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Certain recursion formulas for new hypergeometric functions of four variables

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Abstract Inspired by certain study of recursion formulas involving multivariable hypergeometric functions [14, 15, 16, 17, 18, 22, 23]. In this article, we introduce five new quadruple hypergeometric functions together with their regions of convergence and then we establish certain recursion relations for these new functions. This enriches the theory of special functions. The quadruple hypergeometric functions and results listed here are believed to be new.

1 Introduction

The multivariable multiple hypergeometric functions are highly significant in special functions theory, used in a wide range of applications, such as integral representations, generating functions, recurrence relations, finite and infinite sums, analytic continuation, asymptotic behaviour and some special formulas (linear transformation, quadratic transformation, decomposition, reduction, limit and differentiation (see, e.g., [1, 2, 3, 4, 5, 6, 7, 11, 20, 21, 25] and elsewhere). Besides from these studies, they are also utilized in theories such as perturbation theory (in numerical analysis) and quantum theory (in modern physic) [10, 13]. In [5], Exton presented twenty one complete hypergeometric functions in four variables denoted by the symbols $K_1, K_2, ..., K_{21}$. In [11], Sharma and Parihar defined eighty three complete quadruple hypergeometric functions, namely $F_1^{(4)}, F_2^{(4)}, ..., F_{83}^{(4)}$. Bin-Saad and Younis [4] presented thirty new quadruple hypergeometric functions given by $X_1^{(4)}, X_2^{(4)}, ..., X_{30}^{(4)}$. In [15], the authors discovered the existence of twenty additional complete hypergeometric functions in four variables $X_{31}^{(4)}, X_{32}^{(4)}, \cdots, X_{50}^{(4)}$. Each quadruple hypergeometric function in [4, 5, 13, 17] is of the form:

$$X^{(4)}(.) = \sum_{m,n,p,q=0}^{\infty} \Omega(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!},$$

where $\Omega(m,n,p,q)$ is a certain sequence of complex parameters and there are twelve parameters in each series $X^{(4)}(.)$ (eight $a's$ and four $c's$). The 1st, 2nd, 3rd and 4th parameters in $X^{(4)}(.)$ are connected with the integers m, n, p and q, respectively. Each repeated parameter in the series $X^{(4)}(.)$ points out a term with double parameters in $\Omega(m,n,p,q)$.
For example,

$$X^{(4)}(a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_5),$$

means that: $(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q$ includes the term.
Similarly,

$$X^{(4)}(a_1, a_1, a_1, a_2, a_1, a_1, a_2, a_3),$$

points out the term $(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q$,
and

$$X^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_4),$$

shows the existence of the term $(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_n(a_4)_q$. Thus, it is possible to form various combinations of indices. There seems to be no way of establishing independently the number of distinct Gaussian hypergeometric series for any given integer $n \geq 2$ without stating explicitly all

such series. Thus, in every situation with $n = 4$, one ought to begin by actually constructing the set just as in the case $n = 3$ (see [18]).
By using the conventions and notations above, we define the following quadruple hypergeometric functions:

$$X_{100}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_4; c_1, c_2, c_1, c_2; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_n(a_4)_q}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!};$$
$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |u| < 1\right), \tag{1.1}$$

$$X_{101}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_4; c_1, c_2, c_2, c_1; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_n(a_4)_q}{(c_1)_{m+q}(c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!};$$
$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |u| < 1\right), \tag{1.2}$$

$$X_{102}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_4; c, c, c, c; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_n(a_4)_q}{(c)_{m+n+p+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!};$$
$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |u| < 1\right), \tag{1.3}$$

$$X_{103}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_3; c_1, c_2, c_3, c_4; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_{p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!};$$
$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right), \tag{1.4}$$

$$X_{104}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_3; c_1, c_1, c_2, c_3; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!};$$
$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right), \tag{1.5}$$

where $(a)_n$ is the well-known Pochhammer symbol given by

$$(a)_n := \begin{cases} 1, & (n = 0) \\ a(a+1)...(a+n-1), & (n \in \mathbb{N} := \{1, 2, ...\}) \end{cases}$$
$$= \frac{\Gamma(a+n)}{\Gamma(a)},$$

$\Gamma(a)$ is the well-known Gamma function defined by

$$\Gamma(a) = \int_0^{\infty} e^{-t} t^{a-1} dt, \quad (Re(a) > 0).$$

Many authors have investigated several recursion formulas involving classical hypergeometric functions. In [8], Opps et al. introduced the recursion formulas for Appell's function F_2 and gave its applications to radiation field problems. Wang [16] presented the recursion formulas for Appell functions F_1, F_2, F_3 and F_4. Sahai and Verma [9, 12] established the recursion formulas for Lauricella's triple functions, Srivastava hypergeometric functions in three variables, k-variable Lauricella functions and the Srivastava-Daoust and related multivariable hypergeometric functions. In [10, 11, 17], it was given the recursion formulas for Srivastava general triple hypergeometric function, Exton's triple hypergeometric functions and certain quadruple hypergeometric functions. In this paper, we aim to establish certain recursion formulas related to the functions $X_i^{(4)} (i = 100, 101, ..., 104)$.

In the following, some abbreviated notations are used in this paper. We, for example, write:
•$X_{100}^{(4)}$ for the function $X_{100}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_4; c_1, c_2, c_1, c_2; x, y, z, u\right)$.
•$X_{100}^{(4)}(a_2 + n)$ for $X_{100}^{(4)} \left(a_1, a_1, a_2 + n, a_2 + n, a_1, a_3, a_2 + n, a_4; c_1, c_2, c_1, c_2; x, y, z, u\right)$.
•The notation $X_{100}^{(4)}(a_2 + n, c_2 + n_1)$ stands for:
$X_{100}^{(4)} \left(a_1, a_1, a_2 + n, a_2 + n, a_1, a_3, a_2 + n, a_4; c_1, c_2 + n_1, c_1, c_2 + n_1; x, y, z, u\right)$, etc.

2 Main Results

Here, we establish the recursion formulas for the functions $X_{100}^{(4)}, X_{101}^{(4)}, ..., X_{104}^{(4)}$. Throughout the paper, n denotes a non-negative integer.

Theorem 2.1. *The following recursion formulas hold true for the numerator parameter* a_1, a_2,

a_3, a_4 *of the* $X_{100}^{(4)}$:

$$X_{100}^{(4)}(a_1+n) = X_{100}^{(4)} + \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)X_{100}^{(4)}(a_1+1+n_1, c_1+1) + \frac{ya_3}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_1+n_1, a_3+1, c_2+1), \tag{2.1}$$

$$X_{100}^{(4)}(a_1-n) = X_{100}^{(4)} - \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)X_{100}^{(4)}(a_1+2-n_1, c_1+1) - \frac{ya_3}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_1+1-n_1, a_3+1, c_2+1), \tag{2.2}$$

$$X_{100}^{(4)}(a_2+n) = X_{100}^{(4)} + \frac{2z}{c_1}\sum_{n_1=1}^{n}(a_2+n_1)X_{100}^{(4)}(a_2+1+n_1, c_1+1) + \frac{ua_4}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_2+n_1, a_4+1, c_2+1), \tag{2.3}$$

$$X_{100}^{(4)}(a_2-n) = X_{100}^{(4)} - \frac{2z}{c_1}\sum_{n_1=1}^{n}(a_2+1-n_1)X_{100}^{(4)}(a_2+2-n_1, c_1+1) - \frac{ua_4}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_2+1-n_1, a_4+1, c_2+1), \tag{2.4}$$

$$X_{100}^{(4)}(a_3+n) = X_{100}^{(4)} + \frac{ya_1}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_1+1, a_3+n_1, c_2+1), \tag{2.5}$$

$$X_{100}^{(4)}(a_3-n) = X_{100}^{(4)} - \frac{ya_1}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_1+1, a_3+1-n_1, c_2+1), \tag{2.6}$$

$$X_{100}^{(4)}(a_4+n) = X_{100}^{(4)} + \frac{ua_2}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_2+1, a_4+n_1, c_2+1), \tag{2.7}$$

$$X_{100}^{(4)}(a_4-n) = X_{100}^{(4)} - \frac{ua_2}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_2+1, a_4+1-n_1, c_2+1). \tag{2.8}$$

Proof. From the definition of the hypergeometric function $X_{100}^{(4)}$ and the relation

$$(a_1+1)_{2m+n} = (a_1)_{2m+n}\Big(1+\frac{2m}{a_1}+\frac{n}{a_1}\Big) \tag{2.9}$$

we obtain the following contiguous relation:

$$X_{100}^{(4)}(a_1+1) \tag{2.10}$$

$$= X_{100}^{(4)} + \frac{2x}{c_1}(a_1+1)\,X_{100}^{(4)}(a_1+2,\ c_1+1) + \frac{ya_3}{c_2}X_{100}^{(4)}(a_1+1, a_3+1, c_2+1). \tag{2.11}$$

To obtain a contiguous relation for $X_{100}^{(4)}(a_1+2)$, we replace $a_1 \to a_1+1$ in (2.11) and simplify. This leads to

$$X_{100}^{(4)}(a_1+2) = X_1 + \frac{2x_1}{c_1}\sum_{n_1=1}^{2}(a_1+n_1)\,X_{100}^{(4)}(a_1+n_1+1,\ c_1+1) + \frac{ya_3}{c_2}\sum_{n_1=1}^{2}X_{100}^{(4)}(a_1+n_1, a_3+1, c_2+1). \tag{2.12}$$

Iterating this process n times, we obtain (2.1). For the proof of (2.2), replace the parameter $a_1 \to a_1 - 1$ in (2.11). This gives

$$X_{100}^{(4)}(a_1 - 1) \;=\; X_1 - \frac{2x}{c_1} a_1\, X_{100}^{(4)}(a_1+1\,, c_1+1) - \frac{y a_3}{c_2} X_{100}^{(4)}(a_3+1,\, c_2+1). \tag{2.13}$$

Iteratively, we get (2.2).

The recursion formulas from (2.3)-(2.8) can be proved in a similar manner.

□

Theorem 2.2. *The following recursion formulas hold true for the numerator parameter a_3, a_4 of the $X_{100}^{(4)}$:*

$$\begin{aligned} &X_{100}^{(4)}(a_3+n) \\ &= \sum_{n_1 \le n} \binom{n}{n_1} \frac{(a_1)_{n_1} y^{n_1}}{(c_2)_{n_1}}\, X_{100}^{(4)}(a_1+n_1, a_3+n_1,\, c_2+n_1), \end{aligned} \tag{2.14}$$

$$\begin{aligned} &X_{100}^{(4)}(a_3-n) \\ &= \sum_{n_1 \le n} \binom{n}{n_1} \frac{(a_1)_{n_1} (-y)^{n_1}}{(c_2)_{n_1}}\, X_{100}^{(4)}(a_1+n_1,\, c_2+n_1), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &X_{100}^{(4)}(a_4+n) \\ &= \sum_{n_1 \le n} \binom{n}{n_1} \frac{(a_2)_{n_1} u^{n_1}}{(c_2)_{n_1}}\, X_{100}^{(4)}(a_2+n_1, a_4+n_1,\, c_2+n_1), \end{aligned} \tag{2.16}$$

$$\begin{aligned} &X_{100}^{(4)}(a_4-n) \\ &= \sum_{n_1 \le n} \binom{n}{n_1} \frac{(a_2)_{n_1} (-u)^{n_1}}{(c_2)_{n_1}}\, X_{100}^{(4)}(a_2+n_1,\, c_2+n_1). \end{aligned} \tag{2.17}$$

Proof. Using the definition of $X_{100}^{(4)}$ and the relation

$$(a_3+1)_n = (a_3)_n\Big(1 + \frac{n}{a_2}\Big), \tag{2.18}$$

we get

$$X_{100}^{(4)}(a_3+1) = X_{100}^{(4)} + \frac{a_1 y}{c_2}\, X_{100}^{(4)}(a_1+1, a_3+1, c_2+1). \tag{2.19}$$

Replace $a_3 \to a_3 + 1$ in (2.19) to get

$$\begin{aligned} X_{100}^{(4)}(a_3+2) &= X_{100}^{(4)} + \frac{a_1 y}{c_2}\, X_{100}^{(4)}(a_1+1,\, a_3+1,\, c_1+1) \\ &+ \frac{a_1 y}{c_2}\Big[X_{100}^{(4)}(a_1+1,\, a_3+1,\, c_1+1) + \frac{(a_1+1)y}{(c_2+1)}\, X_{100}^{(4)}(a_1+2,\, a_2+2,\, c_1+2)\Big]. \end{aligned} \tag{2.20}$$

Simplifying, we get

$$\begin{aligned} X_{100}^{(4)}(a_3+2) = X_{100}^{(4)} &+ \frac{2a_1 y}{c_2}\, X_{100}^{(4)}(a_1+1,\, a_3+1,\, c_2+1) \\ &+ \frac{(a_1)_2\, y^2}{(c_2)_2}\, X_{100}^{(4)}(a_1+2,\, a_3+2,\, c_2+2). \end{aligned} \tag{2.21}$$

Iterating this process n times, we get (2.14). Proof of (2.15) is similar.

By using Pascal's identity, the recursion formulas (2.14) and (2.15) can also be proved by an induction method. □

The recursion formulas from (2.16)-(2.17) can be proved in a similar manner.

Theorem 2.3. *The following recursion formulas hold true for the denominator parameter* c_1, c_2 *of the* $X_{100}^{(4)}$*:*

$$\begin{aligned} X_{100}^{(4)}(c_1-n) &= X_{100}^{(4)} \\ &+ (a_1)_2\, x \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_{100}^{(4)}(a_1+2,\, c_1+2-n_1) \\ &+ (a_2)_2 z \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_{100}^{(4)}(a_2+2, c_1+2-n_1), \end{aligned} \tag{2.22}$$

$$\begin{aligned} X_{100}^{(4)}(c_2-n) &= X_{100}^{(4)} \\ &+ a_1 a_3\, y \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_{100}^{(4)}(a_1+1, a_3+1, c_1+2-n_1) \\ &+ a_2 a_4 u \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)}\, X_{100}^{(4)}(a_2+1,\, a_4+1,\, c_2+2-n_1). \end{aligned} \tag{2.23}$$

Proof. Using the definition of the hypergeometric function $X_{100}^{(4)}$ and the relation

$$\frac{1}{(c_1-1)_{m+p}} = \frac{1}{(c_1)_{m+p}} \left(1 + \frac{m}{c_1-1} + \frac{p}{c_1-1}\right), \tag{2.24}$$

we have

$$\begin{aligned} X_{100}^{(4)}(c_1-1) = X_{100}^{(4)} + \frac{(a_1)_2\, x}{c_1(c_1-1)} X_{100}^{(4)}(a_1+2,\, c_1+1) + \\ \frac{(a_2)_2 z}{c_1(c_1-1)} X_{100}^{(4)}(a_2+2,\, c_1+1). \end{aligned} \tag{2.25}$$

Using this contiguous relation to the $X_{100}^{(4)}$ with the parameter c_1-n for n times, we get (2.22). The second recursion formula (2.23) can be proved in a similar manner. □

Now, we present the recursion formulas for other hypergeometric functions from $X_{101}^{(4)} - X_{104}^{(4)}$. We omit the proof of the given below theorems.

Theorem 2.4. *The following recursion formulas hold true for the numerator parameter* a_1, a_2,

a_3, a_4 *of the* $X_{101}^{(4)}$*:*

$$\begin{aligned}
X_{101}^{(4)}(a_1+n) &= X_{101}^{(4)} + \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)X_{101}^{(4)}(a_1+1+n_1,c_1+1)\\
&\quad + \frac{ya_3}{c_2}\sum_{n_1=1}^{n}X_{101}^{(4)}(a_1+n_1,a_3+1,c_2+1),\\
X_{101}^{(4)}(a_1-n) &= X_{101}^{(4)} - \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)X_{101}^{(4)}(a_1+2-n_1,c_1+1)\\
&\quad - \frac{ya_3}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_1+1-n_1,a_3+1,c_2+1),\\
X_{101}^{(4)}(a_2+n) &= X_{101}^{(4)} + \frac{2z}{c_2}\sum_{n_1=1}^{n}(a_2+n_1)X_{101}^{(4)}(a_2+1+n_1,c_2+1)\\
&\quad + \frac{ua_4}{c_1}\sum_{n_1=1}^{n}X_{101}^{(4)}(a_2+n_1,a_4+1,c_1+1),\\
X_{101}^{(4)}(a_2-n) &= X_{101}^{(4)} - \frac{2z}{c_2}\sum_{n_1=1}^{n}(a_2+1-n_1)X_{101}^{(4)}(a_2+2-n_1,c_2+1)\\
&\quad - \frac{ua_4}{c_1}\sum_{n_1=1}^{n}X_{101}^{(4)}(a_2+1-n_1,a_4+1,c_1+1),\\
X_{101}^{(4)}(a_3+n) &= X_{101}^{(4)} + \frac{ya_1}{c_2}\sum_{n_1=1}^{n}X_{101}^{(4)}(a_1+1,a_3+n_1,c_2+1),\\
X_{101}^{(4)}(a_3-n) &= X_{101}^{(4)} - \frac{ya_1}{c_2}\sum_{n_1=1}^{n}X_{101}^{(4)}(a_1+1,a_3+1-n_1,c_2+1),\\
X_{101}^{(4)}(a_4+n) &= X_{101}^{(4)} + \frac{ua_2}{c_1}\sum_{n_1=1}^{n}X_{101}^{(4)}(a_2+1,a_4+n_1,c_1+1),\\
X_{101}^{(4)}(a_4-n) &= X_{101}^{(4)} - \frac{ua_2}{c_1}\sum_{n_1=1}^{n}X_{101}^{(4)}(a_2+1,a_4+1-n_1,c_1+1).
\end{aligned} \tag{2.26}$$

Theorem 2.5. *The following recursion formulas hold true for the numerator parameter* a_3, a_4 *of* *the* $X_{101}^{(4)}$*:*

$$\begin{aligned}
&X_{101}^{(4)}(a_3+n)\\
&=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(a_1)_{n_1}y^{n_1}}{(c_2)_{n_1}}\,X_{101}^{(4)}(a_1+n_1,a_3+n_1,\,c_2+n_1), && (2.27)\\
&X_{101}^{(4)}(a_3-n)\\
&=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-y)^{n_1}}{(c_2)_{n_1}}\,X_{101}^{(4)}(a_1+n_1,\,c_2+n_1), && (2.28)\\
&X_{101}^{(4)}(a_4+n)\\
&=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(a_2)_{n_1}u^{n_1}}{(c_1)_{n_1}}\,X_{101}^{(4)}(a_2+n_1,a_4+n_1,\,c_1+n_1), && (2.29)\\
&X_{101}^{(4)}(a_4-n)\\
&=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(a_2)_{n_1}(-u)^{n_1}}{(c_1)_{n_1}}\,X_{101}^{(4)}(a_2+n_1,\,c_1+n_1), && (2.30)
\end{aligned}$$

Theorem 2.6. *The following recursion formulas hold true for the denominator parameter* c_1, c_2 *of the* $X_{101}^{(4)}$*:*

$$\begin{aligned}
X_{101}^{(4)}(c_1-n) = X_{101}^{(4)}+&\\
&(a_1)_2\,x\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{101}^{(4)}(a_1+2,\,c_1+2-n_1)+\\
&a_2a_4u\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{101}^{(4)}(a_2+1,a_4+1,c_1+2-n_1), \qquad (2.31)\\
X_{101}^{(4)}(c_2-n) = X_{101}^{(4)}+&\\
&a_1a_3\,y\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{101}^{(4)}(a_1+1,a_3+1,c_1+2-n_1)\\
&+(a_2)_2z\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{101}^{(4)}(a_2+2,c_2+2-n_1). \qquad (2.32)
\end{aligned}$$

Theorem 2.7. *The following recursion formulas hold true for the numerator parameter* a_1, a_2,

a_3, a_4 *of the* $X_{102}^{(4)}$*:*

$$
\begin{aligned}
X_{102}^{(4)}(a_1+n) &= X_{102}^{(4)} + \frac{2x}{c}\sum_{n_1=1}^{n}(a_1+n_1)X_{102}^{(4)}(a_1+1+n_1,c+1)\\
&\quad + \frac{ya_3}{c}\sum_{n_1=1}^{n}X_{102}^{(4)}(a_1+n_1,a_3+1,c+1),\\
X_{102}^{(4)}(a_1-n) &= X_{102}^{(4)} - \frac{2x}{c}\sum_{n_1=1}^{n}(a_1+1-n_1)X_{102}^{(4)}(a_1+2-n_1,c+1)\\
&\quad - \frac{ya_3}{c}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_1+1-n_1,a_3+1,c+1),\\
X_{102}^{(4)}(a_2+n) &= X_{102}^{(4)} + \frac{2z}{c}\sum_{n_1=1}^{n}(a_2+n_1)X_{102}^{(4)}(a_2+1+n_1,c+1)\\
&\quad + \frac{ua_4}{c}\sum_{n_1=1}^{n}X_{102}^{(4)}(a_2+n_1,a_4+1,c+1),\\
X_{102}^{(4)}(a_2-n) &= X_{102}^{(4)} - \frac{2z}{c}\sum_{n_1=1}^{n}(a_2+1-n_1)X_{101}^{(4)}(a_2+2-n_1,c+1)\\
&\quad - \frac{ua_4}{c}\sum_{n_1=1}^{n}X_{102}^{(4)}(a_2+1-n_1,a_4+1,c+1),\\
X_{102}^{(4)}(a_3+n) &= X_{102}^{(4)} + \frac{ya_1}{c}\sum_{n_1=1}^{n}X_{102}^{(4)}(a_1+1,a_3+n_1,c+1),\\
X_{102}^{(4)}(a_3-n) &= X_{102}^{(4)} - \frac{ya_1}{c}\sum_{n_1=1}^{n}X_{102}^{(4)}(a_1+1,a_3+1-n_1,c+1),\\
X_{102}^{(4)}(a_4+n) &= X_{102}^{(4)} + \frac{ua_2}{c}\sum_{n_1=1}^{n}X_{102}^{(4)}(a_2+1,a_4+n_1,c+1),\\
X_{102}^{(4)}(a_4-n) &= X_{102}^{(4)} - \frac{ua_2}{c}\sum_{n_1=1}^{n}X_{102}^{(4)}(a_2+1,a_4+1-n_1,c+1).
\end{aligned}
\tag{2.33}
$$

Theorem 2.8. *The following recursion formulas hold true for the numerator parameter* a_3, a_4 *of* *the* $X_{102}^{(4)}$*:*

$$X_{102}^{(4)}(a_3+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(a_1)_{n_1}y^{n_1}}{(c)_{n_1}}X_{102}^{(4)}(a_1+n_1,a_3+n_1,\,c+n_1), \tag{2.34}$$

$$X_{102}^{(4)}(a_3-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-y)^{n_1}}{(c)_{n_1}}X_{102}^{(4)}(a_1+n_1,\,c+n_1), \tag{2.35}$$

$$X_{102}^{(4)}(a_4+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(a_2)_{n_1}u^{n_1}}{(c)_{n_1}}X_{102}^{(4)}(a_2+n_1,a_4+n_1,\,c+n_1), \tag{2.36}$$

$$X_{102}^{(4)}(a_4-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(a_2)_{n_1}(-u)^{n_1}}{(c)_{n_1}}X_{102}^{(4)}(a_2+n_1,\,c+n_1). \tag{2.37}$$

Theorem 2.9. *The following recursion formulas hold true for the denominator parameter c of the* $X_{102}^{(4)}$*:*

$$\begin{aligned} X_{102}^{(4)}(c_1-n) = X_{101}^{(4)} &+ (a_1)_2\,x\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{102}^{(4)}(a_1+2,\,c+2-n_1)\\ &+ a_1a_3y\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{102}^{(4)}(a_1+1,a_3+1,c+2-n_1)\\ &+ (a_2)_2z\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{102}^{(4)}(a_2+2,c+2-n_1)\\ &+ a_2a_4u\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{102}^{(4)}(a_2+1,a_4+1,c+2-n_1). \end{aligned} \tag{2.38}$$

Theorem 2.10. *The following recursion formulas hold true for the numerator parameter a_1, a_2,*

a_3 *of the* $X_{103}^{(4)}$*:*

$$
\begin{aligned}
X_{103}^{(4)}(a_1+n) &= X_{103}^{(4)} + \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)X_{103}^{(4)}(a_1+1+n_1,c_1+1)\\
&\quad + \frac{ya_2}{c_2}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_1+n_1,a_2+1,c_2+1),\\
X_{103}^{(4)}(a_1-n) &= X_{103}^{(4)} - \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)X_{103}^{(4)}(a_1+2-n_1,c_1+1)\\
&\quad - \frac{ya_2}{c_2}\sum_{n_1=1}^{n}X_{100}^{(4)}(a_1+1-n_1,a_2+1,c_2+1),\\
X_{103}^{(4)}(a_2+n) &= X_{103}^{(4)} + \frac{a_1y}{c_2}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_1+1,a_2+n_1,c_2+1)\\
&\quad + \frac{a_3z}{c_3}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+n_1,a_3+1,c_3+1)\\
&\quad + \frac{a_3u}{c_4}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+n_1,a_3+1,c_4+1),\\
X_{103}^{(4)}(a_2-n) &= X_{103}^{(4)} - \frac{a_1y}{c_2}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_1+1,a_2+1-n_1,c_2+1)\\
&\quad - \frac{a_3z}{c_3}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+1-n_1,a_3+1,c_3+1)\\
&\quad - \frac{a_3z}{c_4}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+1-n_1,a_3+1,c_4+1),\\
X_{103}^{(4)}(a_3+n) &= X_{103}^{(4)} + \frac{za_2}{c_3}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+1,a_3+n_1,c_3+1)\\
&\quad + \frac{ua_2}{c_4}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+1,a_3+n_1,c_4+1),\\
X_{103}^{(4)}(a_3-n) &= X_{103}^{(4)} - \frac{za_2}{c_3}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+1,a_3+1-n_1,c_3+1)\\
&\quad - \frac{ua_2}{c_4}\sum_{n_1=1}^{n}X_{103}^{(4)}(a_2+1,a_3+1-n_1,c_4+1).
\end{aligned}
\tag{2.39}
$$

Theorem 2.11. *The following recursion formulas hold true for the numerator parameter* a_2*,* a_3

of the $X_{103}^{(4)}$*:*

$$X_{103}^{(4)}(a_2+n) = \sum_{N_3\le n}\binom{n}{n_1,n_2,n_3}\frac{(a_1)_{n_1}(a_3)_{n_2+n_3}y^{n_1}z^{n_2}u^{n_3}}{(c_2)_{n_1}(c_3)_{n_2}(c_4)_{n_3}}$$
$$\times X_{103}^{(4)}(a_1+n_1, a_2+N_3,\ a_3+n_2+n_3,\ c_2+n_1, c_3+n_2, c_4+n_3), \tag{2.40}$$

$$X_{103}^{(4)}(a_2-n) = \sum_{N_3\le n}\binom{n}{n_1,n_2,n_3}\frac{(a_1)_{n_1}(a_3)_{n_2+n_3}(-y)^{n_1}(-z)^{n_2}(-u)^{n_3}}{(c_2)_{n_1}(c_3)_{n_2}(c_4)_{n_3}}$$
$$\times X_{103}^{(4)}(a_1+n_1, a_3+n_2+n_3,\ c_2+n_1, c_3+n_2, c_4+n_3), \tag{2.41}$$

$$X_{103}^{(4)}(a_3+n) =$$
$$\sum_{n_1\le n}\binom{n}{n_1,n_2}\frac{(a_2)_{N_2}z^{n_1}u^{n_2}}{(c_3)_{n_1}(c_4)_{n_2}}\,X_{103}^{(4)}(a_2+N_2, c_3+n_1, c_4+n_2), \tag{2.42}$$

$$X_{103}^{(4)}(a_3-n) =$$
$$\sum_{n_1\le n}\binom{n}{n_1,n_2}\frac{(a_2)_{N_2}(-z)^{n_1}(-u)^{n_2}}{(c_3)_{n_1}(c_4)_{n_2}}\,X_{103}^{(4)}(a_2+N_2,\ c_3+n_1, c_4+n_2). \tag{2.43}$$

where $N_3 = n_1+n_2+n_3$*,* $N_2 = n_1+n_2$*.*

Theorem 2.12. *The following recursion formulas hold true for the denominator parameters* c_1*,* c_2*,* c_3*,* c_4 *of the* $X_{103}^{(4)}$*:*

$$X_{103}^{(4)}(c_1-n) = X_{103}^{(4)} +$$
$$(a_1)_2x\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{103}^{(4)}(a_1+2-n_1,\ c_1+2-n_1), \tag{2.44}$$

$$X_{103}^{(4)}(c_2-n) = X_{103}^{(4)} +$$
$$a_1a_2y\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)}\,X_{103}^{(4)}(a_1+1, a_2+1, c_2+2-n_1), \tag{2.45}$$

$$X_{103}^{(4)}(c_3-n) = X_{103}^{(4)} +$$
$$a_2a_3z\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{103}^{(4)}(a_2+1, a_3+1, c_3+2-n_1), \tag{2.46}$$

$$X_{103}^{(4)}(c_4-n) = X_{103}^{(4)} +$$
$$a_2a_3u\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,X_{103}^{(4)}(a_2+1, a_3+1, c_4+2-n_1). \tag{2.47}$$

Theorem 2.13. *The following recursion formulas hold true for the numerator parameter* a_1*,* a_2*,*

a_3 *of the* $X_{104}^{(4)}$:

$$X_{104}^{(4)}(a_1+n) = X_{104}^{(4)} + \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)X_{104}^{(4)}(a_1+1+n_1, c_1+1) + \frac{ya_2}{c_1}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_1+n_1, a_2+1, c_1+1), \tag{2.48}$$

$$X_{104}^{(4)}(a_1-n) = X_{104}^{(4)} - \frac{2x}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)X_{104}^{(4)}(a_1+2-n_1, c_1+1) - \frac{ya_2}{c_1}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_1+1-n_1, a_2+1, c_1+1), \tag{2.49}$$

$$X_{104}^{(4)}(a_2+n) = X_{104}^{(4)} + \frac{a_1 y}{c_1}\sum_{n_1=1}^{n} X_{103}^{(4)}(a_1+1, a_2+n_1, c_1+1) + \frac{a_3 z}{c_2}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_2+n_1, a_3+1, c_2+1) + \frac{a_3 u}{c_3}\sum_{n_1=1}^{n} X_{103}^{(4)}(a_2+n_1, a_3+1, c_3+1), \tag{2.50}$$

$$X_{104}^{(4)}(a_2-n) = X_{104}^{(4)} - \frac{a_1 y}{c_1}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_1+1, a_2+1-n_1, c_1+1) - \frac{a_3 z}{c_2}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_2+1-n_1, a_3+1, c_2+1) - \frac{a_3 u}{c_3}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_2+1-n_1, a_3+1, c_3+1), \tag{2.51}$$

$$X_{104}^{(4)}(a_3+n) = X_{104}^{(4)} + \frac{za_2}{c_2}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_2+1, a_3+n_1, c_2+1) + \frac{ua_2}{c_3}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_2+1, a_3+n_1, c_3+1), \tag{2.52}$$

$$X_{104}^{(4)}(a_3-n) = X_{104}^{(4)} - \frac{za_2}{c_2}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_2+1, a_3+1-n_1, c_2+1) - \frac{ua_2}{c_3}\sum_{n_1=1}^{n} X_{104}^{(4)}(a_2+1, a_3+1-n_1, c_3+1). \tag{2.53}$$

Theorem 2.14. *The following recursion formulas hold true for the numerator parameter* a_2, a_3

of the $X_{104}^{(4)}$*:*

$$\begin{aligned} & X_{104}^{(4)}(a_2+n) \\ & = \sum_{N_3 \le n} \binom{n}{n_1, n_2, n_3} \frac{(a_1)_{n_1}(a_3)_{n_2+n_3} y^{n_1} z^{n_2} u^{n_3}}{(c_1)_{n_1}(c_2)_{n_2}(c_3)_{n_3}} \\ & \times X_{104}^{(4)}(a_1+n_1, a_2+N_3,\ a_3+n_2+n_3,\ c_1+n_1, c_2+n_2, c_3+n_3), \end{aligned} \tag{2.54}$$

$$\begin{aligned} & X_{104}^{(4)}(a_2-n) \\ & = \sum_{N_3 \le n} \binom{n}{n_1, n_2, n_3} \frac{(a_1)_{n_1}(a_3)_{n_2+n_3}(-y)^{n_1}(-z)^{n_2}(-u)^{n_3}}{(c_1)_{n_1}(c_2)_{n_2}(c_3)_{n_3}} \\ & \times X_{104}^{(4)}(a_1+n_1, a_3+n_2+n_3,\ c_1+n_1, c_2+n_2, c_3+n_3), \end{aligned} \tag{2.55}$$

$$\begin{aligned} & X_{104}^{(4)}(a_3+n) \\ & = \sum_{N_2 \le n} \binom{n}{n_1, n_2} \frac{(a_2)_{N_2} z^{n_1} u^{n_2}}{(c_2)_{n_1}(c_3)_{n_2}} X_{104}^{(4)}(a_2+N_2, c_2+n_1, c_3+n_2), \end{aligned} \tag{2.56}$$

$$\begin{aligned} & X_{104}^{(4)}(a_3-n) \\ & = \sum_{N_2 \le n} \binom{n}{n_1, n_2} \frac{(a_2)_{N_2}(-z)^{n_1}(-u)^{n_2}}{(c_2)_{n_1}(c_3)_{n_2}} X_{104}^{(4)}(a_2+N_2,\ c_2+n_1, c_3+n_2). \end{aligned} \tag{2.57}$$

Theorem 2.15. *The following recursion formulas hold true for the denominator parameter* c_1*,* c_2*,* c_3 *of the* $X_{104}^{(4)}$*:*

$$\begin{aligned} X_{104}^{(4)}(c_1-n) = X_{104}^{(4)} + & \\ & (a_1)_2 x \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)} X_{104}^{(4)}(a_1+2-n_1,\ c_1+2-n_1) \\ & + a_1 a_2 y \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)} X_{104}^{(4)}(a_1+1, a_2+1\, c_1+2-n_1), \end{aligned} \tag{2.58}$$

$$\begin{aligned} X_{104}^{(4)}(c_2-n) = X_{104}^{(4)} + & \\ & a_2 a_3 y \sum_{n_1=1}^{n} \frac{1}{(c_2-n_1)(c_2+1-n_1)} X_{103}^{(4)}(a_2+1, a_3+1, c_2+2-n_1), \end{aligned} \tag{2.59}$$

$$\begin{aligned} X_{104}^{(4)}(c_3-n) = X_{104}^{(4)} + & \\ & a_2 a_3 u \sum_{n_1=1}^{n} \frac{1}{(c_1-n_1)(c_1+1-n_1)} X_{104}^{(4)}(a_2+1, a_3+1, c_3+2-n_1). \end{aligned} \tag{2.60}$$

3 Particular cases

If we set $z = u = 0$. Then $X_{101}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_4; c_1, c_2, c_2, c_1; x, y, z, u)$ reduces to Horn series $H_4(a_1, a_3; c_1, c_2; x, y)$ defined in [21]. So the recursion formulas of $X_{101}^{(4)}$ reduces to recursion formulas for Horn series $H_4(a_1, a_3; c_1, c_2; x, y)$.

Following abbreviated notations are used in this section. We, for example, write H_4 for the series $H_4(a_1, a_3; c_1, c_2; x, y)$, $H_4(a_1+n)$ for the series $H_4(a_1+n, a_3; c_1, c_2; x, y)$ and $H_4(a_1+n_1, c_1+n_2)$ stands for the series $H_4(a_1+n_1, a_3; c_1+n_2, c_2; x, y)$.

Theorem 3.1. *The following recursion formulas hold true for the numerator parameter* a_1 *of the*

Horn's series H_4 :

$$H_4[a_1+n] = H_4 + \frac{2\,x}{c_1}\sum_{n_1=1}^{n}(a_1+n_1)\,H_4[a_1+1+n_1,\,c_1+1]+$$

$$\frac{a_3\,y}{c_2}\sum_{n_1=1}^{n}H_4[a_1+n_1,\,c_1+1,\,c_2+1], \tag{3.1}$$

$$H_4[a_1-n] = H_4 - \frac{2\,x}{c_1}\sum_{n_1=1}^{n}(a_1+1-n_1)\,H_4[a_1+2-n_1,\,c_1+1]-$$

$$\frac{a_3\,y}{c_2}\sum_{n_1=1}^{n}H_4[a_1+1-n_1,\,c_1+1,\,c_2+1]. \tag{3.2}$$

Theorem 3.2. *The following recursion formulas hold true for the numerator parameter* a_3 *of the Horn's series* H_4 :

$$H_4[a_3+n] = H_4 + \frac{a_1\,y}{c_1}\sum_{n_1=1}^{n}H_4[a_1+1,a_3+n_1,c_1+1], \tag{3.3}$$

$$H_4[a_3-n] = H_4 - \frac{a_1\,y}{c_1}\sum_{n_1=1}^{n}H_4[a_1+1,a_3+1-n_1,c_1+1]. \tag{3.4}$$

Theorem 3.3. *The following recursion formulas hold true for the numerator parameter* a_3 *of the Horn's series* H_4 :

$$H_4[a_3+n] = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}y^{n_1}}{(c_1)_{n_1}}\,H_4[a_1+n_1,a_3+n_1,c_1+n_1], \tag{3.5}$$

$$H_4[a_3-n] = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(-y)^{n_1}}{(c_1)_{n_1}}\,H_4[a_1+n_1,c_1+n_1]. \tag{3.6}$$

Theorem 3.4. *The following recursion formulas hold true for the denominator parameters* c_1, c_2 *of the Horn's series* H_4 :

$$H_4[c_1-n] = H_4 +$$

$$(a_1)_2\,x\sum_{n_1=1}^{n}\frac{1}{(c_1-n_1)(c_1+1-n_1)}\,H_4[a_1+2,\,c_1+2-n_1], \tag{3.7}$$

$$H_4[c_1-n] = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{2n_1}x^{n_1}}{(c_1)_{n_1}(c_1-n)_{n_1}}\,H_4[a_1+2n_1,\,c_1+n_1], \tag{3.8}$$

$$H_4[c_2-n] = H_4 +$$

$$a_1\,a_3y\sum_{n_1=1}^{n}\frac{1}{(c_2-n_1)(c_2+1-n_1)}\,H_4[a_1+1,a_3+1,\,c_2+2-n_1], \tag{3.9}$$

$$H_4[c_2-n] = \sum_{n_1=0}^{n}\binom{n}{n_1}\frac{(a_1)_{n_1}(a_3)_{n_1}y^{n_1}}{(c_2)_{n_1}(c_2-n)_{n_1}}\,H_4[a_1+n_1,a_3+n_1,\,c_2+n_1]. \tag{3.10}$$

Similarly, certain particular cases leading to recursion formulas of certain hypergeometric function of one, two and three variable hypergeometric series.

4 Conclusion

This paper is to obtain recursion formulas for some hypergeometric functions of four variables. Also, some interested particular cases and consequences of our results have been discussed.

Within such a context, new hypergeometric functions of four variables structures emerge with wide possibilities of applications in physics and engineering. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

Declarations

Data availability:
The data used to support the findings of this study are available from the corresponding author upon request.

Competing interests:
The authors declare that they have no competing interests.

Authors' contributions:
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

[1] S. Abreu, R. Britto, C. Duhr, E.Gardi, J. Matthew, *From positive geometries to a coaction on hypergeometric functions*, Journal of High Energy Physics, 2(2020), 1-45.

[2] S.I. Bezrodnykh, *Analytic continuation of Lauricella's functions,* $F_A^{(N)}$, $F_B^{(N)}$ *and*$F_D^{(N)}$, Integral Transforms and Special Functions, 31(2020), 921-940.

[3] S.I. Bezrodnykh, Horn's hypergeometric functions with three variables, Integral Transforms and Special Functions, 32(2021), 207-223.

[4] Y.A. Brychkov, N. Saad, On some formulas for the Appell function $F_2(a, b, b'; c, c'; w; z)$, Integral Transforms and Special Functions, 25(2014), 111-123.

[5] A. Saboor, G. Rahman, H. Ali, K.S. Nisar, T. Abdeljawad, *Properties and Applications of a New Extended Gamma Function Involving Confluent Hypergeometric Function*, Journal of Mathematics, vol. 2021, Article ID 2491248, 12 pages, 2021.

[6] L.C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publishing Company, New York, 1985.

[7] L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, Wiley, New York, 1958.

[8] M.G. Bin-Saad, J.A. Younis, *On connections between certain class of new quadruple and known triple hypergeomtric series*, Tamap Journal of Mathematics and Statistics. Volume 2019.

[9] H. Exton, Multiple Hypergeometric Functions and Applications, Halsted Press, New York, London, Sydney and Toronto, 1976.

[10] J.C. Jaeger, H.R. Hulme, The internal conversion of Gamma rays with the production of electrons and positrons, Proceedings of the Royal Society of London A, 148(1935), 708-728.

[11] G. Lohöfer, Theory of an electromagnetically deviated metal sphere. I: Absorbed power, SIAM J. Appl. Math., Vol. 49, 567-581, 1989.

[12] S. Mubeen, S. Shah, G. Rahman, Kottakkaran S. Nisar, T. Abdeljawad, *Some Generalized Special Functions and their Properties*, Advances in the Theory of Nonlinear Analysis and its Application 6(2022), pp.45-65.

[13] P.O.M. Olsson, A hypergeometric function of two variables of importance in perturbation theory I and II, Arkiv for Fysik 30(1965), 187-191, ibid. 29(1965), 459-465.

[14] S.B. Opps, N. Saad, H.M. Srivastava, Recursion formulas for Appell's hypergeometric function F_2 with some applications to radiation field problems, *Appl. Math. Comput.* 207 (2009), pp. 545-558.

[15] V. Sahai, A. Verma, *Recursion formulas for multivariable hypergeometric functions*, Asian-Eur. J. Math., 8 (2015), no. 4, 1550082, 50 pp.

[16] V. Sahai, A. Verma, *Recursion formulas for Exton's triple hypergeometric functions*, Kyungpook Math. J., 56 (2016),473-506.

[17] V. Sahai, A. Verma, *Recursion formulas for Srivastava's general triple hypergeometric function*, Asian-Eur. J. Math., 9 (2016), no. 3, 1650063, 17 pp.

[18] V. Sahai, A. Verma, *Recursion formulas for the Srivastava-Daoust and related multivariable hypergeometric functions*, Asian-Eur. J. Math., 9 (2016), no. 4, 1650081, 35 pp.

[19] C. Sharma and C.L. Parihar, *Hypergeometric functions of four variables (I)*, J. Indian Acad. Math., 11 (1989), 121-133.

[20] H.M. Srivastava, A. Tassaddiq, G. Rahman, K.S. Nisar and I. Khan, *A New Extension of the τ-Gauss Hypergeometric Function and Its Associated Properties* Mathematics 7 no. 10 (2019): 996.

[21] H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Ltd., Chichester, 1984.

[22] X. Wang, *Recursion formulas for Appell functions*, Integral Transforms Spec. Funct., 23(6) (2012), 421-433.

[23] J.A. Younis, A. Verma, H. Aydi, K.S. Nisar, H. Alsmair, *Recursion formulas for certain quadruple hypergeometric functions* Adv. Diffe, Equ 2021, 407 (2021).

[24] J.A. Younis, M.G. Bin-Saad, *Integral representations involving new hypergeometric functions of four variables*, J. Frac. Calc. Appl., 10 (2019), 77-91.

[25] M. Tarfiq, S. K. Sahoo, J. Nasir, S.K. Awan, *Some Ostrowski type integral inequalities using hypergeometric functions*, J. Frac. Calc. & Nonlinear Sys. 2(1) (2021), 24-41.

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