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LAPLACIAN AND SIGNLESS LAPLACIAN DEGREE PRODUCT DISTANCE ENERGY OF SOME GRAPHS

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Abstract In this paper we define the Laplacian and signless Laplacian degree product distance matrix from the well known degree product distance matrix. Further we define Laplacian degree product distance energy and signless Laplacian degree product distance energy. We obtain the Laplacian degree product distance energy and signless Laplacian degree product distance energy of some graphs of diameter 2.

1 Introduction

All the graphs considered here are finite simple, connected and undirected.Let G be a connected graph of order n,the degree of a vertex is the number of edges incident on it and the distance between two vertices is the length of the shortest path joining them.

The concept of distance valency matrix was introduced in 1999 by O.Ivanciuc [1].The distance valency matrix for a simple graph G of order n is defined by, $Dval(p,q,r) = Dval(p,q,r,G)$ as a square $n \times n$ matrix with entries $[Dval(p,q,r)]_{ij}$ given by,

$$\begin{aligned} [Dval(p,q,r)]_{ij} &= d_{ij}^{p} val_i^{q} val_j^{r}, \text{ if } i \neq j \\ &= 0, \text{ if } i = j \end{aligned}$$

where d_{ij} is the distance between vertex v_i and v_j and val means valency or degree of the vertex with p,q and r being real numbers. Molecular matrices A, D, RD, $Dval(-1,1,1)$, $Dval(-2,1,1)$, $Dval(-2,0,0)$ have been already discussed. We have discussed Dval(1,1,1) in [2] as $DPD(G)$.

The Zagreb index was first introduced by Gutman and Trinajstic as , $M_1(G)$=$\sum_{v\in V} d(v)^2$.For more details one can refer [3],[4], [5].

Several results on Laplacian energy of graph G are reported in the literature [6],[7],[8]. Recently signless Laplacian energy is studied in the literature [9],[10],[11]. We consider the abbreviation for Laplacian degree product distance matrix as L_{DPDM}, Laplacian degree product distance energy as LE_{DPD}, signless Laplacian degree product distance matrix as Q_{DPDM}, and signless Laplacian degree product distance energy as, QE_{DPD}. In this paper we define the Laplacian degree product distance matrix as, $L_{DPD}(G) = D^2(G) - DPDM(G)$ and signless Laplacian degree product distance matrix as, $Q_{DPD}(G) = D^2(G) + DPDM(G)$, where $D^2(G)$ is square of the degree matrix of G.
The following hold for $L_{DPD}(G)$ and $Q_{DPD}(G)$.

- Since the matrix $L_{DPD}(G)$ and $Q_{DPD}(G)$ are real symmetric, their eigenvalues are real.

- The sum of eigenvalues of $L_{DPD}(G)$ or $Q_{DPD}(G)$ is first Zagreb index of G,since trace$L_{DPD}(G)$=trace$Q_{DPD}(G) = \sum_{i=1}^{n} d_i^2$.

- If β_i and γ_i, i=1, 2, 3 . . . n. are eigenvalues of $L_{DPD}(G)$ and $Q_{DPD}(G)$ respectively then they can be arranged in non-increasing order as $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$.

Definition: Let G be graph of order n and size m.If $avd^2(G)$ denotes average square degree of a graph given by, $\frac{\sum_1^n d_i^2(G)}{n}$, then analogous to usual Laplacian and signless Laplacian energy we define the Laplacian and signless Laplacian degree product distance energy as,

$$LE_{DPD}(G) = \sum_{i=1}^{n} |\beta_i - avd^2(G)| = \sum_{i=1}^{n} |\beta_i - \frac{4m^2 - 2\sum_{i<j} d_i d_j}{n}|$$

and

$$QE_{DPD}(G) = \sum_{i=1}^{n} |\gamma_i - avd^2(G)| = \sum_{i=1}^{n} |\gamma_i - \frac{4m^2 - 2\sum_{i<j} d_i d_j}{n}|$$

.

The average square degree of G is $\frac{9}{2} = 4.5$

Example: For the graph G given below,

$L_{DPD}(G) = \begin{bmatrix} 1 & -3 & -4 & -4 \\ -3 & 9 & -6 & -6 \\ -4 & -6 & 4 & -4 \\ -4 & -6 & -4 & 4 \end{bmatrix}$	$Q_{DPD}(G) = \begin{bmatrix} 1 & 3 & 4 & 4 \\ 3 & 9 & 6 & 6 \\ 4 & 6 & 4 & 4 \\ 4 & 6 & 4 & 4 \end{bmatrix}$
Laplacian degree product distance eigenvalues are $\beta_1 = 14.1155$, $\beta_2 = 8$, $\beta_3 = 5.0233$ and $\beta_4 = -9.1388$	signless Laplacian degree product distance eigenvalues are $\gamma_1 = 19.0609$, $\gamma_2 = 1.4842$, $\gamma_3 = 0$ and $\gamma_4 = -2.5451$
$LE_{DPD}(G) = \|14.1155-4.5\|+\|8-4.5\|+\|5.0233-4.5\|+\|4.5+9.1388\| = 27.2776$	$QE_{DPD}(G) = \|19.0609 - 4.5\| + \|4.5 - 1.4842\| + \|0 + 4.5\| + \|4.5 + 2.5451\| = 29.1215$.

2 Bound on largest signless Laplacian degree product distance eigenvalue

Proposition 2.1. *The largest signless Laplacian degree product distance eigenvalue is bounded above by,*
$\gamma_1 \leq \frac{M_1}{n} + \sqrt{\frac{n-1}{n}M + M_1^2(\frac{1}{n^2} - \frac{1}{n}})$*, where*

$$M = \sum_{i=1}^{n} d_i^4 + \sum_{\substack{i=1\\j=1}}^{n} (d_i d_j d_{ij})^2$$

with $d_{ij} = d(v_i, v_j)$, $M_1 = M_1(G)$ *first Zagrab index of G.*

Proof. The trace of $Q_{DPD}(G)$ is $\sum_1^n \gamma_i = \sum d_i^2 = M_1$ so that, $\sum_2^n \gamma_i = M_1 - \gamma_1$, $\sum_1^n \gamma_i^2 = traceL_{DPD}^2 = M$ we have, $\gamma_1^2 + \sum_2^n \gamma_i^2 = M$, $\sum_2^n \gamma_i^2 = M - \gamma_1^2$,
$(\sum_2^n \gamma_i)^2 \leq (n-1)\sum_2^n \gamma_1^2$,
$(M_1 - \gamma_1)^2 \leq (n-1)(M - \gamma_1^2)$,
$M_1^2 - 2M_1\gamma_1 + \gamma_1^2 \leq (n-1)M - (n-1)\gamma_1^2$,
$n\gamma_1^2 - 2M_1\gamma_1 \leq (n-1)M - M_1^2$,

$\gamma_1^2 - \frac{2M_1}{n}\gamma_1 \leq \frac{(n-1)}{n}M - \frac{M_1^2}{n}$,
$\gamma_1^2 - \frac{2M_1}{n}\gamma_1 + \frac{M_1^2}{n^2} \leq \frac{n-1}{n}M - \frac{M_1}{n} + \frac{M_1^2}{n^2}$,
$\gamma_1 - \frac{M_1}{n} \leq \sqrt{\frac{n-1}{n}M + M_1^2(\frac{1}{n^2} - \frac{1}{n})}$
Hence, $\gamma_1 \leq \frac{M_1}{n} + \sqrt{\frac{n-1}{n}M + M_1^2(\frac{1}{n^2} - \frac{1}{n})}$. □

Since the largest Laplacian degree product distance eigenvalue of any matrix is smaller than largest signless Laplacian eigenvalue, the above bound also holds for Laplacian eigenvalue.
For the graph G in figure 1; $n = 4$, $M_1 = 18$ and $M = 514$ giving $\gamma_1 = 15.7852$.
$\frac{M_1}{n} + \sqrt{\frac{n-1}{n}M + M_1^2(\frac{1}{n^2} - \frac{1}{n})} = 22.52083$.
Obviously for a graph with maximum pendent vertices such as star bound is close. For the graph $K_{1.4}$,$n = 1$, $M_1 = 20$ and $M = 436$ giving $\gamma_1 = 20.6788$.
$\frac{M_1}{n} + \sqrt{\frac{n-1}{n}M + M_1^2(\frac{1}{n^2} - \frac{1}{n})} = 20.8760184$.

Corollary 2.2. *If G is a r-regular graph of order n, then*
$\gamma_1 \leq r^2 + \sqrt{\frac{n-1}{n}M + r^4(\frac{1}{n^2} - \frac{1}{n})}$.

For cycle C_n, $r = 2$ bound becomes $\gamma_1 \leq 4 + \sqrt{\frac{n-1}{n}M + 16(\frac{1}{n^2} - \frac{1}{n})}$.

Solving for M_1 we have bound for first Zagrab index in terms of eigenvalues of Q_{DPD} given by,
$M_1 \leq \gamma_1 + \sqrt{(n-1)\sum_2^n \gamma_i^2}$.

Lemma 2.3. *If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of any matrix P of order n then the eigenvalues of the matrix $kI_n \pm P$ are $k \pm \lambda_1, k \pm \lambda_2, .., k \pm \lambda_n$.*

Lemma 2.4. *Let a and b be two arbitrary constants, I is the identity matrix and J is $n \times n$ matrix whose all entries $1's$.If $A = (a - b)I + bJ$ then the characteristic polynomial of A, is*
$|\lambda I - A| = [\lambda - a + b]^{n-1}[\lambda - a - (n-1)b]$.

Using Lemma[2.3] one can directly obtain the Laplacian and signless Laplacian degree product distance eigenvalues from its degree product distance eigenvalues for a regular graph G.The degree product distance energy is already discussed by the present authors in [2].
In general the Laplacian degree product distance energy and signless Laplacian degree product distance energy are equal for a regular graph G.This is consistent with the equality of Laplacian and signless Laplacian energy for regular graph G.

Hence we discuss graphs which are not regular.

3 LE_{DPD} of some graphs of diameter 2

Theorem 3.1. *The Laplacian degree product distance energy of the complete bipartite graph $K_{m,n}$ is, $LE_{DPD}(K_{m,n})$= $|mn - 3n^2(m-1)| + |mn - n^2 + 2n^2(m-1)| + |mn - \beta_1| + |mn - \beta_2|$, where β_1 and β_2 are the roots of the equation, $[\beta^2 - (m^2 - 2m^2(n-1) + n^2 - 2n^2(m-1))\beta + (m^2 - 2m^2(n-1))(n^2 - 2n^2(m-1)) - m^3n^3] = 0$.*

Proof. In $K_{m,n}$, m vertices have degree n and n vertices have degree m. The diameter being 2 the structure of the degree product distance matrix is,
$$L_{DPD}(K_{m,n}) = \begin{pmatrix} n^2 I_m - 2n^2 A(K_m) & -mnJ_{m\times n} \\ -mnJ_{n\times m} & m^2 I_n - 2m^2 A(K_n) \end{pmatrix}$$

where J is matrix of all 1's and A the adjacency matrix. The Laplacian degree product distance polynomial is then given by,

$$|\beta I - L_{DPD}(K_{m,n})| = \begin{vmatrix} (\beta - n^2)I_m + 2n^2A(K_m) & mnJ_{m\times n} \\ mnJ_{n\times m} & (\beta - m^2)I_n + 2m^2A(K_n) \end{vmatrix}$$

Using Lemma 2.4 with, $a = \beta - m^2 - \dfrac{m^3n^2}{\beta - n^2 + 2n^2(m-1)}$ and $b = 2m^2 - \dfrac{m^3n^2}{\beta - n^2 + 2n^2(m-1)}$.
$|\beta I - L_{DPD}(K_{m,n})|$=$[\beta - 3n^2]^{m-1}[\beta - n^2 + 2n^2(m-1)][\beta^2 - (m^2 - 2m^2(n-1) + n^2 - 2n^2(m-1))\beta + (m^2 - 2m^2(n-1))(n^2 - 2n^2(m-1)) - m^3n^3]$.
So that $\beta = m + 2m^2$ $(n-1)$ times, $n - 2n^2$ $(m-1)$ times and remaining two given by roots of,
$[\beta^2 - (m^2 - 2m^2(n-1) + n^2 - 2n^2(m-1))\beta + (m^2 - 2m^2(n-1))(n^2 - 2n^2(m-1)) - m^3n^3] = 0$.
Since the average square degree of $K_{m,n}$ is $\dfrac{mn^2 + nm^2}{m+n} = mn$, theorem follows. □

Corollary 3.2. *If $m = 1$ we get star graph $K_{1,n}$ whose Laplacian degree product distance energy is, $LE_{DPD}(K_{1,n}) = n^2 + |n - \beta_1| + |n - \beta_2|$, where β_1 and β_2 are roots of the equation,* $[\beta^2 - (n^2 - 2n + 3)\beta + (n^2 - 2(n-1)n^2 - n^3)] = 0$.

Consider $K_n + e$ and $K_n - e$ graphs of diameter 2 obtained by adding or deleting an edge e respectively from the complete graph K_n.

Theorem 3.3. *The LE_{DPD} of $K_n + e$ is, $LE_{DPD}(K_n + e)$=*
$|\dfrac{(n-2)(n-1)^2 + n^2 + 1}{n+1} - 2(n-1)^2| \times (n-2) + |\dfrac{(n-2)(n-1)^2 + n^2 + 1}{n+1} - \beta_1| +$
$|\dfrac{(n-2)(n-1)^2 + n^2 + 1}{n+1} - \beta_2| + |\dfrac{(n-2)(n-1)^2 + n^2 + 1}{n+1} - \beta_3|$, *where β_1, β_2 and β_3 are roots of the equation,*$[\beta^3 + ((n-1)^2(n-3) - (n^2+1))\beta^2 - ((n-1)^2(n-3)(n^2+1) + (n^2+4)(n-1)^3)\beta + 9n^2(n-1)^3] = 0$.

Proof. In $K_n + e$ one vertex having n degrees , one vertex having degree 1 remaining having $n-1$ degree then we get the matrix structure as,
$L_{DPD}(K_n + e)$=

$$\begin{bmatrix} n^2 & -n & -n(n-1) & \dots & -n(n-1) & \dots & -n(n-1) \\ -n & 1 & -2(n-1) & \dots & -2(n-1) & \dots & -2(n-1) \\ -n(n-1) & -2(n-1) & (n-1)^2 & \dots & -(n-1)^2 & \dots & -(n-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -n(n-1) & -2(n-1) & -(n-1)^2 & \dots & (n-1)^2 & \dots & -(n-1)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -n(n-1) & -2(n-1) & -(n-1)^2 & \dots & -(n-1)^2 & \dots & (n-1)^2 \end{bmatrix}.$$

$|\beta I - L_{DPD}(K_n + e)|$=

$$\begin{vmatrix} \beta - n^2 & n & n(n-1) & \dots & n(n-1) & \dots & n(n-1) \\ n & \beta - 1 & 2(n-1) & \dots & 2(n-1) & \dots & 2(n-1) \\ n(n-1) & 2(n-1) & \beta - (n-1)^2 & \dots & (n-1)^2 & \dots & (n-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n(n-1) & 2(n-1) & (n-1)^2 & \dots & \beta - (n-1)^2 & \dots & (n-1)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n(n-1) & 2(n-1) & (n-1)^2 & \dots & (n-1)^2 & \dots & \beta - (n-1)^2 \end{vmatrix}.$$

Performing the operations, $C_1 - \dfrac{n(n-1)\sum_3^{n+1} C_i}{\beta - (n-1)^2 + (n-1)^2(n-2)}$

$C_2 - \dfrac{2(n-1)\sum_3^{n+1} C_i}{\beta - (n-1)^2 + (n-1)^2(n-2)}$, where $i = 1, 2, \dots n$.
$|\beta I - L_{DPD}(K_n + e)|$=

$$\begin{vmatrix} \beta - n^2 - \dfrac{n^2(n-1)^2}{\beta-(n-1)+(n-1)^2(n-2)} & n - \dfrac{2n(n-1)^3}{\beta-(n-1)+(n-1)^2(n-2)} \\ n - \dfrac{2n(n-1)^3}{\beta-(n-1)+(n-1)^2(n-2)} & \beta - 1 - \dfrac{4(n-1)^2}{\beta-(n-1)+(n-1)^2(n-2)} \end{vmatrix}.$$

Laplacian degree product distance polynomial of $L_{DPD}(K_n+e)$ is,
$|\beta I - L_{DPD}(K_n+e)|$=$[\beta-2(n-1)^2]^{n-2}[\beta^3+((n-1)^2(n-3)-(n^2+1))\beta^2-((n-1)^2(n-3)(n^2+1)+(n^2+4)(n-1)^3)\beta+9n^2(n-1)^3]$. Extracting the eigenvalues and using the average square degree= $\dfrac{(n-2)(n-1)^2+n^2+1}{n+1}$, theorem follows. □

Theorem 3.4. *The LE_{DPD} of K_n-e is,*
$LE_{DPD}(K_n-e)$= $|\dfrac{(n-2)(n^2-3)}{n} - 2(n+1)^2| \times (n-3) + |\dfrac{(n-2)(n^2-3)}{n} - (n-2)^2| + |\dfrac{(n-2)(n^2-3)}{n} - \beta_1| + |\dfrac{(n-2)(n^2-3)}{n} - \beta_2|$, *where β_1 and β_2 are roots of the equation,*
$[\beta^2-((n-1)^2(n-3)+(n-2)^2-(n-1)^2)\beta+(n-2)^2(n-1)^2(n-4)-2(n-2)^3(n-1)^2]=0.$

Proof. The graph K_n-e is of diameter 2 and has two vertices with distance two remaining at distance one. So that the Laplacian degree product distance polynomial of K_n-e is denoted by, $|\beta I - L_{DPD}(K_n-e)| = [\beta-2(n+1)^2]^{n-3}[\beta-(n-2)^2][\beta^2-((n-1)^2(n-3)+(n-2)^2-(n-1)^2)\beta+(n-2)^2(n-1)^2(n-4)-2(n-2)^3(n-1)^2]=0$. Extracting the eigenvalues and using the average square degree= $\dfrac{(n-2)(n^2-3)}{n}$, theorem follows. □

Noe we consider another pair of graphs of diameter 2.

Let K_n be a complete graph of order n then the vertex coalescence of K_n with K_n will be denoted by $K_nO_vK_n$ and the edge coalescence by $K_nO_eK_n$. $K_nO_vK_n$ has $2n-1$ vertices and $(2n)C_2$ edges whereas $K_nO_eK_n$ has $2n-2$ vertices and $(2n)C_2-1$ edges.

Theorem 3.5. *The LE_{DPD} of $K_nO_vK_n$ is,* $LE_{DPD}(K_nO_vK_n) = |\dfrac{2(n-1)^2+2(n-1)^3}{2n-1} - 2(n-1)^2| \times n + |\dfrac{2(n-1)^2+2(n-1)^3}{2n-1} - 6(n-1)^2| \times (n-3) + |\beta_1 - \dfrac{2(n-1)^2+2(n-1)^3}{2n-1}| + |\beta_2 - \dfrac{2(n-1)^2+2(n-1)^3}{2n-1}|$, *where β_1 and β_2 are the roots of the equation,*
$[\beta^2+(n-1)^2(3n-9)\beta-4(n-1)^4(5n-7)]=0.$

Proof. The graph $K_nO_vK_n$ is of diameter 2 has two sets of vertices one at a distance 2 from each other and other at 1.There is one vertex of degree $(2n-2)$ and remaining $(2n-2)$ of degree $(n-1)$. Since the average degree square of $K_nO_vK_n$ is, $avd^2(K_nO_vK_n)$ =$\dfrac{2(n-1)^2+2(n-1)^3}{n}$. With suitable labeling the L_{DPD} of $K_nO_vK_n$ takes the form,
$L_{DPD}(K_nO_vK_n)$=

$$\begin{vmatrix}
(2n-2)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 \\
-2(n-1)^2 & (n-1)^2 & -(n-1)^2 & \cdots & -(n-1)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 \\
-2(n-1)^2 & -(n-1)^2 & (n-1)^2 & \cdots & -2(n-1)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-2(n-1)^2 & -(n-1)^2 & -(n-1)^2 & \cdots & (n-1)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 \\
-2(n-1)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 & (n-1)^2 & -(n-1)^2 & \cdots & -(n-1)^2 \\
-2(n-1)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 & -(n-1)^2 & (n-1)^2 & \cdots & -(n-1)^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-2(n-1)^2 & -2(n-1)^2 & -2(n-1)^2 & \cdots & -2(n-1)^2 & -(n-1)^2 & -(n-1)^2 & \cdots & (n-1)^2
\end{vmatrix}.$$

So that the Laplacian degree product distance polynomial of $K_nO_vK_n$ is denoted by,
$|\beta I - L_{DPD}(K_nO_vK_n)|$. Simplifying finally we arrive at
$|\beta I - L_{DPD}(K_nO_vK_n)|$=
$[\beta-2(n-1)]^n[\beta-6(n-1)^2]^{n-3}[\beta^2+(n-1)^2(3n-9)\beta-4(n-1)^4(5n-7)]$. Using the average degree= $\dfrac{2(n-1)^2+2(n-1)^3}{2n-1}$, theorem follows.□

On similar lines we get the Laplacian degree product distance energy of $K_nO_eK_n$.

Theorem 3.6. *The LE_{DPD} of the edge coalescence of two complete graphs K_n is given by,* $LE_{DPD}(K_nO_eK_n)= |\frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2} - 2(n-1)^2| \times (2n-6) + |\frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2} - 2(2n-3)^2| + |\frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2} - (n-1)^2 - (n-1)^3| + |\beta_1 - \frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2}| + |\beta_2 - \frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2}|$, *where* β_1 *and* β_2 *are roots of the equation,* $[\beta^2 + (n-1)^2(3n-8)\beta - 4(2n-3)^2(n-1)^2(n-2)] = 0$.

4 QE_{DPD} of some graphs of diameter 2

We now state without proof results on QE_{DPD} which follow on similar lines like LE_{DPD} proved in previous section.

Theorem 4.1. *The signless Laplacian degree product distance energy of the complete bipartite graph* $K_{m,n}$ *is,* $LE_{DPD}(K_{m,n})= |mn+n^2(m-1)| + |mn+m^2(n-1)| + |mn-\gamma_1| + |mn-\gamma_2|$, *where* γ_1 *and* γ_2 *are the roots of the equation,* $[\gamma^2 - (2mn(m+n) - m^2 - n^2)\gamma + m^2n^2(4mn - 2m - 2n + 1) - m^3n^3] = 0$.

Corollary 4.2. *If* $m = 1$ *we get star graph* $K_{1,n}$ *whose signless Laplacian degree product distance energy is,* $QE_{DPD}(K_{1,n}) = |n-3(n-1)| + |n-2(n-1)(n-2)| + |n-\gamma_1| + |n-\gamma_2|$, *where* γ_1 *and* γ_1 *are roots of the equation,* $[\gamma^2 - (n^2-2n-3)\gamma + 3(n^2-n^3)] = 0$.

Theorem 4.3. *The* QE_{DPD} *of* $K_n - e$ *is,* $QE_{DPD}(K_n - e)=$ $|\frac{(n-2)(n^2-3)}{n}| \times (n-3) + |\frac{(n-2)(n^2-3)}{n} + (n-1)^2| + |\gamma_1 - \frac{(n-2)(n^2-3)}{n}| + |\gamma_2 - \frac{(n-2)(n^2-3)}{n}|$, *where* γ_1 *and* γ_2 *are roots of the equation,* $[\gamma^2 - (3(n-2)^2 + (n-1)^2(n-2))\gamma + (n-1)^2(n-2)^3] = 0$.

Theorem 4.4. *The* QE_{DPD} *of* $K_n + e$ *is,* $QE_{DPD}(K_n + e)=$ $|\frac{(n-2)(n-1)^2+n^2+1}{n+1}| \times (n-3) + |\frac{(n-2)(n-1)^2+n^2+1}{n+1} - \gamma_1| + |\frac{(n-2)(n-1)^2+n^2+1}{n+1} - \gamma_2| + |\frac{(n-2)(n-1)^2+n^2+1}{n+1} - \gamma_3|$, *where* γ_1, γ_2 *and* γ_3 *are roots of the equation,* $[\gamma^3 + [(n^2+1) + (n-1)^2(n-2)]\gamma^2 + [(n-1)^2(n-2)(1-n^2)]\gamma + n^2(n-1)^4(n-2)] = 0$.

Theorem 4.5. *The* QE_{DPD} *of* $K_nO_vK_n$ *is,* $QE_{DPD}(K_nO_vK_n)=$ $|\frac{2(n-1)^2+2(n-1)^3}{2n-1}| \times n + |\frac{2(n-1)^2+2(n-1)^3}{2n-1} - (2n-2)^2| + |\gamma_1 - \frac{2(n-1)^2+2(n-1)^3}{2n-1}| + |\gamma_2 - \frac{2(n-1)^2+2(n-1)^3}{2n-1}|$, *where* γ_1 *and* γ_2 *are roots of the equation,* $[\gamma^2 - (n-1)^2(3n+1)\gamma + 4(n-1)^5] = 0$.

Theorem 4.6. *The* QE_{DPD} *of* $K_nO_eK_n$ *is given by,* $QE_{DPD}(K_nO_eK_n) =$ $|\frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2}| \times (2n-5) + |\frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2} - (n-1)^2 + (n-1)^3| + |\gamma_1 - \frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2}| + |\gamma_2 - \frac{2(2n-3)^2+(2n-4)(n-1)^2}{2n-2}|$, *where* γ_1 *and* γ_2 *are the roots of equation,* $[\gamma^2 - (3n^3 - 4n^2 - 9n - 12)\gamma + 2(n-1)^2(n-2)(2n-3)^2] = 0$.

5 Conclusion

We defined the L_{DPD} and Q_{DPD} of a graph G, obtained expressions for energy of some graphs of diameter 2.

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