

Rubel and Yang [22] appear to be the first to study the entire functions that share values with their derivatives. In 1977, they proved the following well-known theorem.

Theorem 1.1. *Let f be a non-constant entire function. If f and $f^{'}$ share two distinct finite numbers a, b CM, then $f \equiv f^{'}$.*

Since then, shared value problems, especially the case of f and $f^{'}$ sharing values, have been studied by many authors and a number of profound results have been obtained (see, eg[9, 13],etc).

In 1979, Mues and Steinmetz [21] proved the following result, which is an improvement of Theorem 1.1.

Theorem 1.2. *Let f be a non-constant entire function. If f and $f^{'}$ share two distinct values a, b IM, then $f \equiv f^{'}$.*

In connection to finding the relation between an entire function with its derivative when they share one value CM, in 1996 in this direction the following famous conjecture was proposed by Brück [9].

Conjecture. Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite, where

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

If f and $f^{'}$ share a finite value a CM, then $\frac{f^{'}-a}{f-a} = c$, where c is a non-zero constant.

In recent years, many results have been published concerning the above conjecture, (see,[2, 3, 4, 6, 8, 10, 11, 17, 18, 19]). Next we recall the following definitions:

Definition 1.3. [16] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid \geq p)$ (resp. $\overline{N}(r, a; f \mid \geq p)$) denotes the counting function (resp. reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r, a; f \mid \leq p)$ (resp. $\overline{N}(r, a; f \mid \leq p)$) denotes the counting function (resp. reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.4. [25] For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p, we define

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + ... + \overline{N}(r, a; f \mid \geq p).$$

Definition 1.5. [25] For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p, we put

$$\delta_p(a, f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Thus

$$0 \leq \delta(a, f) \leq \delta_p(a, f) \leq \delta_{p-1}(a, f) \leq ... \leq \delta_2(a, f) \leq \delta_1(a, f) = \Theta(a, f) \leq 1.$$

Definition 1.6. [4] For two positive integers n, p we define

$$\mu_p = \min\{n, p\} \text{ and } \mu_p^* = p + 1 - \mu_p.$$

Then clearly

$$N_p(r, 0; f^n) \leq \mu_p N_{\mu_p^*}(r, 0; f).$$

Definition 1.7. [8] Let z_0 be a zero of $f - a$ of multiplicity p and a zero of $g - a$ of multiplicity q.

i) We denote by $\overline{N}_L(r, a; f)$, the counting function of those a-points of f and g where $p > q \geq 1$,

ii) by $N_E^{1)}(r, a; f)$, we denote the the counting function of those a-points of f and g where $p = q = 1$ and

iii) by $\overline{N}_E^{(2}(r, a; f)$, we denote the counting function of those a-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once.

Similarly, we can define $\overline{N}_L(r, a; g), N_E^{1)}(r, a; g), \overline{N}_E^{(2}(r, a; g)$.

Definition 1.8. [15] Let k be a non-negative integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. By $E_k(a; f)$, we mean the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k. Thus we note that f and g share a value a-IM (resp. CM) if and only if f and g share $(a, 0)$ (resp.(a, ∞)).

With the notion of weighted sharing of values Lahiri-Sarkar [16] improved the result of Zhang [24]. In [25] Zhang extended the result of Lahiri-Sarkar [16] and replaced the concept of value sharing by small function sharing.

In 2008, Zhang and Lü [26] obtained the following result.

Theorem 1.9. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also, let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0, l)$. If $l = \infty$ and*

$$(3 + k)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > 6 + k - n,$$

or, $l = 0$ and

$$(6 + 2k)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 12 + 2k - n,$$

then $f^n \equiv f^{(k)}$.

At the end of [26] the following question was raised by Zhang and Lü [26].

Question 1.1. What will happen if f^n and $[f^{(k)}]^m$ share a small function ?

In 2010, Chen and Zhang [11] answered the above question. But unfortunately there were some errors in their results. Banerjee-Majumder [8] first pointed out the errors, rectified them and obtained the correct form of the same as follows.

Theorem 1.10. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also, let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \geq 2$ and*

$$(3+k)\Theta(\infty,f) + 2\Theta(0,f) + \delta_{2+k}(0,f) > 6+k-n,$$

or, $l = 1$ and

$$\left(\frac{7}{2}+k\right)\Theta(\infty,f) + \frac{5}{2}\Theta(0,f) + \delta_{2+k}(0,f) > 7+k-n,$$

or, $l = 0$ and

$$(6+2k)\Theta(\infty,f) + 4\Theta(0,f) + \delta_{2+k}(0,f) + \delta_{1+k}(0,f) > 12+2k-n,$$

then $f^n \equiv f^{(k)}$.

Theorem 1.11. *Let $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$ be integers and f be a non-constant meromorphic function. Also, let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^n - a$ and $[f^{(k)}]^m - a$ share $(0,l)$. If $l = 2$ and*

$$(3+2k)\Theta(\infty,f) + 2\Theta(0,f) + 2\delta_{1+k}(0,f) > 7+2k-n, \tag{1.1}$$

or, $l = 1$ and

$$\left(\frac{7}{2}+2k\right)\Theta(\infty,f) + \frac{5}{2}\Theta(0,f) + 2\delta_{1+k}(0,f) > 8+2k-n, \tag{1.2}$$

or, $l = 0$ and

$$(6+3k)\Theta(\infty,f) + 4\Theta(0,f) + 3\delta_{1+k}(0,f) > 13+3k-n, \tag{1.3}$$

then $f^n \equiv [f^{(k)}]^m$.

It can be easily proved that Theorem 1.10 is a better result than Theorem 1.11 for $m = 1$ case. Also, it is observed that in the conditions (1.1) - (1.3) there was no influence of m.

Very recently, in order to improve the results of Zhang [25], Li and Huang [17] obtained the following theorem. In view of Lemma 2.1 proved latter on, we see that the following result obtained in [17] is better than that of Theorem 1.10 for $n = 1$.

Theorem 1.12. *Let $k(\geq 1)$, $l(\geq 0)$ be integers and f be a non-constant meromorphic function. Also, let $a(z)$ $(\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \geq 2$ and*

$$(3+k)\Theta(\infty,f) + \delta_2(0,f) + \delta_{2+k}(0,f) > k+4,$$

or, $l = 1$ and

$$\left(\frac{7}{2}+k\right)\Theta(\infty,f) + \frac{1}{2}\Theta(0,f) + \delta_2(0,f) + \delta_{2+k}(0,f) > k+5,$$

or, $l = 0$ and

$$(6+2k)\Theta(\infty,f) + 2\Theta(0,f) + \delta_2(0,f) + \delta_{1+k}(0,f) + \delta_{2+k}(0,f) > 2k+10,$$

then $f \equiv f^{(k)}$.

Now, we recall the following definition

Definition 1.13. [14] Let $n_{0j}, n_{1j}, ..., n_{kj}$ be nonnegative integers.

- The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d_{Mj} = d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{Mj} = \sum_{i=0}^{k}(i+1)n_{ij}$.

- The sum $H[f] = \sum_{j=1}^{t} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\bar{d}(H) = \max\{d(M_j) : 1 \le j \le t\}$ and weight $\Gamma = \Gamma_P = \max\{\Gamma_{Mj} : 1 \le j \le t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, ..., t$.

- The numbers $\underline{d}(H) = \min\{d(M_j) : 1 \le j \le t\}$ and k (the highest order of the derivative of f in $H[f]$) are called respectively the lower degree and order of $H[f]$.

- $H[f]$ is said to be homogeneous if $\bar{d}(H) = \underline{d}(H)$. Also, we define $Q := \max\{\Gamma_{Mj} - d(M_j) : 1 \le j \le t\}$; and for the sake of convenience for a differential monomial $M[f]$, we denote by $\lambda = \Gamma_M - d_M$.

Since the natural extension of $[f^{(k)}]^m$ is a differential monomial, it will be interesting to see whether Theorem 1.11 can remain true when $[f^{(k)}]^m$ is replaced by $M[f]$. In this direction, very recently Banerjee-Chakraborty [4] have improved Theorem 1.11 in the following way which in turn improve a recent result of Li-Huang [17] as well.

Theorem 1.14. *Let $k(\ge 1)$, $n(\ge 1)$ be integers and f be a non-constant meromorphic function. Also, let $M[f]$ be a differential monomial of degree d_M and weight Γ_M and k is the highest derivative in $M[f]$. Let $a(z)$ $(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $f^n - a$ and $M[f] - a$ share $(0, l)$. If $l \ge 2$ and*

$$(3 + \lambda)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0, f) > 3 + \Gamma_M + \mu_2 - n,$$

or, $l = 1$ and

$$\left(\frac{7}{2} + \lambda\right)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0, f) > 4 + \Gamma_M + \mu_2 - n,$$

or, $l = 0$ and

$$(6 + 2\lambda)\Theta(\infty, f) + 2\Theta(0, f) + \mu_2\delta_{\mu_2^*}(0, f) + d_M\delta_{k+2}(0, f) + d_M\delta_{k+1}(0, f) > 8 + 2\Gamma_M + \mu_2 - n,$$

then $f^n \equiv M[f]$.

In the same paper the following was asked:

Question 1.2. Is it possible to extend Theorem 1.14 upto differential polynomial instead of differential monomial?

To answer the above question, recently Bikash Chakraborty [5] obtained the following theorem:

Theorem 1.15. *Let $k(\ge 1)$, $n(\ge 1)$ be integers and f be a non-constant meromorphic function. Let $H[f]$ be a homogeneous differential polynomial of degree $\bar{d}(H)$ and weight Γ_P such that $\Gamma_P > (k+1)\underline{d}(H) - 2$, where k is the highest derivative in $H[f]$. Also, let $a(z)$ $(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $f^n - a$ and $H[f] - a$ share $(0, l)$. If $l \ge 2$ and*

$$(\Gamma_P - \underline{d}(H) + 3)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(0, f) + \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(0, f) > \Gamma_P + \mu_2 + 3 - n, \quad (1.4)$$

or, $l = 1$ and

$$\left(\Gamma_P - \underline{d}(H) + \frac{7}{2}\right)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \mu_2\delta_{\mu_2^*}(0, f) + \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(0, f) > \Gamma_P + \mu_2 + 4 - n, \quad (1.5)$$

or, $l = 0$ *and*

$$(2(\Gamma_P - \underline{d}(H)) + 6)\Theta(\infty, f) + 2\Theta(0, f) + \mu_2\delta_{\mu_2^*}(0, f) + \underline{d}(H)\delta_{1+\Gamma_P-\underline{d}(H)}(0, f)$$
$$+ \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(0, f) > 2\Gamma_P + \mu_2 + 8 - n, \tag{1.6}$$

then $f^n \equiv H[f]$.

Naturally at the end of the paper the following question was posed by the author in [5].

Question 1.3. Is it is possible to extend Theorem 1.15 up to an arbitrary differential polynomial?

One of our objectives in writing this paper is to solve this Question. Now observing the above results it is Quite natural to place the following Question

Question 1.4. Is it is possible to replace f^n by arbitrary polynomial $P[f] = a_0 f^n + a_1 f^{n-1} + ... + a_n$ in Theorem 1.15 ?

Throughout the paper, we will use the following notations. Let

$$\mathcal{P}(w) = a_{n+m}w^{n+m} + ... + a_n w^n + ... + a_0 = a_{n+m}\prod_{i=1}^{s}(w - w_{p_i})^{p_i}$$

where a_j $(j = 0, 1, 2, ..., n+m-1)$, $a_{n+m} \neq 0$ and w_{p_i} $(i = 1, 2, ..., s)$ are distinct finite complex numbers and $2 \leq s \leq n + m$ and $p_1, p_2, ..., p_s$, $s \geq 2$, n, m and k are all positive integers with $\sum_{i=1}^{s} p_i = n + m$. Also let $p > \max_{p \neq p_i, i=1,...,r}\{p_i\}$, $r = s - 1$, where s and r are two positive integers.

Let $P(w_1) = a_{n+m}\prod_{i=1}^{s-1}(w_1 + w_p - w_{p_i})^{p_i} = b_q w_1^q + b_{q-1}w_1^{q-1} + ... + b_0$, where $a_{n+m} = b_q$, $w_1 = w - w_p$, $q = n + m - p$. Therefore, $\mathcal{P}(w) = w_1^p P(w_1)$.
Next we assume $P(w_1) = b_q\prod_{i=1}^{r}(w_1 - \alpha_i)^{p_i}$, where $\alpha_i = w_{p_i} - w_p$, $(i = 1, 2, ..., r)$, be distinct zeros of $P(w_1)$.

The following theorem is the main result of this paper which gives an affirmative answer of the question 1.4 and also the question of Bikash Chakraborty [5] in a more convenient way.

Theorem 1.16. *Let* $k(\geq 1)$, $n(\geq 1)$, $p(\geq 1)$ *and* $m(\geq 0)$ *be integers and* f *and* $f_1 = f - w_p$ *be two non-constant meromorphic functions and* $H[f]$ *be a non-constant differential polynomial of degree* $\bar{d}(H)$ *and weight* Γ_P *satisfying* $\Gamma_P > (k + 1)\underline{d}(H) - 2$. *Let* $\mathcal{P}(z) = a_{m+n}z^{n+m} + ... + a_n z^n + ... + a_0$, $a_{m+n} \neq 0$, *be a polynomial in* z *of degree* $m + n$ *such that* $\mathcal{P}(f) = f_1^p P(f_1)$. *Also, let* $a(z)$ $(\not\equiv 0, \infty)$ *be a small function with respect to* f. *Suppose* $\mathcal{P}(f) - a$ *and* $H[f] - a$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$(\Gamma_P - \bar{d}(H) + 3)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(0, f) > \Gamma_P + \mu_2 + 3 - p$$
$$+ \bar{d}(H) - \underline{d}(H), \tag{1.7}$$

or, $l = 1$ *and*

$$\left(\Gamma_P - \bar{d}(H) + \frac{7}{2}\right)\Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(0, f) > \Gamma_P + \mu_2$$
$$+ 4 + \frac{(m + n) - 3p}{2} + \bar{d}(H) - \underline{d}(H), \tag{1.8}$$

or, $l = 0$ *and*

$$(2(\Gamma_P - \bar{d}(H)) + 6)\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \underline{d}(H)\left(\sum_{i=1}^{2}\delta_{i+\Gamma_P-\underline{d}(H)}(0, f)\right)$$
$$> 2\Gamma_P + \mu_2 + 8 + 2(m + n) - 3p + 2(\bar{d}(H) - \underline{d}(H)), \tag{1.9}$$

then $\mathcal{P}(f) \equiv H[f]$.

The following Corollary can easily be deduced from the above theorem which is an extension and improvement of the Theorem 1.15. It is clear that for $P(z)=1$ i.e., $m=0$ and $\bar{d}(H)-\underline{d}(H)$ (i.e., Homogeneous differential polynomial), we get exactly Theorem 1.15 from Corollary 1.17.

Corollary 1.17. *Let $k(\geq 1)$, $n(\geq 1)$, and $m(\geq 0)$ be integers and f be a non-constant meromorphic function and $H[f]$ be a non-constant differential polynomial of degree $\bar{d}(H)$ and weight Γ_P satisfying $\Gamma_P>(k+1)\underline{d}(H)-2$. Let $P(z)=a_m z^m+...+a_0$, $a_m\neq 0$, be a polynomial in z of degree m. Also, let $a(z)\ (\not\equiv 0,\infty)$ be a small function with respect to f. Suppose $f^nP(f)-a$ and $H[f]-a$ share $(0,l)$. If $l\geq 2$ and*

$$(\Gamma_P-\bar{d}(H)+3)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(0,f)>\Gamma_P+\mu_2+3-n+\bar{d}(H)-\underline{d}(H), \quad (1.10)$$

or, $l=1$ and

$$\left(\Gamma_P-\bar{d}(H)+\frac{7}{2}\right)\Theta(\infty,f)+\frac{1}{2}\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(0,f)>\Gamma_P+\mu_2+4+\frac{m}{2}-n+\bar{d}(H)-\underline{d}(H), \quad (1.11)$$

or, $l=0$ and

$$(2(\Gamma_P-\bar{d}(H))+6)\Theta(\infty,f)+2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\underline{d}(H)\left(\sum_{i=1}^{2}\delta_{i+\Gamma_P-\underline{d}(H)}(0,f)\right)>2\Gamma_P+\mu_2+8+2m-n+2(\bar{d}(H)-\underline{d}(H)), \quad (1.12)$$

then $f^nP(f)\equiv H[f]$.

The following examples show that the conditions (1.7) - (1.9) in Theorem 1.16 cannot be removed.

Example 1.1. Let $f(z)=Ae^z+Be^{-z}$, $AB\neq 0$. Then $\overline{N}(r,f)=S(r,f)$ and

$$\overline{N}(r,0;f)=\overline{N}\left(r,\frac{-B}{A};e^{2z}\right)\sim T(r,f).$$

Here $m=0$, $p=n=1$, $w_p=0$, $\bar{d}(H)=\underline{d}(H)=1$, $\mu_2=1$ and $\Gamma_P=2$. Again $\Theta(\infty,f)=1$ and $\Theta(0,f)=\delta_q(0,f)=0$. Let $m=0$, hence $\mathcal{P}(f)=f$. Therefore it is clear that $H[f]=2f''$ and $\mathcal{P}(f)$ share $(a,l)(l\geq 0)$ but none of the inequalities (1.7), (1.8) and (1.9) of Theorem 1.16 is satisfied and $\mathcal{P}(f)\not\equiv H[f]$.

Example 1.2. Let $f(z)=-\sin\alpha(z)+a-\frac{a}{\alpha^{4k}}$, $k\in\mathbb{N}$; where $\alpha\neq 0$, $\alpha^{4k}\neq 1$, and $a\in\mathbb{C}-\{0\}$. Let $p=n=1$, $w_p=0$, and $m=0$. Then let $\mathcal{P}(f)=f$. Again let $H[f]=2f^{4k}$. Then $H[f]=-2\alpha^{(4k)}\sin(\alpha(z))$. Here $m=0$, $\mu_2=1$, $\Gamma_P=4k$, $\bar{d}(H)=\underline{d}(H)=1$. Again $\Theta(\infty,f)=1$ and

$$\overline{N}(r,0;f)=\overline{N}\left(r,a-\frac{a}{\alpha^{4k}};\sin\alpha(z)\right)\sim T(r,f).$$

Therefore, $\Theta(0,f)=0=\delta_q(0,f),\forall q\in\mathbb{N}$. Also it is clear that $\mathcal{P}(f)$ and $H[f]$ share $(a,l)(l\geq 0)$ but none of the inequalities (1.7), (1.8) and (1.9) of Theorem 1.16 is satisfied and $\mathcal{P}(f)\not\equiv H[f]$.

2 Preliminary Lemmas

In this section, we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by Δ the following function.

$$\Delta=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \quad (2.1)$$

Lemma 2.1. *([4]) If f is a non-constant meromorphic function, then*

$$1+\delta_2(0,f)\geq 2\Theta(0,f).$$

Lemma 2.2. *([8]) If F and G share $(1,l)$, $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)$ and $\Delta\not\equiv 0$, then*

$$\begin{aligned}N(r,\infty;\Delta)\leq{}&\overline{N}(r,\infty;F)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\&+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.3. *([4]) Let F and G share $(1,l)$. Then*

$$\overline{N}_L(r,1;F)\leq\frac{1}{2}\overline{N}(r,\infty;F)+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)\ \text{when}\ l\geq 1,$$

and

$$\overline{N}_L(r,1;F)\leq\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+S(r,F)\ \text{when}\ l=0.$$

Similar expressions also hold for G.

Lemma 2.4. *([4]) Let F and G share $(1,l)$ and $\Delta\not\equiv 0$. Then*

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)\leq{}&N(r,\infty;\Delta)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)\\&+S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.5. *Let f be non-constant meromorphic function and $a(z)$ be a small function of f. Let $F=\frac{\mathcal{P}(f)}{a}=\frac{f_1^pP(f_1)}{a}$ and $G=\frac{H[f]}{a}$. such that F and G shares $(1,\infty)$. Then one of the following cases holds:*

(i) $T(r)\leq N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+S(r)$,

(ii) $F\equiv G$,

(iii) $FG\equiv 1$,

where $T(r)=\max\{T(r,F),T(r,G)\}$ and $S(r)=o(T(r))$, $r\in I$, I is a set of infinite linear measure of $r\in(0,\infty)$.

Proof. Let z_0 be a pole of f which is not a pole or zero of $a(z)$. Then z_0 is a pole of F and G simultaneously. Thus F and G share those pole of f which is not zero or pole of $a(z)$. Clearly

$$\begin{aligned}N(r,\infty;\Delta)\leq{}&\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_L(r,\infty;F)+\overline{N}_L(r,\infty;G)+\overline{N}_0(r,0;F^{'})\\&+\overline{N}_0(r,0;G^{'})+S(r,F)+S(r,G).\end{aligned}$$

Rest of the proof can be carried out in the line of proof of Lemma 2.13 of [1]. So we omit the details

Lemma 2.6. *([18]) The inequality $N(r,\infty;H[f])\leq\bar{d}(H)N(r,\infty;f)+(\Gamma_P-\bar{d}(H))\overline{N}(r,\infty;f)$ holds.*

Lemma 2.7. *([20]) Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{i=0}^{n}a_if^i}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$ where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=pT(r,f)+S(r,f),$$

where $p=\max\{n,m\}$.

Lemma 2.8. *([7, 12]) Let f be a meromorphic function and $H[f]$ be a differential polynomial. Then*

$$m\left(r, \frac{H[f]}{f^{\bar{d}(H)}}\right) \leq (\bar{d}(H) - \underline{d}(H))m\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 2.9. *([2, 3]) Let $H[f]$ be a differential polynomial generated by a non-constant meromorphic function f. Then*

$$\begin{aligned} N\left(r,\infty; \frac{H[f]}{f^{\bar{d}(H)}}\right) &\leq (\Gamma_P - \bar{d}(H))\overline{N}(r,\infty;f) + (\bar{d}(H) - \underline{d}(H))N(r,0;f \mid\geq k+1) \\ &+ Q\overline{N}(r,0;f\mid\geq k+1) + \bar{d}(H)N(r,0;f\mid\leq k) + S(r,f). \end{aligned}$$

Lemma 2.10. *Let f be a non-constant meromorphic function and $a(z)$ be a small function in f. Let us define $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{H[f]}{a}$. Then $FG \not\equiv 1$.*

Proof. On contrary, assume that $FG \equiv 1,$ i.e., $f_1^p P(f_1)H[f] = (a(z))^2$. Then

$$N(r,0;f\mid\geq k+1) = S(r,f).$$

Now applying Lemmas 2.8, 2.9 and the first fundamental theorem, we get

$$\begin{aligned} (m+n+\bar{d}(H))T(r,f) &= T\left(r, \frac{a^2}{f_1^p P(f_1) f^{\bar{d}(H)}}\right) + S(r,f) = T\left(r, \frac{H[f]}{f^{\bar{d}(H)}}\right) + S(r,f) \\ &\leq m\left(r, \frac{H[f]}{f^{\bar{d}(H)}}\right) + N\left(r, \frac{H[f]}{f^{\bar{d}(H)}}\right) + S(r,f) \\ &\leq (\bar{d}(H) - \underline{d}(H))[T(r,f) - \{N(r,0;f\mid\leq k) + N(r,0;f\mid\geq k+1)\}] \\ &+ (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty;f) + (\bar{d}(H) - \underline{d}(H))N(r,0;f\mid\geq k+1) \\ &+ Q\overline{N}(r,0;f\mid\geq k+1) + \bar{d}(H)N(r,0;f\mid\leq k) + S(r,f) \\ &\leq (\bar{d}(H) - \underline{d}(H))T(r,f) + \underline{d}(H)N(r,0;f\mid\leq k) \\ &+ (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty;f) + S(r,f) \\ &\leq \bar{d}(H)T(r,f) + (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty; f_1^p P(f_1)) + S(r,f) \\ &\leq \bar{d}(H)T(r,f) + (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty;(a(z))^2) + S(r,f) \\ &\leq \bar{d}(H)T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction.

Lemma 2.11. *([5]) For the differential polynomial $H[f]$,*

$$\begin{aligned} N(r,0;H[f]) &\leq (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty;f) + \underline{d}(H)\overline{N}(r,0;f) \\ &+ (\bar{d}(H) - \underline{d}(H))\left(m\left(r,\frac{1}{f}\right) + T(r,f)\right) + S(r,f). \end{aligned}$$

Lemma 2.12. *([5]) Let j and p be two positive integers satisfying $j \geq p+1$ and $\Gamma_P > (k+1)\underline{d}(H) - (p+1)$. Then for the differential polynomial $H[f]$,*

$$\overline{N}_{(j+\Gamma_P-\underline{d}(H)}(r,0;\underline{d}(H)) \leq \overline{N}_{(j}(r,0;H[f]).$$

Lemma 2.13. *Let j and p be two positive integers satisfying $j \geq p+1$ and $\Gamma_P > (k+1)\underline{d}(H) - (p+1)$. Then for a differential polynomial $H[f]$,*

Proof. From Lemmas 2.11 and 2.12 we have

$$
\begin{aligned}
N_P(r,0;H[f]) &\leq (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty;f) + \overline{N}(r,0;f^{\underline{d}(H)}) + (\bar{d}(H)-\underline{d}(H))\\
&\left(m\left(r,\frac{1}{f}\right)+T(r,f)\right) - \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,0;H[f]) + S(r,f)\\
&\leq (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty;f) + N_{P+\Gamma_P-\underline{d}(H)}(r,0;f^{\underline{d}(H)})\\
&+ (\bar{d}(H)-\underline{d}(H))\left(m\left(r,\frac{1}{f}\right)+T(r,f)\right)\\
&+ \sum_{j=p+\Gamma_P-\underline{d}(H)+1}^{\infty}\overline{N}_{(j}(r,0;f^{\underline{d}(H)}) - \sum_{j=p+1}^{\infty}\overline{N}_{(j}(r,0;H[f]) + S(r,f)\\
&\leq (\Gamma_p - \bar{d}(H))\overline{N}(r,\infty;f) + N_{P+\Gamma_P-\underline{d}(H)}(r,0;f^{\underline{d}(H)})\\
&+ (\bar{d}(H)-\underline{d}(H))\left(m\left(r,\frac{1}{f}\right)+T(r,f)\right) + S(r,f).
\end{aligned}
$$

This completes the proof.

3 Proof of Theorem

Proof of Theorem 1.1. Let $F = \frac{\mathcal{P}(f)}{a} = \frac{f_1^p P(f_1)}{a}$ and $G = \frac{H[f]}{a}$. Then $F-1 = \frac{f_1^p P(f_1)-a}{a}$ and $G-1 = \frac{H[f]-a}{a}$. Since $\mathcal{P}(f)$ and $H[f]$ share (a,l), it follows that F and G share $(1,l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. First we assume that $\Delta \not\equiv 0$.

Subcase 1.1. If $l \geq 1$, then using the second fundamental Theorem and Lemmas 2.2 and 2.4, we get

$$
\begin{aligned}
T(r,F)+T(r,G) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+N(r,\infty;\Delta)\\
&+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)-\overline{N}_0(r,0;F^{'})\\
&-\overline{N}_0(r,0;G^{'})+S(r,f).\\
&\leq 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)\\
&+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f). \qquad (3.1)
\end{aligned}
$$

Subcase 1.1.1. If $l \geq 2$, then using the inequality (3.1), we get

$$
\begin{aligned}
T(r,F)+T(r,G) &\leq 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)+\overline{N}_E^{(2}(r,1;F)\\
&+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}(r,1;G)+S(r,f)\\
&\leq 2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\mu_2 N_{\mu_2^*}(r,w_p;f)+(m+n-p)T(r,f)\\
&+N_2(r,0;G)+N(r,1;G)+S(r,f).
\end{aligned}
$$

i.e., for any $\epsilon > 0$, in view of Lemma 2.13, the above inequality becomes

$$
\begin{aligned}
(m+n)T(r,f) &\leq (\Gamma_P-\bar{d}(H)+3)\overline{N}(r,\infty;f)+\mu_2 N_{\mu_2^*}(r,w_p;f)+(m+n-p)T(r,f)\\
&+N_{2+\Gamma_P-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2(\bar{d}(H)-\underline{d}(H))T(r,f)+S(r,f)\\
&\leq \{(\Gamma_P-\bar{d}(H)+3)-(\Gamma_P-\bar{d}(H)+3)\Theta(\infty,f)+\mu_2-\mu_2\delta_{\mu_2^*}(w_p,f)\\
&+\underline{d}(H)-\underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(r,0;f)+(m+n-p)+2(\bar{d}(H)-\underline{d}(H))+\epsilon\}T(r,f)\\
&+S(r,f).
\end{aligned}
$$

i.e.,

$$(\Gamma_P - \bar{d}(H) + 3)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(r, 0; f) \le \Gamma_P + \mu_2 + 3 - p + \bar{d}(H) - \underline{d}(H),$$

which contradicts to the condition (1.7) of Theorem 1.16.

Subcase 1.1.2. If $l = 1$, then using the inequality (3.1) and Lemma 2.3, we get

$$\begin{aligned} T(r,F) + T(r,G) &\le 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) + \overline{N}_E^{(2}(r,1;F) \\ &+ 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,f) \\ &\le \frac{5}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) + \mu_2 N_{\mu_2^*}(r,w_p;f) \\ &+ (m+n-p)T(r,f) + N_2(r,0;G) + N(r,1;G) + S(r,f). \end{aligned}$$

i.e., for any $\epsilon > 0$, in view of Lemma 2.13, the above inequality becomes

$$\begin{aligned} (m+n)T(r,f) &\le (\Gamma_P - \bar{d}(H) + \frac{7}{2})\overline{N}(r,\infty;f) + \frac{1}{2}\overline{N}(r,w_p;f) + \frac{1}{2}(m+n-p)T(r,f) \\ &+ \mu_2 N_{\mu_2^*}(r,w_p;f) + (m+n-p)T(r,f) + N_{2+\Gamma_P-\underline{d}(H)}(r,0;f^{\underline{d}(H)}) \\ &+ 2(\bar{d}(H) - \underline{d}(H))T(r,f) + S(r,f) \\ &\le \{(\Gamma_P - \bar{d}(H) + \frac{7}{2}) - (\Gamma_P - \bar{d}(H) + \frac{7}{2})\Theta(\infty,f) + \frac{1}{2} - \frac{1}{2}\Theta(w_p,f) \\ &+ \mu_2 - \mu_2\delta_{\mu_2^*}(w_p,f) + \underline{d}(H) - \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(r,0;f) + \frac{3(m+n-p)}{2} \\ &+ 2(\bar{d}(H) - \underline{d}(H)) + \epsilon\}T(r,f) + S(r,f). \end{aligned}$$

i.e.,

$$\begin{aligned} (\Gamma_P - \bar{d}(H) + \frac{7}{2})\Theta(\infty,f) + \frac{1}{2}\Theta(w_p,f) + \mu_2\delta_{\mu_2^*}(w_p,f) + \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(r,0;f) \\ \le \Gamma_P + \mu_2 + 4 + \frac{m+n-3p}{2} + \bar{d}(H) - \underline{d}(H), \end{aligned}$$

which contradicts to the condition (1.8) of Theorem 1.16.

Subcase 1.2. If $l = 0$, then applying the second fundamental theorem and Lemmas 2.2, 2.3, 2.4 we get

$$\begin{aligned} T(r,F) + T(r,G) &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) \\ &+ \overline{N}(r,1;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + N(r,\infty;\Delta) \\ &+ \overline{N}_E^{(2}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) - \overline{N}_0(r,0;F') \\ &- \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ &\le 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) + \overline{N}_E^{(2}(r,1;F) \\ &+ 2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,f) \\ &\le 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \mu_2 N_{\mu_2^*}(r,w_p;f) + (m+n-p)T(r,f) \\ &+ N_2(r,0;G) + 2(\overline{N}(r,\infty;F) + \overline{N}(r,0;F)) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) \\ &+ \overline{N}_E^{(2}(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G) + S(r,f) \\ &\le 4\overline{N}(r,\infty;F) + \mu_2 N_{\mu_2^*}(r,w_p;f) + (m+n-p)T(r,f) + N_2(r,0;G) \\ &+ 2\overline{N}(r,\infty;G) + \overline{N}(r,0;G) + 2\overline{N}(r,0;F) + T(r,G) + S(r,f) \end{aligned}$$

i.e., for any $\epsilon > 0$, in view of Lemma 2.13, the above inequality becomes

$$
\begin{aligned}
(m+n)T(r,f) \leq{} & (2(\Gamma_P - \bar{d}(H)) + 6)\overline{N}(r,\infty;f) + \mu_2 N_{\mu_2^*}(r,w_p;f) + (m+n-p)T(r,f) \\
& + N_{2+\Gamma_P-\underline{d}(H)}(r,0;f^{\underline{d}(H)}) + 2(\bar{d}(H)-\underline{d}(H))T(r,f) + N_{1+\Gamma_P-\underline{d}(H)}(r,0;f^{\underline{d}(H)}) \\
& + 2\overline{N}(r,w_p;f) + 2(m+n-p)T(r,f) + S(r,f) \\
& \leq \{(2(\Gamma_P - \bar{d}(H)) + 6) - (2(\Gamma_P - \bar{d}(H)) + 6)\Theta(\infty,f) + \mu_2 - \mu_2\delta_{\mu_2^*}(w_p,f) \\
& + \underline{d}(H) - \underline{d}(H)\delta_{2+\Gamma_P-\underline{d}(H)}(r,0;f) + \underline{d}(H) - \underline{d}(H)\delta_{1+\Gamma_P-\underline{d}(H)}(r,0;f) + 2- \\
& 2\Theta(w_p,f) + (m+n-p) + 2(\bar{d}(H)-\underline{d}(H)) + \epsilon\}T(r,f) + S(r,f).
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(2(\Gamma_P - \bar{d}(H)) + 6)\Theta(\infty,f) + 2\Theta(w_p,f) + \mu_2\delta_{\mu_2^*}(w_p,f) + \underline{d}(H)\left(\sum_{i=1}^{2}\delta_{i+\Gamma_P-\underline{d}(H)}(r,0;f)\right) \\
\leq 2\Gamma_P + \mu_2 + 8 + 2(m+n) - 3p + 2(\bar{d}(H)-\underline{d}(H)),
\end{aligned}
$$

which contradicts to the condition (1.9) of Theorem 1.16.

Case 2. Next we assume that $\Delta \equiv 0$. Then on integration of (2.1), we get,

$$
\frac{1}{G-1} \equiv \frac{A}{F-1} + B, \tag{3.2}
$$

where $A(\neq 0)$ and B are complex constants. Clearly F and G share $(1,\infty)$. Also, by construction of F and G, F and G share $(\infty,0)$. So using Lemma 2.13 and condition (1.7) of Theorem 1.16, we obtain

$$
\begin{aligned}
& N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_L(r,\infty;F) + \overline{N}_L(r,\infty;G) + S(r) \\
& \leq \mu_2 N_{\mu_2^*}(r,w_p;f) + (m+n-p)T(r,f) + N_{2+\Gamma_P-\underline{d}(H)}(r,0;f^{\underline{d}(H)}) \\
& + (\Gamma_P - \underline{d}(H) + 3)\overline{N}(r,\infty;f) + 2(\bar{d}(H)-\underline{d}(H))T(r,f) + S(r) \\
& \leq \{\Gamma_P + 3 + \mu_2 + m + n - p + 2(\bar{d}(H)-\underline{d}(H)) - (\Gamma_P + \mu_2 + 3 - p + 2\bar{d}(H) - 2\underline{d}(H)) + \epsilon\} \\
& T(r,f) + S(r) \\
& \leq (m+n)T(r,f) + S(r) \\
& < T(r,F) + S(r).
\end{aligned}
$$

Hence inequality (1) of Lemma 2.5 does not hold. Again in view of Lemma 2.10, we get $FG \not\equiv 1$. Therefore $F \equiv G$ i.e., $\mathcal{P}(f) \equiv H[f]$.

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