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Recurrence Relation and Integral Representation of p - k Mittag-Leffler Function

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Abstract In this paper we evaluate second order differential recurrence relation and four different integral representation of p-k Mittag-Leffler function defined by [7]. Also point out some special cases.

1 Introduction

The different Mittag-Liffler function has been given by different authors in last century are, the Mittag-Leffler function $E_\alpha(z)$ introduced by Gosta Mittag-Leffler [5], in 1903, defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{1.1}$$

Here $z \in C, \alpha \geqslant 0$.
Wiman [3], generalized $E_\alpha(z)$ in 1905 and gave $E_{\alpha,\beta}(z)$, known as Wiman function, defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. \tag{1.2}$$

Here $z, \alpha, \beta \in C; Re(\alpha) > 0, Re(\beta) > 0$.
Prabhakar [11], in 1971, gave next generalization of Mittag-Leffler function and denoted as $E^{\gamma}_{\alpha,\beta}(z)$ and defined as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}. \tag{1.3}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$.
Shukla and Prajapati [2], in 2007, gave second generalization of Mittag-Leffler function and denoted it as $E^{\gamma,q}_{\alpha,\beta}(z)$ and defined as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}. \tag{1.4}$$

Here $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$.
The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all z if $q < Re(\alpha) + 1$ and for $|z| < 1$ if $q = Re(\alpha) + 1$. It is entire function of order $\frac{1}{Re(\alpha)}$.

K.S.Gehlot [8], introduce Generalized k- Mittag-Leffler function in 2012, denoted as $GE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k \in R^+; z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$, as,

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\, z^n}{\mathit{\Gamma}_k(n\alpha + \beta)(n!)}, \tag{1.5}$$

where$(\gamma)_{nq,k}$ is the k- pochhammer symbol and $\Gamma_k(x)$ is the k-gamma function given by [10]. The generalized Pochhammer symbol is given as,

$$(\gamma)_{nq} = \frac{\Gamma(\gamma + nq)}{\Gamma(\gamma)} = q^{qn} \prod_{r=1}^{q} (\frac{\gamma + r - 1}{q})_n, \; if \; q \in N. \tag{1.6}$$

Recentely K.S.Gehlot [7], introduce p-k Mittag-Leffler function,
Let $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$.
The p - k Mittag-Leffler function denoted by ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined as

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha + \beta)} \frac{z^n}{n!}. \tag{1.7}$$

Where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by [6].
The two parameter pochhammer symbol is recently introduce by [6], equation (2.1), the p - k Pochhammer Symbol, ${}_p(x)_{n,k}$ is given by,

$${}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k} + p)(\frac{xp}{k} + 2p).........(\frac{xp}{k} + (n-1)p). \tag{1.8}$$

Where $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$.
Two Parameter Gamma Function is given by [6], equation (2.6), (2.7) and (2.14), the p - k Gamma Function ${}_p\Gamma_k(x)$ is given by,

$${}_p\Gamma_k(x) = \frac{1}{k} \lim_{n \to \infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \tag{1.9}$$

or

$${}_p\Gamma_k(x) = \frac{1}{k} \lim_{n \to \infty} \frac{n! p^{n+1} (np)^{\frac{x}{k} - 1}}{{}_p(x)_{n,k}}. \tag{1.10}$$

Where $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$.
The integral representation of p - k Gamma Function is given by

$${}_p\Gamma_k(x) = \int_0^{\infty} e^{-\frac{t^k}{p}} t^{x-1} dt, \; (k, p \in R^+ - \{0\}; Re(x) > 0). \tag{1.11}$$

Main Results

This section contains two subsection, first subsection contains recurrence relations of p-k Mittag-Leffler function and second subsection we evaluate integral representations of p-k Mittag-Leffler function.

2 Recurrence Relations of ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$.

In this section we evaluate the second order differential recurrence relation of p-k Mittag-Leffler function.

Theorem 2.1. *For* $k, p \in R^+ - \{0\}$; $\alpha + r, \beta + s + k, \gamma \in C/kZ^-$; $R(\alpha + r) > 0, R(\beta + s + k) > 0, R(\gamma) > 0, q \in (0,1) \cup N$, *we get*

$${}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z) - p\; {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z) = \frac{p^2}{k^2}[(\alpha + r)^2 z^2\, {}_p\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)$$
$$+[(\alpha + r)^2 + (\alpha + r)(2\beta + 2s + 2k)]z\, {}_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)$$
$$+(\beta + s)(\beta + s + 2k)\, {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)], \tag{2.1}$$

where ${}_p\dot{E}^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{d}{dz}\, {}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ *and* ${}_p\ddot{E}^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{d^2}{dz^2}\, {}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$.

Proof. The p-k Mittag-Leffler function, from equation (1.7),

$$ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+k)(n!)}, $$

using ([6], Equation 2.22), we have

$$ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z) = \sum_{n=0}^{\infty} \frac{k}{p} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s)\}(n!)}, \tag{2.2} $$

and

$$ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+2k)(n!)}, \tag{2.3} $$

using ([6], Equation 2.22), we have

$$ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)(n!)} $$

$$ \times \frac{k^2}{p^2} \frac{1}{\{n(\alpha+r)+\beta+s)\}\{n(\alpha+r)+\beta+s+k)\}}, $$

$$ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z) = \sum_{n=0}^{\infty} \frac{k}{p^2}\left[\frac{1}{(n(\alpha+r)+\beta+s)} - \frac{1}{(n(\alpha+r)+\beta+s+k)}\right] $$

$$ \times \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)(n!)}, $$

$$ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z) = \frac{k}{p^2}\left[\frac{p}{k}\, {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z) - S\right], $$

$$ S = \frac{p}{k}\, {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(z) - \frac{p^2}{k}\, {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(z), \tag{2.4} $$

where

$$ S = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k)\}(n!)}, \tag{2.5} $$

applying the simple identity $\frac{1}{u} = \frac{k}{u(u+k)} + \frac{1}{u+k}$; for $u = n(\alpha+r)+\beta+s+k$ to (2.5),we obtain,

$$ S = \sum_{n=0}^{\infty} \frac{k\, {}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)(n!)} \times \frac{1}{\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}} $$

$$ + \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+2k)\}(n!)}, $$

$$ S = \sum_{n=0}^{\infty} \frac{k\{n(\alpha+r)+\beta+s\}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)(n!)} $$

$$ \times \frac{1}{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}} $$

$$ + \sum_{n=0}^{\infty} \frac{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)(n!)} $$

$$ \times \frac{1}{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}\{n(\alpha+r)+\beta+s+2k\}}, $$

using ([6], Equation 2.22), we obtain

$$S = \sum_{n=0}^{\infty} \frac{k\{n(\alpha+r)+\beta+s\}_p(\gamma)_{nq,k}\, z^n}{\frac{k^3}{p^3}\ {}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)}$$

$$+\sum_{n=0}^{\infty} \frac{\{n(\alpha+r)+\beta+s\}\{n(\alpha+r)+\beta+s+k\}_p(\gamma)_{nq,k}\, z^n}{\frac{k^3}{p^3}\ {}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)},$$

$$S\frac{k^3}{p^3} = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+p)+\beta+s+3k)(n!)}$$

$$\times[n^2(\alpha+r)^2+2n(\alpha+r)(\beta+s+k)+(\beta+s)(\beta+s+2k)]. \tag{2.6}$$

We now express each summation in the right hand side of (2.6) as follows:

$$\frac{d}{dz}[z\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)] = \sum_{n=0}^{\infty} \frac{(n+1)\ {}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)},$$

$$z\ {}_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) +\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) = \sum_{n=0}^{\infty} \frac{(n+1)\ {}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)},$$

$$z\ {}_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) = \sum_{n=0}^{\infty} \frac{n\ {}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)}. \tag{2.7}$$

Again

$$\frac{d^2}{dz^2}[z^2\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)] = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)\ {}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)}, \tag{2.8}$$

and

$$\frac{d^2}{dz^2}[z^2\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)]$$

$$= z^2\ {}_p\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + 4z\ {}_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + 2\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z), \tag{2.9}$$

from equation (2.8) and (2.9) we have

$$\sum_{n=0}^{\infty} \frac{\{n^2\}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)} = z^2\ {}_p\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z)$$

$$+4z\ {}_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) - 3\sum_{n=0}^{\infty} \frac{\{n\}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)},$$

using equation (2.7), we have

$$\sum_{n=0}^{\infty} \frac{\{n^2\}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n(\alpha+r)+\beta+s+3k)(n!)}$$

$$= z^2\ {}_p\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + z\ {}_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z), \tag{2.10}$$

applying equation (2.4), (2.7) and (2.10) to (2.6), we get

$$\frac{k^3}{p^3}S = (\alpha+r)^2z^2\ {}_p\ddot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + [(\alpha+r)^2+(\alpha+r)(2\beta+2s+2k)]z$$

$$\times\ {}_p\dot{E}^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z) + (\beta+s)(\beta+s+2k)\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+3k}(z),$$

Hence.

Theorem 2.2. *For* $r \in N; k, p \in R^+ - \{0\};\ \alpha, \beta, \gamma \in C/kZ^-;\ R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0; q \in (0,1) \cup N$, *we get*

$$ {}_p(\gamma)_{qr,k} \times {}_pE^{\gamma+qrk,q}_{k,\alpha,\beta+\alpha r}(z) = \frac{d^r}{dz^r}[{}_pE^{\gamma,q}_{k,\alpha,\beta}(z) - \sum_{n=0}^{r-1} \frac{{}_p(\gamma)_{qn,k}\ z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)}]. \quad (2.11) $$

Proof. Consider the right hand side,

$$ A \equiv \frac{d^r}{dz^r}[{}_pE^{\gamma,q}_{k,\alpha,\beta}(z) - \sum_{n=0}^{r-1} \frac{{}_p(\gamma)_{qn,k}\ z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)}], $$

$$ A \equiv \frac{d^r}{dz^r}[\sum_{n=r}^{\infty} \frac{{}_p(\gamma)_{qn,k}\ z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)}], $$

$$ A \equiv \frac{d^r}{dz^r}[\sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+r)q,k}\ z^{n+r}}{{}_p\Gamma_k((n+r)\alpha+\beta)(n+r)!}], $$

using ([6],equation (2.34)), we have

$$ A \equiv \frac{d^r}{dz^r}[\sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{rq,k}\ {}_p(\gamma+qrk)_{nq,k}\ z^{n+r}}{{}_p\Gamma_k(n\alpha+\beta+\alpha r)(n+r)!}] $$

$$ A \equiv {}_p(\gamma)_{qr,k} \times E^{\gamma+qrk,q}_{k,\alpha,\beta+\alpha r}(z). $$

Hence.

3 Integral Representation of ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$.

In this section we evaluate four different integral representation of p-k Mittag-Liffler function.

Theorem 3.1. *For* $k, p \in R^+ - \{0\}; \alpha+r, \beta+s+k, \gamma \in C/kZ^-;\ R(\alpha+r) > 0, R(\beta+s+k) > 0, R(\gamma) > 0, q \in (0,1) \cup N$, *we get*

$$ \int_0^1 t^{\beta+s+k-1}\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s}(t^{\alpha+r})dt = \frac{p}{k}\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(1) - \frac{p^2}{k}\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(1). \quad (3.1) $$

Proof. Put $z = 1$ in equation (2.4) and (2.5), we have

$$ S = \frac{p}{k}\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+k}(1) - \frac{p^2}{k}\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s+2k}(1) $$

$$ = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k)\}(n!)}, \quad (3.2) $$

now consider the integral,

$$ A \equiv \int_0^1 t^{\beta+s+k-1}\ {}_pE^{\gamma,q}_{k,\alpha+r,\beta+s}(t^{\alpha+r})dt, $$

using the equation (1.7), we have

$$ A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)(n!)} \int_0^1 t^{n(\alpha+r)+\beta+s+k-1}dt, $$

$$ A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n(\alpha+r)+\beta+s)\{n(\alpha+r)+\beta+s+k)\}(n!)}, $$

from equation (3.2), we have the desired result.

Theorem 3.2. *For* $k, p \in R^{+} - \{0\}; \alpha, \beta, \gamma, \delta \in C; R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0, R(\delta) > 0$ *and* $q \in (0,1) \cup N$, *then*

$$p^{\delta}\ {}_pE^{\gamma,q}_{k,\alpha,\beta+\delta k}(z) = \frac{1}{\Gamma(\delta)} \int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\delta-1}\ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z\, u^{\frac{\alpha}{k}})du. \tag{3.3}$$

Proof. Consider the right side integral and using equation (1.7), we have

$$A \equiv \frac{1}{\Gamma(\delta)} \int_0^1 u^{\frac{\beta}{k}-1}(1-u)^{\delta-1}\ {}_pE^{\gamma,q}_{k,\alpha,\beta}(z\, u^{\frac{\alpha}{k}})du,$$

$$A \equiv \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)} \int_0^1 u^{\frac{\alpha n+\beta}{k}-1}(1-u)^{\delta-1}du,$$

using the definition of Beta function, we have

$$A \equiv \frac{1}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)} \frac{\Gamma(\frac{\alpha n+\beta}{k})\Gamma(\delta)}{\Gamma(\frac{\alpha n+\beta}{k}+\delta)},$$

applying ([6],equation (2.19)), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{p^{\delta}\ {}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta+\delta k)(n!)} = p^{\delta}\ {}_pE^{\gamma,q}_{k,\alpha,\beta+\delta k}(z),$$

Hence.

Theorem 3.3. *For* $k, p \in R^{+} - \{0\}; \beta, \gamma \in C; R(\beta) > 0, R(\gamma) > 0$ *and* $\alpha, q \in N$, *then*

$${}_pE^{\gamma,q}_{k,k\alpha,\beta}(z) = \frac{1}{{}_p\Gamma_k(\beta)} \prod_{i=1}^{q}\prod_{j=1}^{\alpha} \frac{\Gamma(b_j)}{\Gamma(a_i)\Gamma(b_j-a_i)} \int_0^1 u^{a_i-1}(1-u)^{b_j-a_i-1} e^{(\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})uz}du. \tag{3.4}$$

Where $a_i = \dfrac{\frac{\gamma}{k}+i-1}{q}$ *and* $b_j = \dfrac{\frac{\beta}{k}+j-1}{\alpha}$.

Proof. Using definition of p-k Mittag- Leffler function, from equation (1.7),

$$A \equiv\ {}_pE^{\gamma,q}_{k,k\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(nk\alpha+\beta)(n!)},$$

using relation ([6], equation 2.22),we have

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{(\beta)_{n\alpha,k}\ {}_p\Gamma_k(\beta)(n!)} = \sum_{n=0}^{\infty} D \frac{z^n}{{}_p\Gamma_k(\beta)(n!)}. \tag{3.5}$$

Where $D \equiv \dfrac{{}_p(\gamma)_{nq,k}}{{}_p(\beta)_{n\alpha,k}}$,
using (equation 2.20, [6]), we have

$$D \equiv \frac{{}_p(\gamma)_{nq,k}}{{}_p(\beta)_{n\alpha,k}} = \frac{p^{qn}(\frac{\gamma}{k})_{qn}}{p^{\alpha n}(\frac{\beta}{k})_{\alpha n}},$$

using the relation given by equation (1.6), we have

$$D \equiv \frac{p^{(q-\alpha)n} q^{qn} \prod_{i=1}^{q}(\frac{\frac{\gamma}{k}+i-1}{q})_n}{\alpha^{\alpha n} \prod_{j=1}^{\alpha}(\frac{\frac{\beta}{k}+j-1}{\alpha})_n},$$

$$let\ a_i = \frac{\frac{\gamma}{k}+i-1}{q}\ and\ b_j = \frac{\frac{\beta}{k}+j-1}{\alpha},$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{j=1}^{\alpha} \frac{(a_i)_n}{(b_j)_n},$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{j=1}^{\alpha} \frac{\Gamma(a_i+n)\Gamma(b_j)}{\Gamma(b_j+n)\Gamma(a_i)},$$

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{j=1}^{\alpha} \frac{\Gamma(b_j)}{\Gamma(b_j-a_i)\Gamma(a_i)} \frac{\Gamma(a_i+n)\Gamma(b_j-a_i)}{\Gamma(b_j-a_i+a_i+n)},$$

using the definition of Beta function, we have

$$D \equiv (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n \prod_{i=1}^{q}\prod_{j=1}^{\alpha} \frac{\Gamma(b_j)}{\Gamma(b_j-a_i)\Gamma(a_i)} \int_0^1 u^{a_i+n-1}(1-u)^{b_j-a_i-1}du, \tag{3.6}$$

from equation (3.5) and (3.6), we have

$$A \equiv \frac{1}{{}_p\Gamma_k(\beta)} \prod_{i=1}^{q}\prod_{j=1}^{\alpha} \frac{\Gamma(b_j)}{\Gamma(b_j-a_i)\Gamma(a_i)} \int_0^1 u^{a_i-1}(1-u)^{b_j-a_i-1} \sum_{n=0}^{\infty} \frac{(uz)^n}{n!} (\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})^n du,$$

$$A \equiv \frac{1}{{}_p\Gamma_k(\beta)} \prod_{i=1}^{q}\prod_{j=1}^{\alpha} \frac{\Gamma(b_j)}{\Gamma(a_i)\Gamma(b_j-a_i)} \int_0^1 u^{a_i-1}(1-u)^{b_j-a_i-1} e^{(\frac{p^{(q-\alpha)}q^q}{\alpha^\alpha})uz} du.$$

Hence.

Theorem 3.4. *For* $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C; R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0$ *and* $q \in (0,1) \cup N$, *then*

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{1}{\Gamma(\frac{\gamma}{k})} \int_0^\infty e^{-t}\, t^{(\frac{\gamma}{k}-1)}\, {}_pE^{1,0}_{k,\alpha,\beta}(zt^q p^q)dt. \tag{3.7}$$

Proof. Using definition of p-k Mittag-Leffler function, equation (1.7), we have

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)},$$

using ([6], equation (2.20)), we have

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)} \frac{p^{qn}\Gamma(\frac{\gamma}{k}+qn)}{\Gamma(\frac{\gamma}{k})},$$

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)} \frac{p^{qn}}{\Gamma(\frac{\gamma}{k})} \int_0^\infty e^{-t}\, t^{(\frac{\gamma}{k}+qn-1)}dt,$$

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{1}{\Gamma(\frac{\gamma}{k})} \int_0^\infty e^{-t}\, t^{(\frac{\gamma}{k}-1)} \sum_{n=0}^{\infty} \frac{z^n\, p^{qn}\, t^{qn}}{{}_p\Gamma_k(n\alpha+\beta)(n!)}dt,$$

$${}_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \frac{1}{\Gamma(\frac{\gamma}{k})} \int_0^\infty e^{-t}\, t^{(\frac{\gamma}{k}-1)}\, {}_pE^{1,0}_{k,\alpha,\beta}(zt^q k^q)dt.$$

Hence.

4 Particular cases

For some particular values of the parameters p, q and k, we can obtain certain known results for different Mittag-Leffler functions,

[A]

By putting $p = k$,
[i] in equation (2.1), then it reduces to the result earlier given by ([9], Theorem 2.1, page 178).
[ii] in equation (2.11), then it reduces to the result earlier given by ([9], Theorem 2.2, page 181).
[iii] in equation (3.1), then it reduces to the result earlier given by ([9], Theorem 3.1, page 182).
[iv] in equation (3.3), then it reduces to the result earlier given by ([9], Theorem 3.2, page 182).
[v] in equation (3.4), then it reduces to the result earlier given by ([9], Theorem 3.3, page 183).
[vi] in equation (3.7), then it reduces to the result earlier given by ([9], Theorem 3.4, page 185).

[B]

By putting $p = k$ and $q = 1$,
[i] in equation (2.1), then we get the new recurrence relation for k-Mittag-Leffler function given by [4] .
[ii] in equation (2.11), then we get the new recurrence relation for k-Mittag-Leffler function given by [4].
[iii] in equation (3.1), then we get the new Integral representation for k-Mittag-Leffler function given by [4].
[iv] in equation (3.3), then we get the new Integral representation for k-Mittag-Leffler function given by [4].
[v] in equation (3.4), then we get the new Integral representation for k-Mittag-Leffler function given by [4].
[vi] in equation (3.7), then we get the new Integral representation for k-Mittag-Leffler function given by [4].

[C]

By putting $p = q = k = 1$,
[i] in equation (2.1), then we get the well known result obtained by ([1], equation 2.1, page 134).
[ii] in equation (2.11), then we get the new recurrence relation for Mittag-Leffler function defined by [2].
[iii] in equation (3.1), then we get the well known result obtained by ([1], equation 3.1, page 137).
[iv] in equation (3.3), then we get the well known result obtained by ([2], equation 2.4.1, page 803)..
[v] in equation (3.4), then we get the new Integral representation for Mittag-Leffler function defined by [2].
[vi] in equation (3.7), then we get the new Integral representation for Mittag-Leffler function defined by [2].

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