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SOME OSCILLATION THEOREMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSIVE EFFECT

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Abstract The present article deals with the oscillatory nature of nonlinear impulsive fractional differential equations of order α, where $\alpha \in (2, 3)$. Here, some oscillation results are established using sufficient parts of the differential inequalities established via differential inequality methods. Also, abstract results are illustrated by an example.

1 Introduction

Fractional differential equations are the generalization of integer order classical differential equations to non-integer orders. Due to the wide applicability of these differential equations in many science and engineering areas, fractional calculus deserves an independent study parallel to the well-known theory of ordinary differential equations. The accountable number of scientific and engineering problems, including fractional derivatives, is already very large and growing. Fractional order differential equations came into existence because ordinary differential equations cannot formulate many physical problems. Also, fractional differentials and integrals provide a more accurate model of the system under consideration.

Initially, the existence, uniqueness, approximations of solutions, and controllability of fractional differential equations have been studied by many authors. Some such work can be found in papers [20, 22, 23]. Basic results and definitions of fractional differential equations are discussed in [5, 11, 19]. In the last few decades, various types of models on fractional derivatives have been studied by many authors [20, 21, 24, 25]. In recent years, a huge interest in studying oscillatory and non-oscillatory behavior of solutions for different types of fractional differential equations developed [14, 24, 25, 26, 27], etc. In papers [2, 3, 10, 16, 18, 26], authors studied the oscillatory behavior of different classes of fractional differential equations without impulses. Oscillation criteria for different orders of neutral differential equations have been discussed in [1].

The physical problem, where parameters are subject to short-term perturbations, can be modeled as impulsive differential equations. The differential equations with impulsive effect can be used to simulate those discontinuous processes in which impulses occur. So, it becomes an important tool to handle the natural process of mathematical models and phenomena such as in optimal control, electric circuit, biotechnology, population dynamics, fractals, neural network, viscoelasticity, chemical technology. For more details on impulsive differential equations, we refer to the paper [12]. Investigation on oscillation theory for impulsive differential equations started in 1989 [9] and is at the initial stage of its development. Later on, authors in papers [4, 6, 7, 15] have extended the study of oscillation to parabolic and hyperbolic impulsive partial differential equations. In the past few years, many researchers have shown great interest in studying the oscillatory behavior of solutions of fractional differential equations with impulsive conditions. For these studies, we refer to [6, 7, 8, 13, 15, 17, 21, 24, 25] and references cited in these papers. In [24], Sadhasivam and Deepa derived some sufficient conditions for oscillatory solutions of hybrid partial differential equations with impulses. New oscillation criteria (known as Philos type) were obtained for a class of second-order differential equations with impulses in [25].

Our work is motivated by the work of [21], in which oscillatory properties are studied for a noninteger order differential equation with impulsive conditions. In this paper, we extended the study of oscillatory behavior of solutions to a class of impulsive fractional differential equations of order $2 < \alpha < 3$. We investigated how the impulses affect the oscillation of solutions. The mathematical formulation of the considered problem is given by:

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}u(z,t) + a_1(t)\mathfrak{D}^{\alpha-1}_{+,t}u(z,t) + a_2(t)\mathfrak{D}^{\alpha-2}_{+,t}u(z,t) \\ \qquad\qquad +a_3(t)g\left(\int_t^{\infty}(s-t)^{-(\alpha-2)}u(z,s)ds\right) = 0, \quad t \neq t_j, \\ \mathfrak{D}^{\alpha-1}_{+,t}u(z,t_j^+) - \mathfrak{D}^{\alpha-1}_{+,t}u(z,t_j^-) = \rho_1(z,t_j)\mathfrak{D}^{\alpha-1}_{+,t}u(z,t_j), \\ \mathfrak{D}^{\alpha-2}_{+,t}u(z,t_j^+) - \mathfrak{D}^{\alpha-2}_{+,t}u(z,t_j^-) = \rho_2(z,t_j)\mathfrak{D}^{\alpha-2}_{+,t}u(z,t_j), \\ \qquad\qquad j = 1,2,3,\ldots, \quad (z,t) \in D \times \mathbb{R}_+ = \Omega, \end{cases} \tag{1.1}$$

where a_1, a_2, a_3, g are piecewise continuous functions defined from $[t_0, \infty)$ into $\mathbb{R}_+$ with discontinuities at $t = t_j$, $j = 1, 2, \ldots$ but continuous from left at $t = t_j$. g also satisfies $zg(z) > 0$, for $z \neq 0$. $\mathfrak{D}^{\alpha}_{+,t}$ is the fractional derivative of Riemann-Liouville type, where $\alpha \in (2,3)$, $D \subset \mathbb{R}^n$ is bounded with a smooth boundary ∂D and $\bar{D} = D \cup \partial D$.

The organization of the rest of this paper is as follows. Section 2 contains some basic lemmas and assumptions which are required for the next sections. In section 3, some oscillation criteria are obtained for the problem (1.1) by using differential inequality methods.

2 Preliminaries and Assumptions

Throughout the paper, we assume the following assumptions:

(H1) The functions $\rho_i : \bar{D} \times \mathbb{R}_+ \to \mathbb{R}_+ \quad (i = 1, 2)$ are such that

$$\rho_1(z,t_j) \leq \beta_j, \; \rho_2(z,t_j) \leq \gamma_j.$$

(H2) The given numbers

$$0 < t_1 < \cdots < t_j < \cdots$$

are such that

$$\lim_{j\to\infty} t_j = +\infty.$$

(H3) The solution $u(z,t)$ of the problem (1.1), $\mathfrak{D}^{\alpha-1}_{+,t}u(z,t)$ and $\mathfrak{D}^{\alpha-2}_{+,t}u(z,t)$ are piecewise continuous with discontinuities of first kind only at $t = t_j$, and left continuous at $t = t_j$ i.e., $u(z,t_j^-) = u(z,t_j)$, $\mathfrak{D}^{\alpha-1}u(z,t_j^-) = \mathfrak{D}^{\alpha-1}u(z,t_j)$, $\mathfrak{D}^{\alpha-2}u(z,t_j^-) = \mathfrak{D}^{\alpha-2}u(z,t_j)$.

Lemma 2.1. *[21] For any function $U : \mathbb{R}_+ \to \mathbb{R}$, $(\mathfrak{D}^{\alpha}_{+,t}U)(t) = (\mathfrak{D}^{\alpha-1}_{+,t}U)'(t)$ and $(\mathfrak{D}^{\alpha}_{+,t}U)(t) = (\mathfrak{D}^{\alpha-2}_{+,t}U)''(t)$.*

Lemma 2.2. *[14] For any function $U : \mathbb{R}_+ \to \mathbb{R}$, let*

$$G(t) = \int_0^t (t-\tau)^{-(\alpha-2)}U(\tau)d\tau, \quad \textit{where} \;\; \alpha \in (2,3) \;\; \textit{and} \;\; t \geq 0, \; \textit{then}$$

$$G'(t) = \Gamma(3-\alpha)(\mathfrak{D}^{\alpha-2}_{+,t}U)(t).$$

Lemma 2.3. *[12] Assume that*

$$\begin{aligned} V'(t) &\leq g_1(t)V(t) + g_2(t), \quad t \neq t_i, \; t \geq t_0 \\ V(t_i^+) &\leq (1+a_i)V(t_i), \quad i = 1,2,3,\ldots, \end{aligned}$$

where $\{t_i\}$ *is an increasing sequence such that* $\lim_{i\to\infty} t_i = +\infty$, $V \in PC^1[\mathbb{R}_+, \mathbb{R}_+]$, $g_1, g_2 \in PC[\mathbb{R}_+, \mathbb{R}_+]$ *and* a_i *are constants. Then*

$$\begin{aligned} V(t) &\leq V(t_0) \prod_{t_0<t_i<t} (1+a_i) \exp\left(\int_{t_0}^t g_1(s)ds\right) \\ &\quad + \int_{t_0}^t \prod_{s<t_i<t} (1+a_i) \exp\left(\int_s^t g_1(\sigma)d\sigma\right) g_2(s)ds, \quad t \geq t_0. \end{aligned}$$

3 Main Results

Theorem 3.1. *If each solution of the following system of inequalities*

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}U(t) + a_1(t)\mathfrak{D}^{\alpha-1}_{+,t}U(t) + a_2(t)\mathfrak{D}^{\alpha-2}_{+,t}U(t) + a_3(t)g(G(t)) \leq 0 \\ \mathfrak{D}^{\alpha-1}_{+,t}U(t_j^+) \leq (1+\beta_j)\mathfrak{D}^{\alpha-1}_{+,t}U(t_j) \\ \mathfrak{D}^{\alpha-2}_{+,t}U(t_j^+) \leq (1+\gamma_j)\mathfrak{D}^{\alpha-2}_{+,t}U(t_j) \end{cases} \tag{3.1}$$

is eventually negative and each solution of the following system of inequalities

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}U(t) + a_1(t)\mathfrak{D}^{\alpha-1}_{+,t}U(t) + a_2(t)\mathfrak{D}^{\alpha-2}_{+,t}U(t) + a_3(t)g(G(t)) \geq 0 \\ \mathfrak{D}^{\alpha-1}_{+,t}U(t_j^+) \geq (1+\beta_j)\mathfrak{D}^{\alpha-1}_{+,t}U(t_j) \\ \mathfrak{D}^{\alpha-2}_{+,t}U(t_j^+) \geq (1+\gamma_j)\mathfrak{D}^{\alpha-2}_{+,t}U(t_j) \end{cases} \tag{3.2}$$

is eventually positive, then all nonzero solutions of (1.1) are oscillatory in the domain Ω, *where* $G(t) = \int_t^\infty (s-t)^{-(\alpha-2)}U(s)ds$.

Proof. Let on contrary, we assume that $u(z,t) \neq 0$ be a non oscillatory solution of (1.1) and for some $\tau \geq t_0$, $u(z,t_0) > 0$ for $t \geq \tau$.

Case 1: $t \neq t_j$. Integrating the first equation of (1.1) with respect to z over the domain D, we have

$$\begin{aligned} \mathfrak{D}^{\alpha}_{+,t} \int_D u(z,t)dz + a_1(t)\mathfrak{D}^{\alpha-1}_{+,t} \int_D u(z,t)dz + a_2(t)\mathfrak{D}^{\alpha-2}_{+,t} \int_D u(z,t)dz \\ = -a_3(t) \int_D g\left(\int_t^\infty (s-t)^{-(\alpha-2)}u(z,s)ds\right) dz. \end{aligned} \tag{3.3}$$

Using Jensen's inequality, we get

$$\begin{aligned} &\int_D g\left(\int_t^\infty (s-t)^{-(\alpha-2)}u(z,s)ds\right) dz \\ &\qquad \geq g\left(\int_D \left(\int_t^\infty (t-s)^{-(\alpha-2)}u(z,s)ds\right) dz\right) \\ &\qquad \geq \left(\int_D dz\right) g\left[\int_t^\infty (t-s)^{-(\alpha-2)} \left(\int_D u(z,s)dz\right)\left(\int_D dz\right)^{-1} ds\right]. \end{aligned}$$

Let $U(t) = \dfrac{\int_D u(z,t)dz}{\int_D dz}$. Using the above inequality in (3.3), we get

$$\mathfrak{D}^{\alpha}_{+,t}U(t) + a_1(t)\mathfrak{D}^{\alpha-1}_{+,t}U(t) + a_2(t)\mathfrak{D}^{\alpha-2}_{+,t}U(t) + a_3(t)g(G(t)) \leq 0. \tag{3.4}$$

Case 2: $t = t_j$. From last two equations of (1.1), we have

$$\begin{aligned} \mathfrak{D}^{\alpha-1}_{+,t} \int_D u(z,t_j^+)dz \leq (1+\beta_j)\mathfrak{D}^{\alpha-1}_{+,t} \int_D u(z,t_j)dz \\ \mathfrak{D}^{\alpha-2}_{+,t} \int_D u(z,t_j^+)dz \leq (1+\gamma_j)\mathfrak{D}^{\alpha-2}_{+,t} \int_D u(z,t_j)dz. \end{aligned}$$

Dividing both equations by $\int_D dz$, we get

$$\mathfrak{D}^{\alpha-1}_{+,t}U(t_j^+) \leq (1+\beta_j)\mathfrak{D}^{\alpha-1}_{+,t}U(t_j) \tag{3.5}$$

$$\mathfrak{D}^{\alpha-2}_{+,t}U(t_j^+) \leq (1+\gamma_j)\mathfrak{D}^{\alpha-2}_{+,t}U(t_j). \tag{3.6}$$

Thus the equations (3.4), (3.5) and (3.6) show that the function $U(t) = \frac{\int_D u(z,t)dz}{\int_D dz}$ is an eventually positive solution of (3.1) which is a contradiction.

Secondly, in the case of eventually negative solutions of (1.1), the arguments are similar. □

Lemma 3.2. *For an eventually positive solution $U(t)$ of (3.1) such that $(\mathfrak{D}^{\alpha}_{+,t}U)(t) > 0$ for $t \geq \tau > 0$, and*

$$\int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right) ds = \infty,$$

then $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t) > 0$ for $t \geq \tau$.

Proof. If we take $u_1(t) = \exp\left(\int_{t_0}^t \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right)$, then

$$\begin{aligned}
[(\mathfrak{D}^{\alpha-2}_{+,t}U)(t)u_1(t)]' &= (\mathfrak{D}^{\alpha-1}_{+,t}U)(t)u_1(t) + \frac{a_2(t)}{a_1(t)}(\mathfrak{D}^{\alpha-2}_{+,t}U)(t)u_1(t) \\
&= \frac{1}{a_1(t)}[a_1(t)(\mathfrak{D}^{\alpha-1}_{+,t}U)(t) + a_2(t)(\mathfrak{D}^{\alpha-2}_{+,t}U)(t)]u_1(t) \\
&\leq -\frac{1}{a_1(t)}[(\mathfrak{D}^{\alpha}_{+,t}U)(t) + a_3(t)g(G(t))]u_1(t) < 0.
\end{aligned}$$

This implies that $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t)u_1(t)$ is strictly decreasing for $t \geq \tau$ and is eventually of constant sign. Since $u_1(t) > 0$, we see that $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t)$ is eventually of constant sign. We now claim that $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t) > 0$ for $t \geq \tau$, otherwise $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t) < 0$ for $t \geq \tau$. Since $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t)u_1(t)$ is strictly decreasing for $t \geq \tau$, it follows that

$$(\mathfrak{D}^{\alpha-2}_{+,t}U)(t)u_1(t) < (\mathfrak{D}^{\alpha-2}_{+,t}U)(\tau)u_1(\tau) = C < 0, \quad t \geq \tau.$$

From Lemma 2.2, we get

$$\frac{G'(t)}{\Gamma(3-\alpha)} = (\mathfrak{D}^{\alpha-2}_{+,t}U)(t) \quad < \quad C\exp\left(-\int_{t_0}^t \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right), \quad t \geq \tau.$$

On integration from τ to t, we have

$$G(t) \quad < \quad G(\tau) + C\Gamma(3-\alpha)\int_\tau^t \exp\left(-\int_{t_0}^s \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right) ds.$$

By taking limit, we get $\lim_{t\to\infty} G(t) = -\infty$, which contradicts the fact that $G(t) > 0$. Hence $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t) > 0$ for $t \geq \tau$. □

Following the process of the above lemma, we have:

Lemma 3.3. *For an eventually negative solution of (3.2) such that $(\mathfrak{D}^{\alpha}_{+,t}U)(t) < 0$ for $t \geq \tau > 0$, and*

$$\int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right) ds = \infty,$$

then $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t) < 0$ for $t \geq \tau$.

Theorem 3.4. *If all the conditions of Lemma 3.2 and Lemma 3.3 are satisfied and further, we assume that for some* $\tau \in [t_0, \infty)$,

$$\liminf_{t\to\infty} \frac{W(t)}{\prod_{t_0<t_j<t}(1+\gamma_j)} = -\infty, \tag{3.7}$$

and

$$\limsup_{t\to\infty} \frac{W(t)}{\prod_{t_0<t_j<t}(1+\gamma_j)} = \infty, \tag{3.8}$$

where

$$W(t) = \int_\tau^t \prod_{t_0<t_j<s}(1+\beta_j) \prod_{s<t_j<t}(1+\gamma_j) \exp\left(-\int_{t_0}^s a_1(\sigma)d\sigma\right) ds,$$

then each nonzero solution of (1.1) oscillates in domain Ω.

Proof. To complete the proof, it is sufficient to show that every solution of (3.1) is eventually negative and every solution of (3.2) is eventually positive. Let on contrary, first we assume that (3.1) has an eventually positive solution $U(t)$ (say). Let $v(t) = (\mathfrak{D}^{\alpha-2}_{+,t}U)(t)$, then using Lemma 2.1, we have $v'(t) = (\mathfrak{D}^{\alpha-1}_{+,t}U)(t)$ and $v''(t) = (\mathfrak{D}^{\alpha}_{+,t}U)(t)$. From (3.1), we have

$$\begin{cases} v''(t) + a_1(t)v'(t) \le 0, \\ v'(t_j^+) \le (1+\beta_j)v'(t_j), \\ v(t_j^+) \le (1+\gamma_j)v(t_j). \end{cases} \tag{3.9}$$

Putting $v'(t) = u(t)$ and $v''(t) = u'(t)$, we obtain from above inequality

$$\begin{cases} u'(t) \le -a_1(t)u(t), \quad t \ne t_j, \\ u(t_j^+) \le (1+\beta_j)u(t_j). \end{cases}$$

Using Lemma 2.3, we get

$$u(t) \le u(t_0) \prod_{t_0<t_j<t}(1+\beta_j) \exp\left(-\int_{t_0}^t a_1(\sigma)d\sigma\right).$$

Putting $u(t) = v'(t)$ and using inequality (3.9), we have

$$\begin{cases} v'(t) \le v'(t_0) \prod_{t_0<t_j<t}(1+\beta_j) \exp\left(-\int_{t_0}^t a_1(\sigma)d\sigma\right), \\ v(t_j^+) \le (1+\gamma_j)v(t_j). \end{cases}$$

Again using Lemma 2.3, we have

$$\begin{aligned} v(t) &\le v(t_0) \prod_{t_0<t_j<t}(1+\gamma_j) + v'(t_0)W(t) \\ \Rightarrow \frac{v(t)}{\prod_{t_0<t_j<t}(1+\gamma_j)} &\le v(t_0) + v'(t_0)\frac{W(t)}{\prod_{t_0<t_j<t}(1+\gamma_j)}. \end{aligned}$$

Taking limit infimum and using the condition (3.7), we have

$$\liminf_{t\to\infty} \frac{v(t)}{\prod_{t_0<t_j<t}(1+\gamma_j)} = -\infty$$

which contradicts our assumption that $v(t) > 0$.

Secondly, suppose on contrary, (3.2) has an eventually negative solution $\tilde{U}(t)$. Then, for some $\tau_1 \in [t_0, \infty)$, we have $\tilde{U}(t) < 0$, $t \geq \tau_1$. Let $\tilde{v}(t) = (\mathfrak{D}^{\alpha-2}_{+,t})\tilde{U}(t)$, $t \geq \tau_1$. Using Lemma 3.3 in (3.2), we have

$$\begin{cases} \tilde{v}''(t) + a_1(t)\tilde{v}'(t) \geq 0, \quad t \neq t_j, \\ \tilde{v}'(t_j^+) \geq (1+\beta_j)\tilde{v}'(t_j), \\ \tilde{v}(t_j^+) \geq (1+\gamma_j)\tilde{v}(t_j). \end{cases} \tag{3.10}$$

Let $\tilde{v}'(t) = -\tilde{u}(t)$ and $\tilde{v}''(t) = -\tilde{u}'(t)$, then we have

$$\begin{cases} \tilde{u}'(t) \leq -a_1(t)\tilde{u}(t), \quad t \neq t_j, \\ \tilde{u}(t_j^+) \leq (1+\beta_j)\tilde{u}(t_j). \end{cases}$$

Using Lemma 2.3, we get

$$\tilde{u}(t) \leq \tilde{u}(t_0) \prod_{t_0<t_j<t} (1+\beta_j) \exp\left(-\int_{t_0}^{t} a_1(\sigma)d\sigma\right).$$

Using (3.10), we have

$$\begin{cases} -\tilde{v}'(t) \leq -\tilde{v}'(t_0) \displaystyle\prod_{t_0<t_j<t} (1+\beta_j) \exp\left(-\int_{t_0}^{t} a_1(\sigma)d\sigma\right), \\ -\tilde{v}(t_j^+) \leq -(1+\gamma_j)\tilde{v}(t_j). \end{cases}$$

Again using Lemma 2.3, we get

$$-\tilde{v}(t) \leq -\tilde{v}(t_0) \prod_{t_0<t_j<t} (1+\gamma_j) - \tilde{v}'(t_0)W(t)$$

$$\Rightarrow \frac{\tilde{v}(t)}{\prod_{t_0<t_j<t} (1+\gamma_j)} \geq \tilde{v}(t_0) + \tilde{v}'(t_0)\frac{W(t)}{\prod_{t_0<t_j<t} (1+\gamma_j)}.$$

Taking limit supremum and using the condition (3.8), we have

$$\limsup_{t\to\infty} \frac{\tilde{v}(t)}{\prod_{t_0<t_j<t} (1+\gamma_j)} = \infty,$$

which contradicts our assumption that $\tilde{v}(t) < 0$. □

Theorem 3.5. *Let (H1)-(H3) hold. We further assume that*

(i) $(\mathfrak{D}^{\alpha-1}_{+,t}U)(t) > 0$, $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t) > 0$, $\quad U(t) > 0,\ t \geq \tau_1$.

(ii) $(\mathfrak{D}^{\alpha-1}_{+,t}U)(t) < 0$, $(\mathfrak{D}^{\alpha-2}_{+,t}U)(t) < 0$, $\quad U(t) < 0,\ t \geq \tau_2$.

If

$$\liminf_{t\to\infty} \prod_{t_0<t_j<t} \left(\frac{1+\beta_j}{1+\gamma_j}\right) \exp\left(-\int_{t_0}^{t} a_1(\sigma)d\sigma\right) = -\infty \tag{3.11}$$

and

$$\limsup_{t\to\infty} \prod_{t_0<t_j<t} \left(\frac{1+\beta_j}{1+\gamma_j}\right) \exp\left(-\int_{t_0}^{t} a_1(\sigma)d\sigma\right) = \infty, \tag{3.12}$$

then all the nonzero solutions of (1.1) oscillate.

Proof. Let on contrary $U(t)$ be an eventually positive solution of (3.1) and $v(t) = (\mathfrak{D}^{\alpha-2}_{+,t}U)(t)$, then using Lemma 2.1, we have $v'(t) = (\mathfrak{D}^{\alpha-1}_{+,t}U)(t)$ and $v''(t) = (\mathfrak{D}^{\alpha}_{+,t}U)(t)$. Using condition (i) in (3.1), for $t \geq \tau_1$, we have

$$\begin{cases} v''(t) + a_1(t)v'(t) \leq 0, \quad t \neq t_j, \\ v'(t_j^+) \leq (1+\beta_j)v'(t_j), \\ v(t_j^+) \leq (1+\gamma_j)v(t_j). \end{cases} \tag{3.13}$$

If we define

$$Z(t) = \frac{v'(t)}{v(t)},$$

then

$$Z'(t) = \frac{v''(t)}{v(t)} - \left[\frac{v'(t)}{v(t)}\right]^2.$$

Using (3.13), we have

$$Z'(t) \leq -a_1(t)Z(t) - [Z(t)]^2 \leq -a_1(t)Z(t).$$

Using the last two inequalities of (3.13), we have

$$\begin{cases} Z'(t) \leq -a_1(t)Z(t), \\ Z(t_j^+) \leq \left(\frac{1+\beta_j}{1+\gamma_j}\right) Z(t_j). \end{cases}$$

Using Lemma 2.3, we get

$$Z(t) \leq Z(t_0) \prod_{t_0<t_j<t} \left(\frac{1+\beta_j}{1+\gamma_j}\right) \exp\left(-\int_{t_0}^t a_1(\sigma)d\sigma\right).$$

Taking limit infimum, we get $\liminf_{t\to\infty} Z(t) = -\infty$, which contradicts our assumption that $Z(t)$ is an eventually positive.

By using similar arguments, we get a contradiction if we assume $U(t)$ is an eventually negative solution of (3.2). □

Theorem 3.6. *Assume that all the assumptions of Theorem 3.5 hold and further, we assume that there exist real valued continuously differentiable functions $\Psi(t,s), \phi(t,s)$ with domain $D_1 = \{(t,s)|t \geq s \geq t_0 > 0\}$, with conditions*

(A1) $\Psi(t,t) = 0, \quad t \geq t_0,$ *and* $\Psi(t,s) > 0, \quad t > s \geq t_0$;

(A2) $\dfrac{\partial}{\partial t}\Psi(t,s) \geq 0, \quad \dfrac{\partial}{\partial s}\Psi(t,s) \leq 0$;

(A3) $\phi(t,s) = \dfrac{\partial \Psi(t,s)}{\partial s} - a_1(t)\Psi(t,s)$.

If

$$\liminf_{t\geq t_0}\left[\frac{1}{\Psi(t,t_0)}\int_{t_0}^t \prod_{t_0\leq t_j<s}\left(\frac{1+\beta_j}{1+\gamma_j}\right)^{-1}\frac{\phi^2(t,s)}{\Psi(t,s)}ds\right] = -\infty,$$

then each nonzero solution of (1.1) is oscillatory in Ω.

Proof. To complete the proof, it is sufficient to show that each solution of (3.1) is eventually negative. If we assume that (3.1) has an eventually positive solution, then following the proof of Theorem 3.5, we have

$$\begin{cases} Z'(t) + a_1(t)Z(t) + [Z(t)]^2 \leq 0, \\ Z(t_j^+) \leq \left(\frac{1+\beta_j}{1+\gamma_j}\right) Z(t_j). \end{cases} \tag{3.14}$$

Define

$$H(t) = \prod_{t_0 \le t_j < t} \left(\frac{1+\beta_j}{1+\gamma_j} \right)^{-1} Z(t).$$

Using second inequality of (3.14), we get

$$H(t_j^+) = \prod_{t_0 \le t_i \le t_j} \left(\frac{1+\beta_j}{1+\gamma_j} \right)^{-1} Z(t_j^+) \le \prod_{t_0 \le t_i < t_j} \left(\frac{1+\beta_j}{1+\gamma_j} \right)^{-1} Z(t_j) = H(t_j)$$

which implies that $H(t)$ is continuous on $[t_0, \infty)$.
From (3.14), we have

$$H'(t) + a_1(t)H(t) + \prod_{t_0 \le t_j < t} \left(\frac{1+\beta_j}{1+\gamma_j} \right) H^2(t) \le 0.$$

Multiplying by $\Psi(t,s)$, and integrating from t_0 to t, we get

$$\int_{t_0}^{t} \Psi(t,s)H'(s)ds + \int_{t_0}^{t} \Psi(t,s)a_1(s)H(s)ds + \int_{t_0}^{t} \prod_{t_0 \le t_j < s} \left(\frac{1+\beta_j}{1+\gamma_j} \right) \Psi(t,s)H^2(s)ds \le 0.$$

Since

$$\int_{t_0}^{t} \Psi(t,s)H'(s)ds = -\Psi(t,t_0)H(t_0) - \int_{t_0}^{t} \frac{\partial \Psi(t,s)}{\partial s} H(s)ds.$$

Therefore, we have

$$\int_{t_0}^{t} \prod_{t_0 \le t_j < s} \left(\frac{1+\beta_j}{1+\gamma_j} \right) \Psi(t,s)H^2(s)ds + \int_{t_0}^{t} \Psi(t,s)a_1(s)H(s)ds - \Psi(t,t_0)H(t_0) - \int_{t_0}^{t} \frac{\partial \Psi(t,s)}{\partial s} H(s)ds \le 0.$$

Using condition (A3), we get

$$\int_{t_0}^{t} \prod_{t_0 \le t_j < s} \left(\frac{1+\beta_j}{1+\gamma_j} \right) \Psi(t,s)H^2(s)ds - \int_{t_0}^{t} \phi(t,s)H(s)ds \le \Psi(t,t_0)H(t_0).$$

Using the inequality $\lambda AB^{\lambda-1} - A^{\lambda} \le (\lambda-1)B^{\lambda},\ \lambda \ge 1$, we get

$$-\int_{t_0}^{t} \prod_{t_0 \le t_j < s} \left(\frac{1+\beta_j}{1+\gamma_j} \right)^{-1} \frac{\phi^2(t,s)}{\Psi(t,s)} ds \le 4\Psi(t,t_0)H(t_0).$$

$$\Rightarrow \liminf_{t \ge t_0} \left[\frac{1}{\Psi(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \le t_j < s} \left(\frac{1+\beta_j}{1+\gamma_j} \right)^{-1} \frac{\phi^2(t,s)}{\Psi(t,s)} ds \right] \ge -4H(t_0)$$

which leads to a contradiction. □

4 Application

In this section, we consider the following example to illustrate the main results:

Example 4.1. Consider the following system of fractional impulsive differential equations:

$$\begin{cases} \mathfrak{D}^{\frac{11}{5}}_{+,t}u(z,t) + \frac{1}{t^2}\mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t) + t\mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t) \\ \qquad + \frac{1}{t}g\left(\int_t^\infty (s-t)^{-\frac{1}{5}}u(z,s)ds\right) = 0, \quad t \neq t_j, \\ \mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t_j^+) - \mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t_j^-) = t_j{}^{-5}\sin(z)\mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t_j), \\ \mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t_j^+) - \mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t_j^-) = t_j{}^{-3}\sin(z)\mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t_j), \\ \qquad j = 1,2,3,\ldots, \quad (z,t) \in (0,\frac{\pi}{2}) \times \mathbb{R}_+ = D. \end{cases} \tag{4.1}$$

Here, we easily see that

$$\int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right)ds = \int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{1}{\sigma}d\sigma\right)ds = \int_\tau^\infty \frac{t_0}{s}ds = \infty.$$

Furthermore, we have

$$\limsup_{t\to\infty} \frac{\int_{t_0}^t \prod_{t_0<t_j<s}(1+\beta_j)\prod_{s<t_j<t}(1+\gamma_j)\exp\left(-\int_{t_0}^s a_1(\sigma)d\sigma\right)ds}{\prod_{t_0<t_j<t}(1+\gamma_j)} = \infty$$

and

$$\liminf_{t\to\infty} \frac{\int_{t_0}^t \prod_{t_0<t_j<s}(1+\beta_j)\prod_{s<t_j<t}(1+\gamma_j)\exp\left(-\int_{t_0}^s a_1(\sigma)d\sigma\right)ds}{\prod_{t_0<t_j<t}(1+\gamma_j)} = -\infty.$$

Thus all the conditions of Theorem 3.4 are satisfied. Therefore each nonzero solution of the problem (4.1) oscillates.

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