Palestine Journal of Mathematics

Vol. 10(1)(2021) , 273–278

ON A SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING RUSCHEWEYH OPERATOR

Pardeep Kaur and Sukhwinder Singh Billing

Communicated by S. P. Goyal

MSC 2010 Classifications: Primary 30C45; Secondary 30C80.

Keywords and phrases: Analytic function, convex function, Ruscheweyh operator, starlike function.

Authors are thankful to Prof. S. P. Goyal for his valuable comments

Abstract. In the present paper, we study certain differential inequalities involving Ruscheweyh operator. As particular cases to our main result, we derive certain results for starlike and convex functions.

1 Introduction

Let $\mathcal{H}$ denote the class of functions f, analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$, consisting of functions f, analytic in the open unit disk $\mathbb{E}$ and normalized by the conditions $f'(0) = 0 = f'(0) - 1$. A function $f \in \mathcal{A}$ is said to be starlike of order α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \le \alpha < 1,\ z \in \mathbb{E}.$$

The class of such functions is denoted by $\mathcal{S}^*(\alpha)$. A function $f \in \mathcal{A}$ is said to be convex of order α in $\mathbb{E}$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \le \alpha < 1,\ z \in \mathbb{E}.$$

Let $\mathcal{K}(\alpha)$ denote the class of all those functions $f \in \mathcal{A}$ that are convex of order α in $\mathbb{E}$. Let f and g be two analytic functions in open unit disk $\mathbb{E}$. Then we say f is subordinate to g in $\mathbb{E}$ written as $f \prec g$ if there exists a Schwarz function w, analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.
The Taylor's series expansions of $f,\ g \in \mathcal{A}$ are given as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the convolution/Hadamard product of f and g is denoted by $f * g$, and defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Ruscheweyh [5] introduced a differential operator R^λ, (Known as Ruscheweyh differential operator) for $f \in \mathcal{A}$ is defined as follows

$$R^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z),\ \lambda \ge -1,\ z \in \mathbb{E}. \tag{1.1}$$

For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$R^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!},\ z \in \mathbb{E}.$$

Lecko et al. [3] observed that for $\lambda \geq -1$, the expression given in (1.1) becomes

$$R^{\lambda}f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)(\lambda+2)\dots(\lambda+k-1)}{(k-1)!} a_k z^k,\ z \in \mathbb{E},$$

and for every $\lambda > -1$

$$R^1R^{\lambda}f(z) = z(R^{\lambda}f)'(z) = z\left(\frac{z}{(1-z)^{\lambda+1}} * f(z)\right)'$$
$$= \frac{z}{(1-z)^{\lambda+1}} * (zf'(z)) = R^{\lambda}(zf'(z)) = R^{\lambda}R^1f(z),\ z \in \mathbb{E}.$$

We notice that

$$R^{-1}f(z) = z,\ R^0f(z) = f(z),\ R^1f(z) = zf'(z) \text{ and } R^2f(z) = zf'(z) + \frac{z^2}{2}f''(z),$$

and so on. For $\lambda \geq -1$ and for $z \in \mathbb{E}$, we have

$$z(R^{\lambda}f)'(z) = (\lambda+1)R^{\lambda+1}f(z) - \lambda R^{\lambda}f(z). \tag{1.2}$$

Ruscheweyh [5] introduced the class K_{λ} defined as

$$K_{\lambda} = \left\{ f \in \mathcal{A} :\ \Re\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right) > \frac{1}{2},\ \lambda \in \mathbb{N}_0,\ z \in \mathbb{E} \right\}.$$

He gave coefficient estimates and determined some special elements of K_{λ}. In 1980, Al-Amiri [1] introduced and studied the class $\mathcal{S}_{\lambda}(\alpha,\ \beta),\ \lambda \in \mathbb{N}_0$ defined as :

$$\mathcal{S}_{\lambda}(\alpha,\ \beta) = \{f \in \mathcal{A} : \Re P_{\lambda}(f(z); \alpha,\ \beta) > 0,\ \lambda \in \mathbb{N}_0,\ z \in \mathbb{E}\}$$

where

$$P_{\lambda}(f(z); \alpha,\ \beta) = \left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} - \frac{1}{2}\right)^{\alpha} \left(\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - \frac{1}{2}\right)^{\beta},$$

and $\alpha,\ \beta$ are real numbers. He observed that for every $\lambda \in \mathbb{N}_0$, $\mathcal{S}_{\lambda}(\alpha,\ \beta)$ contains many interesting classes of univalent functions; $\mathcal{S}_{\lambda}(1,\ 0) = K_{\lambda},\ \mathcal{S}_{\lambda}(0,\ 1) = K_{\lambda+1}$ and $\mathcal{S}_0(\alpha,\ 0)$ is contained in strongly starlike class when $|\alpha| \geq 1$. Owa et al. [4] introduced and studied the following classes:

$$\mathcal{S}^*_{\lambda} = \{f \in \mathcal{A} :\ R^{\lambda}f(z) \in \mathcal{S}^*,\ \lambda \geq -1\}$$

and

$$\mathcal{K}_{\lambda} = \{f \in \mathcal{A} :\ R^{\lambda}f(z) \in \mathcal{K},\ \lambda \geq -1\}.$$

They established several interesting properties of $\mathcal{S}^*_{\lambda}$ and $\mathcal{K}_{\lambda}$.
The results of above nature motivated us for the work of present paper. We, here, study the following differential inequality

$$\left|\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)} - 1\right|^{\alpha} \left|\frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} - 1\right|^{\beta} < M(\alpha,\ \beta,\ \mu,\ \lambda),\ z \in \mathbb{E},$$

where $\alpha \geq 0,\ \beta \geq 0,\ 0 \leq \mu < 1$ and $\lambda \geq 0$ and obtain certain results for starlike and convex functions as special cases of our main result.

2 Preliminary

To prove our main result, we shall make use of the following lemma due to Jack [2].

Lemma 2.1. *Suppose $w(z)$ be a non-constant analytic function in $\mathbb{E}$, with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in \mathbb{E}$ on the circle $|z| = r < 1$, then $z_0w'(z_0) = kw(z_0)$, where $k \geq 1$, is some real number.*

3 Main Results

Theorem 3.1. *Let* $\alpha,\ \beta$ *and* μ *be real numbers such that* $\alpha \geq 0,\ \beta \geq 0,\ 0 \leq \mu < 1$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)} - 1\right|^{\alpha} \left|\frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} - 1\right|^{\beta} < M(\alpha,\ \beta,\ \mu,\ \lambda),\ z \in \mathbb{E}, \tag{3.1}$$

where

$$M(\alpha,\ \beta,\ \mu,\ \lambda) = \begin{cases} (1-\mu)^{\alpha+\beta}\left(1+\frac{1}{2(1+\lambda-\mu)}\right)^{\beta}, & 0 \leq \mu \leq 1/2, \\ (1-\mu)^{\alpha+\beta}\left(1+\frac{\mu}{\mu+\lambda}\right)^{\beta}, & 1/2 \leq \mu < 1. \end{cases}$$

then

$$\Re\left(\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)}\right) > \mu,\ \lambda \geq 0,\ z \in \mathbb{E}.$$

Proof. Case(i) Let $0 \leq \mu \leq \frac{1}{2}$. Define

$$\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)} = \frac{1+(1-2\mu)w(z)}{1-w(z)},\quad z \in \mathbb{E}. \tag{3.2}$$

Here $w(z)$ is analytic in $\mathbb{E},\ w(0) = 0$ and $w(z) \neq 1$ in $\mathbb{E}$. Using (1.2), we get

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} = \frac{(1+\lambda)+(1-2\mu-\lambda)w(z)}{(1+\lambda)(1-w(z))}$$

On differentiating above equation logarithmically, we get

$$\frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} - \frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)} = \frac{2(1-\mu)zw'(z)}{[(1+\lambda)+(1-2\mu-\lambda)w(z)][1-w(z)]} \tag{3.3}$$

By making use of (3.2), the above equation reduces to

$$\frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} = \frac{1+(1-2\mu)w(z)}{1-w(z)} + \frac{2(1-\mu)zw'(z)}{[(1+\lambda)+(1-2\mu-\lambda)w(z)][1-w(z)]}.$$

Now, from (3.1), we have

$$\begin{aligned} &\left|\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)} - 1\right|^{\alpha} \left|\frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} - 1\right|^{\beta} \\ &= \left|\frac{2(1-\mu)w(z)}{1-w(z)}\right|^{\alpha} \left|\frac{2(1-\mu)w(z)}{1-w(z)} + \frac{2(1-\mu)zw'(z)}{[(1+\lambda)+(1-2\mu-\lambda)w(z)][1-w(z)]}\right|^{\beta} \\ &= \left|\frac{2(1-\mu)w(z)}{1-w(z)}\right|^{\alpha+\beta} \left|1 + \frac{zw'(z)}{w(z)[(1+\lambda)+(1-2\mu-\lambda)w(z)]}\right|^{\beta} \end{aligned}$$

We need to prove $|w(z)| < 1$ for all $z \in \mathbb{E}$. If $|w(z)| \not< 1$ then there exists a point $z_0 \in \mathbb{E}$ such that $\max\limits_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$. Then by Lemma 2.1, we have $w(z_0) = e^{i\theta},\ 0 < \theta \leq 2\pi$ and $z_0 w'(z_0) = kw(z_0),\ k \geq 1$. Hence

$$\begin{aligned} &\left|\frac{z_0(R^{\lambda}f(z_0))'}{R^{\lambda}f(z_0)} - 1\right|^{\alpha} \left|\frac{z_0(R^{\lambda+1}f(z_0))'}{R^{\lambda+1}f(z_0)} - 1\right|^{\beta} \\ &= \left|\frac{2(1-\mu)w(z_0)}{1-w(z_0)}\right|^{\alpha+\beta} \left|1 + \frac{z_0 w'(z_0)}{w(z_0)[(1+\lambda)+(1-2\mu-\lambda)w(z_0)]}\right|^{\beta} \\ &= \frac{2^{\alpha+\beta}(1-\mu)^{\alpha+\beta}}{|1-e^{i\theta}|^{\beta+\alpha}} \left|1 + \frac{k}{(1+\lambda)+(1-2\mu-\lambda)e^{i\theta}}\right|^{\beta} \\ &\geq (1-\mu)^{\alpha+\beta}\left(1+\frac{k}{2(1+\lambda-\mu)}\right)^{\beta} \geq (1-\mu)^{\alpha+\beta}\left(1+\frac{1}{2(1+\lambda-\mu)}\right)^{\beta} \end{aligned}$$

which contradicts (3.1) for $0 \le \mu \le \frac{1}{2}$. Thus, we must have $|w(z)| < 1$ for all $z \in \mathbb{E}$ and hence the result follows.

Case(ii) For $\frac{1}{2} \le \mu < 1$, define w as

$$\frac{z(R^\lambda f(z))'}{R^\lambda f(z)} = \frac{\mu}{\mu - (1-\mu)w(z)}, \; z \in \mathbb{E}, \tag{3.4}$$

where $w(z) \ne \frac{\mu}{1-\mu}$ in $\mathbb{E}$. Then $w(z)$ is analytic in $\mathbb{E}$ and $w(0) = 0$. In view of (1.2) and proceeding as in Case(i), we obtain

$$\left|\frac{z(R^\lambda f(z))'}{R^\lambda f(z)} - 1\right|^\alpha \left|\frac{z(R^{\lambda+1} f(z))'}{R^{\lambda+1} f(z)} - 1\right|^\beta$$

$$= \left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)}\right|^\alpha \left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)} + \frac{\mu(1-\mu)zw'(z)}{[\mu(1+\lambda)-\lambda(1-\mu)w(z)][\mu-(1-\mu)w(z)]}\right|^\beta$$

$$= \left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)}\right|^{\alpha+\beta} \left|1 + \frac{\mu zw'(z)}{w(z)[\mu(1+\lambda)-\lambda(1-\mu)w(z)]}\right|^\beta .$$

We need to show that $|w(z)| < 1$ for all $z \in \mathbb{E}$. On the contrary, suppose that there exists a point $z_0 \in \mathbb{E}$ such that $\max\limits_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1$. Then by Lemma 2.1, we have $w(z_0) = e^{i\theta}$, $0 < \theta \le 2\pi$ and $z_0 w'(z_0) = kw(z_0)$, $k \ge 1$. Therefore

$$\left|\frac{z_0(R^\lambda f(z_0))'}{R^\lambda f(z_0)} - 1\right|^\alpha \left|\frac{z_0(R^{\lambda+1} f(z_0))'}{R^{\lambda+1} f(z_0)} - 1\right|^\beta$$

$$= \left|\frac{(1-\mu)w(z_0)}{\mu-(1-\mu)w(z_0)}\right|^{\alpha+\beta} \left|1 + \frac{\mu k w(z_0)}{w(z_0)[\mu(1+\lambda)-\lambda(1-\mu)w(z_0)]}\right|^\beta$$

$$\ge (1-\mu)^{\alpha+\beta}\left(1 + \frac{k\mu}{\mu+\lambda}\right)^\beta$$

$$\ge (1-\mu)^{\alpha+\beta}\left(1 + \frac{\mu}{\mu+\lambda}\right)^\beta ,$$

which contradicts (3.1) for $\frac{1}{2} \le \mu < 1$. Thus, we must have $|w(z)| < 1$ for all $z \in \mathbb{E}$, hence we conclude in view of (3.4) that

$$\Re\left(\frac{z(R^\lambda f(z))'}{R^\lambda f(z)}\right) > \mu, \; z \in \mathbb{E}.$$

□

Setting $\lambda = 0$ in Theorem 3.1, we obtain result of Singh et al. [6]:

Corollary 3.2. *Let α, β and μ be real numbers such that $\alpha \ge 0$, $\beta \ge 0$, and $0 \le \mu < 1$, with $\beta + \alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^\alpha \left|\frac{zf''(z)}{f'(z)}\right|^\beta < \begin{cases} (1-\mu)^\alpha(\frac{3}{2}-\mu)^\beta, & 0 \le \mu \le 1/2, \\ (1-\mu)^{\alpha+\beta}2^\beta, & 1/2 \le \mu < 1, \end{cases}$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \mu, \; z \in \mathbb{E}.$$

Hence $f \in \mathcal{S}^*(\mu)$.

Taking $\lambda = 0$ and replacing $f(z)$ with $zf'(z)$ in Theorem 3.1, we get the following result of Singh et al.[6]:

Corollary 3.3. *Let $\alpha,\ \beta$ and μ be real numbers such that $\alpha \geq 0,\ \beta \geq 0,\ 0 \leq \mu < 1$ with $\beta + \alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}\left|\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right|^{\beta} < \begin{cases}(1-\mu)^{\alpha}(\frac{3}{2}-\mu)^{\beta}, & 0 \leq \mu \leq 1/2,\\ (1-\mu)^{\alpha+\beta}2^{\beta}, & 1/2 \leq \mu < 1,\end{cases}$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \mu,\ z \in \mathbb{E}.$$

Hence $f \in \mathcal{K}(\mu)$.

Setting $\lambda = 1$ in Theorem 3.1, we obtain:

Corollary 3.4. *Let $\alpha,\ \beta$ and μ be real numbers such that $\alpha \geq 0,\ \beta \geq 0,\ 0 \leq \mu < 1$ with $\beta + \alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}\left|\frac{3zf''(z)+z^2f'''(z)}{2f'(z)+zf''(z)}\right|^{\beta} < \begin{cases}(1-\mu)^{\alpha+\beta}\left(\frac{5-2\mu}{4-2\mu}\right)^{\beta}, & 0 \leq \mu \leq 1/2,\\ (1-\mu)^{\alpha+\beta}\left(\frac{2\mu+1}{\mu+1}\right)^{\beta}, & 1/2 \leq \mu < 1,\end{cases}$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \mu,\ z \in \mathbb{E}.$$

ı.e. $f \in \mathcal{K}(\mu)$.

Selecting $\alpha = 0$ and $\beta = 1$ in Theorem 3.1, we have:

Corollary 3.5. *Let μ be a real number such that $0 \leq \mu < 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} \prec \begin{cases}1+(1-\mu)z+\frac{(1-\mu)z}{2(1+\lambda-\mu)}, & 0 \leq \mu \leq 1/2,\\ 1+(1-\mu)z+\frac{\mu(1-\mu)z}{\mu+\lambda}, & 1/2 \leq \mu < 1.\end{cases}$$

then

$$\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)} \prec \frac{1+(1-2\mu)z}{1-z},\ z \in \mathbb{E},$$

hence

$$\Re\left(\frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)}\right) > \mu.$$

References

[1] H. S. A-Amiri, On Ruscheweyh derivative, *A*nn. Polo. Math. **38**, 87–94 (1980).

[2] I. S. Jack, Functions starlike and convex of order μ, *J*. London Math. Soc. **3**, 469–474 (1971).

[3] A. Lecko, M. Lecko and T. Yaguchi, Subclasses of typically real functions defined by Rusheweyh derivative,*D*emonstr. Math. **41** (8), 823–832 (2008).

[4] S. Owa, S. Fukui, X. Sakaguchi and S. Ogawa, An application of the Ruscheweyh derivatives, *I*nt. J. Math. Sci. **9**(4), 721–730 (1986).

[5] S. Ruscheweyh, New criteria for univalent functions, *P*roc. Amer. Math. Soc. **49**, 109–115 (1975).

[6] S. Singh, S. Gupta and S. Singh, On a class of multivalent functions defined by a multiplier transformation, *M*atem. Ves. **60**, 87–94 (2008).

Author information

Pardeep Kaur, Department of Applied Sciences, Baba Banda Singh Bahadur Engineering College, Fatehgarh Sahib-140407, Punjab, INDIA.
E-mail: aradhitadhiman@gmail.com

Sukhwinder Singh Billing, Department of Mathematics, Sri Guru Granth Sahib World University, Fatehgarh Sahib-140407, Punjab, INDIA..
E-mail: ssbilling@gmail.com

Received: April 12, 2019.
Accepted: May 22, 2019.