Palestine Journal of Mathematics

Vol. 10(1)(2021) , 279–289

Binomial Sum Formulas From the Exponential Generating Functions of $(p, q)-$Fibonacci and $(p, q)-$Lucas Quaternions

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Communicated by Jose Louis Lopez-Bonilla

MSC 2010 Classifications: Primary 11B39; Secondary 11B83, 11R52

Keywords and phrases: Generalized Fibonacci numbers, generalized Lucas numbers, generalized quaternions, generating functions.

Abstract In this paper, we obtain some binomial sum formulas for (p, q)-Fibonacci and (p, q)-Lucas quaternions, by using the exponential generating functions. By this way we examine formulas for Q_{2n+r}, K_{2n+r}, p^nQ_{n+r}, p^nK_{n+r}, Q_{mn}, K_{mn} and far more than these, where Q_n and K_n are the n-th (p, q)-Fibonacci and (p, q)-Lucas quaternions, respectively with p, q arbitrary nonzero real numbers.

1 Introduction

The (p, q)-Fibonacci sequence $(U_n) = (U_n(p, q))$ and (p, q)-Lucas sequence $(V_n) = (V_n(p, q))$ are defined by

$$U_0 = 0, U_1 = 1, U_n = pU_{n-1} + qU_{n-2} \tag{1.1}$$

and

$$V_0 = 2, V_1 = p, V_n = pV_{n-1} + qV_{n-2} \tag{1.2}$$

for $n \geq 2$, where p, q are arbitrary nonzero real numbers. The terms U_n and V_n are called the n-th (p, q)-Fibonacci and Lucas numbers, respectively. One can find properties of these sequences in [5, 6, 7, 19]. The roots of the characteristic equation $x^2 - px - q = 0$ are $\alpha = \dfrac{p+\sqrt{\Delta}}{2}$ and $\beta = \dfrac{p-\sqrt{\Delta}}{2}$, where $\Delta = p^2 + 4q$. Thus we can give the following properties; $\alpha^2 - q = p\alpha$, $\beta^2 - q = p\beta$, $\alpha + \beta = p$, $\alpha - \beta = \sqrt{\Delta}$ and $\alpha\beta = -q$. Moreover it can be seen that $\alpha^n = \alpha\, U_n + qU_{n-1}$ and $\beta^n = \beta U_n + qU_{n-1}$ for all $n \in \mathbb{Z}$.

If $\Delta \neq 0$, then Binet's formulas for U_n and V_n can be given by

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}} \text{ and } V_n = \alpha^n + \beta^n$$

for all $n \geq 0$. William R. Hamilton observed quaternions firstly in 1843. The group of quaternions are denoted as $\mathbb{H}$. Although addition is closed and commutative, the quaternion multiplication is not commutative over $\mathbb{H}$. A quaternion q is a hyper complex number of the form

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$$

with real components a, b, c, d and basis $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$, where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},\ \mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \text{ and } \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The conjugate of quaternion q is defined as $\overline{q} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$ in [8].

$(p, q)-$Fibonacci quaternion Q_n is defined by

$$Q_n = U_n + \mathbf{i}U_{n+1} + \mathbf{j}U_{n+2} + \mathbf{k}U_{n+3},\ n \geq 0$$

where U_n is the n-th $(p, q)-$Fibonacci number and the $(p, q)-$Lucas quaternion K_n is defined by

$$K_n = V_n + \mathbf{i}V_{n+1} + \mathbf{j}V_{n+2} + \mathbf{k}V_{n+3},\ n \geq 0$$

where V_n is the n-th $(p,q)-$Lucas number in [7, 9]. Moreover it is easy to see that

$$Q_{n+1} = pQ_n + qQ_{n-1} \tag{1.3}$$

and

$$K_{n+1} = pK_n + qK_{n-1}, \tag{1.4}$$

for $n \geq 1$.

Binet's formulas for the $(p,q)-$Fibonacci quaternions Q_n and $(p,q)-$ Lucas quaternions K_n were discovered by Iakin in [10]. For all $n \in \mathbb{Z}$,

$$Q_n = \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta}$$

and

$$K_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n$$

where $\widehat{\alpha} = \mathbf{1} + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$ and $\widehat{\beta} = \mathbf{1} + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$.

Quaternions are widely used in many researches as theoretical and application. For example, in number theory, quaternions are used to prove Lagrange's theorem which says that every positive integer is sum of at most four squares [13]. Quaternions are used to express the Lorentz transform in special and general relativity in physics [3].

In computer graphics, programmers prefer to use spherical linear interpolation technique based on all unit quaternions that form a unit sphere. By this method they can define the sequence of rotations which affects the smoothness of the animation in computer games, that prevents unnatural view, stucking or swinging widely [14]. Also quaternions are used in mechanics to calculate movement and rotations.

One another application of Hamiltonian quaternion algebra is multi antenna radio transmission. Moreover the quaternions are widely used in control systems that guide aircraft and rockets, in modern navigation programmes, besides in electric toothbrushes.

In this study we deal with the combinatorial part of quaternion algebra such as deriving some sum formulas from exponential generating functions of generalized Fibonacci and Lucas quaternions.

Ramirez proved Cassini's identity for k-Fibonacci quaternions and gave a conjecture for Catalan identity of k-Fibonacci quaternions, in [18]. Then Polatlı and Kesim proved Ramirez's conjecture in [16]. Taking $p = k$ and $q = 1$ in the components of (p,q)-Fibonacci quaternion Q_n and (p,q)-Lucas quaternion K_n, it is obvious that Q_n and K_n turn to $k-$Fibonacci quaternion and $k-$Lucas quaternion, respectively.

In [2], Demirtürk Bitim and Topal established Cassini's identity for (p,q)-Fibonacci quaternions and (p,q)-Lucas quaternion counterparts of Cassini's identity. Moreover the authors gave generating functions of Q_{mn+r} and K_{mn+r}. These are the more general forms of the theorems given in [18] by Ramirez and given in [16, 17] by Polatlı and Kesim.

From this point of view, in this paper, firstly we obtained exponential generating functions of $(p,q)-$Fibonacci quaternion Q_n and $(p,q)-$Lucas Quaternion K_n. Actually Ipek in [11], Patel and Ray in [15] gave these exponential generating functions. In [1], Çimen and Ipek considered Pell quaternions QP_n and Pell-Lucas quaternions QPL_n which are the special cases of $(p,q)-$Fibonacci quaternion Q_n and $(p,q)-$Lucas Quaternion K_n, by taking $p = 2$ and $q = 1$ in the identities (1.3) and (1.4). Moreover the authors proved sum formulas $\sum_{i=1}^{n} QP_i$, $\sum_{i=1}^{n} QP_{2i}$ and $\sum_{i=1}^{n} QP_{2i-1}$.

Then in [2], Demirtürk Bitim and Topal established the general formula for $\sum_{i=1}^{n} Q_{mi+r}$ and $\sum_{i=1}^{n} K_{mi+r}$. These formulas are also derived by Patel and Ray in [15]. In addition to this, they gave the formulas

$$Q_{mn} = \sum_{j=0}^{n} \binom{n}{j} q^{n-j} {F_m}^j {F_{m-1}}^{n-j} Q_j$$

and

$$K_{mn} = \sum_{j=0}^{n} \binom{n}{j} q^{n-j} {F_m}^j {F_{m-1}}^{n-j} K_j,$$

where $m, k \in \mathbb{Z}$, $n > m \geq 0$ and F_m is the $(p,q)-$Fibonacci number, by using induction. We will prove these binomial sum formulas by using the exponential generating functions of $(p,q)-$Fibonacci and $(p,q)-$Lucas quaternions. Moreover we obtain new sum formulas for Q_{mn+r} and K_{mn+r}.

Furthermore Ipek proved $\sum_{i=0}^{n} \binom{n}{i} q^{n-i} Q_{2i+k}$ and $\sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{n-i} Q_{2i+k}$ sum formulas in [11]. In this study we will give the following more general formulas

$$\sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i Q_{i+r}, \sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i K_{i+r},$$

$$\sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} Q_{2i+r}, \sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} K_{2i+r},$$

and,

$$\sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i Q_i, \sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i K_i.$$

2 Some Identities of Exponential Generating Functions

Firstly, in 1718, Abraham De Moivre used generating functions to determine the Fibonacci recurrence relations. In 1948, J. Ginzburg proved Lucas's (1876) formula

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1,$$

by using the generating functions. In 1967, V. E. Hoggatt, Jr. and D. A. Lind listed 18 various generating functions of Fibonacci and Lucas numbers and some of their powers.

Then, R. T. Hansen [4] investigated

$$\sum_{n=0}^{\infty} F_{m+n} x^n = \frac{F_m + F_{m-1}x}{1 - x - x^2}$$

and

$$\sum_{n=0}^{\infty} L_{m+n} x^n = \frac{L_m + L_{m-1}x}{1 - x - x^2}$$

in 1972.

In this paper we consider exponential generating functions of $(p,q)-$Fibonacci and $(p,q)-$Lucas quaternions. In short, we know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Moreover it is easy to see that $e^{ax} = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!}$.

Thus we can start by giving the known identities from [12], in Lemma 1 and Lemma 2.

Lemma 2.1. *Let* $n \in \mathbb{Z}$. *Then* $\frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$ *generates the numbers* $\frac{F_n}{n!}$ *and*

$e^{\alpha x} + e^{\beta x}$ generates the numbers $\frac{L_n}{n!}$.

Proof. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we derive the expected formula

$$\frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n,$$

by using the Binet's formulas. In a similar way, it follows that

$$e^{\alpha x} + e^{\beta x} = \sum_{n=0}^{\infty} \frac{L_n}{n!} x^n.$$

□

Lemma 2.2. *Let* $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ *and* $B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$, *where* (a_n) *and* (b_n) *are any number sequences, then we obtain*

$$A(x) B(x) = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} a_n b_{n-i} \right] \frac{x^n}{n!} \tag{2.1}$$

and

$$A(x) B(-x) = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_n b_{n-i} \right] \frac{x^n}{n!}. \tag{2.2}$$

Lemma 2.3. *Let* $m, n, r \in \mathbb{Z}$ *and* x *be any nonzero real number. Then the exponential generating functions of the quaternions* Q_{mn+r} *and* K_{mn+r} *are given by*

$$\sum_{n=0}^{\infty} Q_{mn+r} \frac{x^n}{n!} = \frac{\hat{\alpha}\alpha^r e^{\alpha^m x} - \hat{\beta}\beta^r e^{\beta^m x}}{\alpha - \beta} \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} K_{mn+r} \frac{x^n}{n!} = \hat{\alpha}\alpha^r e^{\alpha^m x} + \hat{\beta}\beta^r e^{\beta^m x}. \tag{2.4}$$

Proof. If we consider Binet's formulas, then we get

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{mn+r} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \frac{\hat{\alpha}\alpha^{mn+r} - \hat{\beta}\beta^{mn+r}}{\alpha - \beta} \frac{x^n}{n!} \\
&= \frac{\hat{\alpha}\alpha^r}{\alpha - \beta} \sum_{n=0}^{\infty} (\alpha^m)^n \frac{x^n}{n!} - \frac{\hat{\beta}\beta^r}{\alpha - \beta} \sum_{n=0}^{\infty} \left(\beta^m\right)^n \frac{x^n}{n!} \\
&= \frac{\hat{\alpha}\alpha^r}{\alpha - \beta} e^{\alpha^m x} - \frac{\hat{\beta}\beta^r}{\alpha - \beta} e^{\beta^m x} \\
&= \frac{\hat{\alpha}\alpha^r e^{\alpha^m x} - \hat{\beta}\beta^r e^{\beta^m x}}{\alpha - \beta}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} K_{mn+r} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left[\hat{\alpha}\alpha^{mn+r} + \hat{\beta}\beta^{m+r}\right] \frac{x^n}{n!} \\
&= \hat{\alpha}\alpha^r \sum_{n=0}^{\infty} (\alpha^m)^n \frac{x^n}{n!} + \hat{\beta}\beta^r \sum_{n=0}^{\infty} (\beta^m)^n \frac{x^n}{n!} \\
&= \hat{\alpha}\alpha^r e^{\alpha^m x} + \hat{\beta}\beta^r e^{\beta^m x}.
\end{aligned}$$

□

Theorem 2.4. *Let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then it follows that*

$$\sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i Q_{i+r} = Q_{2n+r} \tag{2.5}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i K_{i+r} = K_{2n+r}. \tag{2.6}$$

Proof. If we use the fact that $p\alpha + q = \alpha^2$ and $p\beta + q = \beta^2$ with Binet's formulas, then we get

$$\begin{aligned}
e^{qx} \frac{\alpha^r \hat{\alpha} e^{p\alpha x} - \beta^r \hat{\beta} e^{p\beta x}}{\alpha - \beta} &= \frac{\alpha^r \hat{\alpha} e^{\alpha^2 x} - \beta^r \hat{\beta} e^{\beta^2 x}}{\alpha - \beta} = \frac{\hat{\alpha}\alpha^r}{\alpha - \beta} e^{\alpha^2 x} - \frac{\hat{\beta}\beta^r}{\alpha - \beta} e^{\beta^2 x} \\
&= \frac{\hat{\alpha}\alpha^r}{\alpha - \beta} \sum_{n=0}^{\infty} \left(\alpha^2\right)^n \frac{x^n}{n!} - \frac{\hat{\beta}\beta^r}{\alpha - \beta} \sum_{n=0}^{\infty} \left(\beta^2\right)^n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\hat{\alpha}\alpha^{2n+r} - \hat{\beta}\beta^{2n+r}}{\alpha - \beta} \frac{x^n}{n!} = \sum_{n=0}^{\infty} Q_{2n+r} \frac{x^n}{n!}.
\end{aligned} \tag{2.7}$$

Beside this, we have

$$e^{qx} \frac{\alpha^r \hat{\alpha} e^{p\alpha x} - \beta^r \hat{\beta} e^{p\beta x}}{\alpha - \beta} = \left[\sum_{n=0}^{\infty} q^n \frac{x^n}{n!}\right] \left[\sum_{n=0}^{\infty} p^n \frac{\alpha^r \hat{\alpha} \alpha^n - \beta^r \hat{\beta} \beta^n}{\alpha - \beta} \frac{x^n}{n!}\right]$$

$$= \left[\sum_{n=0}^{\infty} q^n \frac{x^n}{n!}\right] \left[\sum_{n=0}^{\infty} p^n Q_{n+r} \frac{x^n}{n!}\right] = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i Q_{i+r}\right] \frac{x^n}{n!} \tag{2.8}$$

by taking $a_n = q^n$ and $b_n = p^n Q_{n+r}$ in (2.1). If we consider the equalities (2.7) and (2.8), then we get

$$\sum_{n=0}^{\infty} Q_{2n+r} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i Q_{i+r}\right] \frac{x^n}{n!},$$

which gives (2.5). On the other hand, calculating

$$\begin{aligned}
e^{qx} \left[\alpha^r \hat{\alpha} e^{p\alpha x} + \beta^r \hat{\beta} e^{p\beta x}\right] &= \alpha^r \hat{\alpha} e^{\alpha^2 x} + \beta^r \hat{\beta} e^{\beta^2 x} \\
&= \hat{\alpha}\alpha^r \sum_{n=0}^{\infty} \left(\alpha^2\right)^n \frac{x^n}{n!} + \hat{\beta}\beta^r \sum_{n=0}^{\infty} \left(\beta^2\right)^n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left[\hat{\alpha}\alpha^{2n+r} + \hat{\beta}\beta^{2n+r}\right] \frac{x^n}{n!} = \sum_{n=0}^{\infty} K_{2n+r} \frac{x^n}{n!}
\end{aligned}$$

and

$$\begin{aligned}
e^{qx} \left[\alpha^r \hat{\alpha} e^{p\alpha x} + \beta^r \hat{\beta} e^{p\beta x}\right] &= \left[\sum_{n=0}^{\infty} q^n \frac{x^n}{n!}\right] \left[\sum_{n=0}^{\infty} p^n \left[\hat{\alpha}\alpha^{n+r} + \hat{\beta}\beta^{n+r}\right] \frac{x^n}{n!}\right] \\
&= \left[\sum_{n=0}^{\infty} q^n \frac{x^n}{n!}\right] \left[\sum_{n=0}^{\infty} p^n K_{n+r} \frac{x^n}{n!}\right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i K_{i+r}\right] \frac{x^n}{n!},
\end{aligned}$$

we achieve (2.6). ∎

Theorem 2.5. *Let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then it follows that*

$$\sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} Q_{2i+r} = p^n Q_{n+r} \tag{2.9}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} K_{2i+r} = p^n K_{n+r}. \tag{2.10}$$

Proof. If we use the facts $\alpha^2 - q = p\alpha$ and $\beta^2 - q = p\beta$ with Binet's formulas, then we get

$$\begin{aligned} e^{-qx} \frac{\alpha^r \hat{\alpha} e^{\alpha^2 x} - \beta^r \hat{\beta} e^{\beta^2 x}}{\alpha - \beta} &= \frac{\hat{\alpha}\alpha^r}{\alpha - \beta} e^{p\alpha x} - \frac{\hat{\beta}\beta^r}{\alpha - \beta} e^{p\beta x} \\ &= \frac{\hat{\alpha}\alpha^r}{\alpha - \beta} \sum_{n=0}^{\infty} p^n \alpha^n \frac{x^n}{n!} - \frac{\hat{\beta}\beta^r}{\alpha - \beta} \sum_{n=0}^{\infty} p^n \beta^n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} p^n \frac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha - \beta} \frac{x^n}{n!} = \sum_{n=0}^{\infty} p^n Q_{n+r} \frac{x^n}{n!}. \end{aligned}$$

Moreover, taking $a_n = (-q)^n$ and $b_n = Q_{2n+r}$ in equality (2.1), we have

$$\begin{aligned} e^{-qx} \frac{\alpha^r \hat{\alpha} e^{\alpha^2 x} - \beta^r \hat{\beta} e^{\beta^2 x}}{\alpha - \beta} &= \left[\sum_{n=0}^{\infty} (-q)^n \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} \frac{\hat{\alpha}\alpha^{2n+r} - \hat{\beta}\beta^{2n+r}}{\alpha - \beta} \frac{x^n}{n!} \right] \\ &= \left[\sum_{n=0}^{\infty} (-q)^n \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} Q_{2n+r} \frac{x^n}{n!} \right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} Q_{2i+r} \right] \frac{x^n}{n!}. \end{aligned}$$

Thus it follows that

$$\sum_{n=0}^{\infty} p^n Q_{n+r} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} Q_{2i+r} \right] \frac{x^n}{n!},$$

which gives (2.9). On the other hand, we obtain

$$\begin{aligned} e^{-qx} \left[\alpha^r \hat{\alpha} e^{\alpha^2 x} + \beta^r \hat{\beta} e^{\beta^2 x} \right] &= \alpha^r \hat{\alpha} e^{p\alpha x} + \beta^r \hat{\beta} e^{p\beta x} \\ &= \hat{\alpha}\alpha^r \sum_{n=0}^{\infty} p^n \alpha^n \frac{x^n}{n!} + \hat{\beta}\beta^r \sum_{n=0}^{\infty} p^n \beta^n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} p^n \left[\hat{\alpha}\alpha^{n+r} + \hat{\beta}\beta^{n+r} \right] \frac{x^n}{n!} = \sum_{n=0}^{\infty} p^n K_{n+r} \frac{x^n}{n!} \end{aligned}$$

and considering $a_n = (-q)^n$ and $b_n = K_{2n+r}$ in equality (2.1), it is seen that

$$\begin{aligned} e^{-qx} \left[\alpha^r \hat{\alpha} e^{\alpha^2 x} + \beta^r \hat{\beta} e^{\beta^2 x} \right] &= \left[\sum_{n=0}^{\infty} (-q)^n \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} \left(\hat{\alpha}\alpha^{2n+r} + \hat{\beta}\beta^{2n+r} \right) \frac{x^n}{n!} \right] \\ &= \left[\sum_{n=0}^{\infty} (-q)^n \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} K_{2n+r} \frac{x^n}{n!} \right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} K_{2i+r} \right] \frac{x^n}{n!}. \end{aligned}$$

Thus it follows that

$$\sum_{n=0}^{\infty} p^n K_{n+r} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (-q)^{n-i} K_{2i+r} \right] \frac{x^n}{n!},$$

which gives the equality (2.10). □

Theorem 2.6. *Let $n \in \mathbb{N}$. We have*

$$\sum_{i=0}^{n} \binom{n}{i} V_{n-i} Q_i = \sum_{i=0}^{n} \binom{n}{i} Q_{n-i} V_i = 2^n Q_n + p^n Q_0 \tag{2.11}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} U_{n-i} K_i = \sum_{i=0}^{n} \binom{n}{i} K_{n-i} U_i = 2^n Q_n - p^n Q_0. \tag{2.12}$$

Proof. Using the fact that $\alpha + \beta = p$ with Binet's formulas, it follows that

$$\begin{aligned} \left(\frac{\hat{\alpha} e^{\alpha x} - \hat{\beta} e^{\beta x}}{\alpha - \beta} \right) \left(e^{\alpha x} + e^{\beta x} \right) &= \frac{\hat{\alpha} e^{2\alpha x} - \hat{\beta} e^{2\beta x}}{\alpha - \beta} + \frac{\hat{\alpha} e^{p x} - \hat{\beta} e^{p x}}{\alpha - \beta} \\ &= \sum_{n=0}^{\infty} 2^n \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} \frac{x^n}{n!} + \sum_{n=0}^{\infty} p^n \frac{\hat{\alpha} - \hat{\beta}}{\alpha - \beta} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[2^n Q_n + p^n Q_0 \right] \frac{x^n}{n!}. \end{aligned}$$

Moreover taking $a_n = Q_n$ and $b_n = V_n$ in equality (2.1), we get

$$\begin{aligned} \left(\frac{\hat{\alpha} e^{\alpha x} - \hat{\beta} e^{\beta x}}{\alpha - \beta} \right) \left(e^{\alpha x} + e^{\beta x} \right) &= \left[\sum_{n=0}^{\infty} Q_n \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} V_n \frac{x^n}{n!} \right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} V_{n-i} Q_i \right] \frac{x^n}{n!}. \end{aligned}$$

Thus equality (2.11) follows from

$$\sum_{n=0}^{\infty} \left[2^n Q_n + p^n Q_0 \right] \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} V_{n-i} Q_i \right] \frac{x^n}{n!}.$$

Similarly, we have

$$\begin{aligned} \left(\hat{\alpha} e^{\alpha x} + \hat{\beta} e^{\beta x} \right) \left(\frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} \right) &= \frac{\hat{\alpha} e^{2\alpha x} - \hat{\beta} e^{2\beta x}}{\alpha - \beta} - \frac{e^{p x} \left(\hat{\alpha} - \hat{\beta} \right)}{\alpha - \beta} \\ &= \sum_{n=0}^{\infty} 2^n \left[\frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha\beta} \right] \frac{x^n}{n!} - \frac{\hat{\alpha} - \hat{\beta}}{\alpha - \beta} \sum_{n=0}^{\infty} p^n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[2^n Q_n - p^n Q_0 \right] \frac{x^n}{n!}. \end{aligned}$$

If we take $a_n = K_n$ and $b_n = U_n$ in equality (2.1), then we have

$$\begin{aligned} \left(\hat{\alpha} e^{\alpha x} + \hat{\beta} e^{\beta x} \right) \left(\frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} \right) &= \left[\sum_{n=0}^{\infty} K_n \frac{x^n}{n!} \right] \left[\sum_{n=0}^{\infty} U_n \frac{x^n}{n!} \right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} U_{n-i} K_i \right] \frac{x^n}{n!}. \end{aligned}$$

Then it follows that

$$\sum_{n=0}^{\infty}[2^nQ_n - p^nQ_0] = \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}\binom{n}{i}U_{n-i}K_i\right]\frac{x^n}{n!},$$

which gives the equality (2.12). □

Theorem 2.7. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{i=0}^{n}\binom{n}{i}U_{n-i}Q_i = \sum_{i=0}^{n}\binom{n}{i}Q_{n-i}U_i = \frac{2^nK_n - p^nK_0}{\Delta}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}V_{n-i}K_i = \sum_{i=0}^{n}\binom{n}{i}K_{n-i}V_i = 2^nK_n + p^nK_0.$$

Proof. Using the fact that $\alpha + \beta = p$ with Binet's formulas, it follows that

$$\begin{aligned}
\left[\frac{\hat{\alpha}e^{\alpha x} - \hat{\beta}e^{\beta x}}{\alpha - \beta}\right]\left[\frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}\right] &= \frac{\hat{\alpha}e^{2\alpha x} + \hat{\beta}e^{2\beta x}}{(\alpha-\beta)^2} - \frac{\hat{\alpha} + \hat{\beta}}{(\alpha-\beta)^2}e^{px} \\
&= \sum_{n=0}^{\infty}\left[\frac{2^nK_n}{\Delta}\right]\frac{x^n}{n!} - \sum_{n=0}^{\infty}\left[\frac{p^nK_0}{\Delta}\right]\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\left[\frac{2^nK_n - p^nK_0}{\Delta}\right]\frac{x^n}{n!}.
\end{aligned}$$

Also taking $a_n = Q_n$ and $b_n = U_n$ in equality (2.1), then we have

$$\left[\frac{\hat{\alpha}e^{\alpha x} - \hat{\beta}e^{\beta x}}{\alpha - \beta}\right]\left[\frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}\right] = \left[\sum_{n=0}^{\infty}Q_n\frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty}U_n\frac{x^n}{n!}\right] = \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}\binom{n}{i}U_{n-i}Q_i\right]\frac{x^n}{n!}.$$

Thus it follows that

$$\sum_{n=0}^{\infty}\left[\frac{2^nK_n - p^nK_0}{\Delta}\right]\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}\binom{n}{i}U_{n-i}Q_i\right]\frac{x^n}{n!}.$$

On the other hand,

$$\begin{aligned}
\left[\hat{\alpha}e^{\alpha x} + \hat{\beta}e^{\beta x}\right]\left[e^{\alpha x} + e^{\beta x}\right] &= \hat{\alpha}e^{2\alpha x} + \hat{\beta}e^{2\beta x} + \hat{\alpha}e^{px} + \hat{\beta}e^{px} \\
&= \sum_{n=0}^{\infty}2^n\ K_n\frac{x^n}{n!} + \sum_{n=0}^{\infty}p^n\left[\hat{\alpha} + \hat{\beta}\right]\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}[2^nK_n + p^nK_0]\frac{x^n}{n!}
\end{aligned}$$

and taking $a_n = K_n$ and $b_n = V_n$ in equality (2.1), we obtain

$$\left[\hat{\alpha}e^{\alpha x} + \hat{\beta}e^{\beta x}\right]\left[e^{\alpha x} + e^{\beta x}\right] = \left[\sum_{n=0}^{\infty}K_n\frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty}V_n\frac{x^n}{n!}\right] = \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}\binom{n}{i}V_{n-i}K_i\right]\frac{x^n}{n!}.$$

Thus it follows that

$$\sum_{n=0}^{\infty}[2^nK_n + p^nK_0]\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}\binom{n}{i}V_{n-i}K_i\right]\frac{x^n}{n!}$$

□

Theorem 2.8. *Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then we have*

$$\sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i Q_i = Q_{mn}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i K_i = K_{mn}.$$

Proof. Considering the facts $U_m\alpha + qU_{m-1} = \alpha^m$ and $U_m\beta + qU_{m-1} = \beta^m$ with Binet's formulas, we get

$$e^{qU_{m-1}x} \frac{\hat{\alpha}e^{U_m\alpha x} - \hat{\beta}e^{U_m\beta x}}{\alpha - \beta} = \frac{\hat{\alpha}e^{(U_m\alpha + qU_{m-1})x} - \hat{\beta}e^{(U_m\beta + qU_{m-1})x}}{\alpha - \beta}$$

$$= \frac{\hat{\alpha}e^{\alpha^m x} - \hat{\beta}e^{\beta^m x}}{\alpha - \beta} = \frac{\hat{\alpha}}{\alpha - \beta}\sum_{n=0}^{\infty} (\alpha^m)^n \frac{x^n}{n!} - \frac{\hat{\beta}}{\alpha - \beta}\sum_{n=0}^{\infty} (\beta^m)^n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{\hat{\alpha}\alpha^{mn} - \hat{\beta}\beta^{mn}}{\alpha - \beta}\right] \frac{x^n}{n!} = \sum_{n=0}^{\infty} Q_{mn} \frac{x^n}{n!}$$

and taking $a_n = (qU_{m-1})^n$ and $b_n = U_m^n Q_n$ in equality (2.1), we have

$$\begin{aligned} e^{qU_{m-1}x} \frac{\hat{\alpha}e^{U_m\alpha x} - \hat{\beta}e^{U_m\beta x}}{\alpha - \beta} &= \left[\sum_{n=0}^{\infty} (qU_{m-1})^n \frac{x^n}{n!}\right] \left[\sum_{n=0}^{\infty} U_m^n Q_n \frac{x^n}{n!}\right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i Q_i\right] \frac{x^n}{n!}. \end{aligned}$$

Thus it follows that

$$\sum_{n=0}^{\infty} Q_{mn} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i Q_i\right] \frac{x^n}{n!},$$

which gives the formula

$$\sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i Q_i = Q_{mn}.$$

On the other hand, we have

$$\begin{aligned} e^{qU_{m-1}x} \left[\hat{\alpha}e^{U_m\alpha x} + \hat{\beta}e^{U_m\beta x}\right] &= \left[\hat{\alpha}e^{(U_m\alpha + qU_{m-1})x} + \hat{\beta}e^{(U_m\beta + qU_{m-1})x}\right] \\ &= \left[\hat{\alpha}e^{\alpha^m x} + \hat{\beta}e^{\beta^m x}\right] \\ &= \sum_{n=0}^{\infty} \hat{\alpha} (\alpha^m)^n \frac{x^n}{n!} + \sum_{n=0}^{\infty} \hat{\beta} (\beta^m)^n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\hat{\alpha}\alpha^{mn} + \hat{\beta}\beta^{mn}\right] \frac{x^n}{n!} = \sum_{n=0}^{\infty} K_{mn} \frac{x^n}{n!}, \end{aligned}$$

and taking $a_n = (qU_{m-1})^n$ and $b_n = U_m^n K_n$ in equality (2.1), we get

$$e^{qU_{m-1}x} \left[\hat{\alpha}e^{U_m\alpha x} + \hat{\beta}e^{U_m\beta x}\right] = \left[\sum_{n=0}^{\infty} (qU_{m-1})^n \frac{x^n}{n!}\right] \left[\sum_{n=0}^{\infty} U_m^n K_n \frac{x^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{n} \binom{n}{i} (qU_{m-1})^{n-i} U_m^i K_i\right] \frac{x^n}{n!}.$$

Then it follows that

$$\sum_{n=0}^{\infty} K_{mn}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n}\binom{n}{i}(qU_{m-1})^{n-i}U_m^i K_i\right]\frac{x^n}{n!},$$

which gives the formula

$$\sum_{i=0}^{n}\binom{n}{i}(qU_{m-1})^{n-i}U_m^i K_i = K_{mn}.$$

□

Furthermore, starting with the expressions

$$e^{qU_{m-1}x}\frac{\hat{\alpha}\alpha^r e^{U_m\alpha x} - \hat{\beta}\beta^r e^{U_m\beta x}}{\alpha-\beta}$$

and

$$e^{qU_{m-1}x}\left(\hat{\alpha}\alpha^r e^{U_m\alpha x} + \hat{\beta}\beta^r e^{U_m\beta x}\right)$$

we obtain the most general formulas

$$\sum_{i=0}^{n}\binom{n}{i}(qU_{m-1})^{n-i}U_m^i Q_{i+r} = Q_{mn+r}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(qU_{m-1})^{n-i}U_m^i K_{i+r} = K_{mn+r},$$

respectively.

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Received: April 2, 2019.
Accepted: August 9, 2019.