Palestine Journal of Mathematics

Vol. 10(1)(2021) , 299–307

V-SEMi-SLANT SUBMERSIONS FROM ALMOST PRODUCT RIEMANNIAN MANIFOLDS

Sezin Aykurt Sepet and Hülya Gün Bozok

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53C43; Secondary 53C15.

Keywords and phrases: Riemannian submersion, almost product Riemannian manifold, v-semi-slant submersion.

Abstract In the present paper, we study v-semi-slant submersions from almost product Riemannian manifolds onto Riemannian manifolds. We investigate the integrability of distributions, the geometry of fibers. We also deal with the condition for such maps to be totally geodesic and give some examples.

1 Introduction

In differential geometry, the notion of Riemannian submersion was first studied by O'Neill [19] and Gray [12]. Then Watson defined almost Hermitian submersions between Hermitian manifolds. Also he showed that the base manifold and each fiber have the same kind of structure as the total space in most case [29]. Recently, according to the different conditions on Riemannian submersion, many authors have carried out studies (see [1, 2, 3, 4, 10, 13, 14, 16, 17, 22, 26, 27, 28]). Sahin investigated slant submersions from almost Hermitian manifolds onto Riemannian manifolds [24]. As a generalization of slant submersions, semi invariant submersions, anti-invariant submersions, Park defined semi-slant submersions from Hermitian manifolds onto Riemannian manifolds [21]. Also, in [20], Park studied v-semi-slant submersions from Hermitian manifolds onto Riemannian manifolds and obtained some characterizations. On the other hand it is well-known that Riemannian submersions are related with physics and have their applications in the Yang Mills theory [8], Kaluza Klein theory [9], supergravity and superstring theories [15] etc. Other applications for Riemannian submersions are statistica machine learning process, medical imaging [18], statistical analysis on manifolds [7] and robotic theory [5].

In the paper we study v-semi-slant submersions from almost product Riemannian manifolds onto Riemannian manifolds. We investigate the integrability of distributions and the geometry of fibers. Also we obtain necessary and sufficient conditions for such maps to be totally geodesic and give some examples.

2 Preliminaries

In this section, we will give some notions for almost product Riemannian manifold and v-semi-slant submersion.

2.1 Almost product Riemannian manifolds

Let M be a m-dimensional manifold with a tensor F of a type $(1,1)$ such that

$$F^2 = I,\ (F \neq I) \tag{2.1}$$

Then it is said that M is an almost product manifold with almost product structure F. Here it can be written

$$P = \frac{1}{2}(I+F),\ Q = \frac{1}{2}(I-F).$$

Thus it is obtained following equations

$$P+Q=I,\ P^2=P,\ PQ=QP=0,\ F=P-Q.$$

P and Q define two complementary distributions. It can be easily seen that the eigevalues of F are $+1$ or -1. If there is a Riemannian metric g on almost product manifold M such that

$$g(FX,FY)=g(X,Y) \tag{2.2}$$

for any vector fields X and Y on M, then (M,g,F) is called an almost product Riemannian manifold. An almost product Riemannian manifold (M,g,F) is said to be a locally product Riemannian manifold if it satisfies

$$\nabla_X F=0,\ X\in\Gamma(TM). \tag{2.3}$$

where ∇ is the Levi-Civita connection on M with respect to g [30].

2.2 Riemannian submersions

Let (M,g) and (N,g') be Riemannian manifolds and $\pi:(M,g)\to(N,g')$ a C^∞ map. The map π is called a C^∞-submersion if π is surjective and the differential $(\pi_*)_p$ has maximal rank for any $p\in M$. A C^∞ submersion π is said to be a Riemannian submersion if the differential $(\pi_*)_p$ preserves the lengths of horizontal vectors for each $p\in M$.
For any $q\in N$, $\pi^{-1}(q)$ is a $(m-n)$-dimensional submanifold of M, so-called fiber. If a vector field on M is always tangent (resp. orthogonal) to fibers, then it is called vertical (resp. horizontal) [23]. A vector field X on M is said to be basic if it is horizontal and π-related to a vector field X_* on N, i.e., $\pi_*X_p=X_{*\pi(p)}$ for all $p\in M$. The fundamental tensors of a Riemannian submersion are defined by the following formulas

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F+\mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.4}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F+\mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.5}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of (M,g) [19, 11].

Lemma 2.1. *([11]) Let $\pi:(M,g)\to(N,g')$ be a Riemannian submersion between Riemannian manifolds. If X and Y are basic vector fields of M, then*

- *$g(X,Y)=g'(X_*,Y_*)\circ\pi$,*
- *the horizontal part $[X,Y]^{\mathcal{H}}$ of $[X,Y]$ is a basic vector field and correspond to $[X_*,Y_*]$ i.e., $\pi_*\left([X,Y]^{\mathcal{H}}\right)=[X_*,Y_*]$,*
- *$[V,X]$ is vertical for any vertical vector field V,*
- *$(\nabla_X Y)^{\mathcal{H}}$ is the basic vector field corresponding to $\nabla'_{X_*}Y_*$,*

where ∇ and ∇' are the Levi-Civita connections on M and N, respectively [19].

Also, using (2.4) and (2.5), we have

$$\nabla_V W=\mathcal{T}_V W+\hat{\nabla}_V W \tag{2.6}$$

$$\nabla_V X=\mathcal{H}\nabla_V X+\mathcal{T}_V X \tag{2.7}$$

$$\nabla_X V=\mathcal{A}_X V+\mathcal{V}\nabla_X V \tag{2.8}$$

$$\nabla_X Y=\mathcal{H}\nabla_X Y+\mathcal{A}_X Y \tag{2.9}$$

for $X,Y\in\Gamma((\ker\pi_*)^\perp)$ and $V,W\in\Gamma(\ker\pi_*)$, where $\hat{\nabla}_V W=\mathcal{V}\nabla_V W$. The tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy the equations

$$\mathcal{T}_U W=\mathcal{T}_W U, \tag{2.10}$$

$$\mathcal{A}_X Y=-\mathcal{A}_Y X=\frac{1}{2}\mathcal{V}[X,Y] \tag{2.11}$$

for $U, W \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma\left((\ker \pi_*)^{\perp}\right)$. On the other hand, it can be easily said that a Riemannian submersion $\pi : M \to N$ has totally geodesic fibers if and only if $\mathcal{T}$ identically vanishes. Let (M, g) and (N, g') be Riemannian manifolds and suppose that $\pi : M \to N$ is a smooth mapping between them. Then the second fundamental form of π is given by

$$\nabla \pi_*(X, Y) = \nabla^{\pi}_X \pi_*(Y) - \pi_*(\nabla_X Y) \tag{2.12}$$

for $X, Y \in \Gamma(TM)$, where ∇^{π} is the pullback connection and ∇ the Riemannian connections of the metrics g and g' [6]. For a Riemannian submersion π, we recall that

$$(\nabla \pi_*)(X, Y) = 0 \tag{2.13}$$

where $X, Y \in \Gamma\left((\ker \pi_*)^{\perp}\right)$. A smooth map $\pi : (M, g) \to (N, g')$ is said to be a totally geodesic map if

$$(\nabla \pi_*)(X, Y) = 0 \tag{2.14}$$

for $X, Y \in \Gamma(TM)$.

We call the map π a slant submersion if π is a Riemannian submersion and the angle $\theta = \theta(X)$ between FX and the space $(\ker \pi_*)_p$ is constant for nonzero $X \in (\ker \pi_*)$ and $p \in M$. We call the angle θ a slant angle [13, 24].

Also, a Riemannian submersion π is said to be a semi-slant submersion if there is a distribution $D_1 \subset \Gamma(\ker \pi_*)$ such that

$$\ker \pi_* = D_1 \oplus D_2, \quad F(D_1) = D_1$$

and the angle $\theta = \theta(X)$ between FX and the space $(D_2)_p$ is constant for nonzero $X \in (D_2)_p$ and $p \in M$, where D_2 is the orthogonal complement of D_1 in $\Gamma(\ker \pi_*)$. We call the angle θ a semi-slant angle [21].

Now, by using [20], we define the v-semi-slant submersions from an almost product Riemannian manifold (M, g, F) onto a Riemannian manifold (N, g').

3 V-Semi-Slant Submersions

Definition 3.1. Let π be a Riemannian submersion from an almost product Riemannian manifold (M, g, F) onto a Riemannian manifold (N, g'). π is called a v-semi-slant submersion if there are two orthogonal complementary distributions D_1 and D_2 of horizontal distribution $(\ker \pi_*)^{\perp}$ such that

$$(\ker \pi_*)^{\perp} = D_1 \oplus D_2, \quad F(D_1) = D_1$$

and the angle $\theta(X)$ between FX and the space D_2 is constant for non-zero $X \in (D_2)_p$ at each point $p \in M$. The angle θ is called a v-semi-slant angle.

If $D_2 = (\ker \pi_*)^{\perp}$, then the map F is said a v-slant submersion and the angle θ v-slant angle. Also, if $\theta = \frac{\pi}{2}$, then the map F is called a v-semi-invariant submersion [20]. Let $F : (M, g, F) \to (N, g')$ be a v-semi-slant submersion. Then for $X \in \Gamma(\ker \pi_*)$, we can write

$$FX = \phi X + \omega X \tag{3.1}$$

where $\phi X \in \Gamma(\ker \pi_*)$ and $\omega X \in \Gamma\left((\ker \pi_*)^{\perp}\right)$. For $Z \in \Gamma\left((\ker \pi_*)^{\perp}\right)$ we obtain

$$FZ = BZ + CZ \tag{3.2}$$

where $BZ \in \Gamma(\ker \pi_*)$ and $CZ \in \Gamma\left((\ker \pi_*)^{\perp}\right)$. For $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U$$

where $\mathcal{V}U \in \Gamma(\ker \pi_*)$ and $\mathcal{H}U \in \Gamma\left((\ker \pi_*)^{\perp}\right)$. Then

$$\ker \pi_* = BD_2 \oplus \mu,$$

where μ is the orthogonal complement of BD_2 in $\ker\pi_*$ and is invariant under F. On the other hand we obtain

$$CD_1 = D_1,\ BD_1 = 0,\ CD_2 \subset D_2,\ \phi^2 + Bw = id,$$
$$C^2 + \omega B = id,\ \omega\phi + C\omega = 0,\ BC + \phi B = 0.$$

Lemma 3.2. *Let (M, g, F) be almost product Riemannian manifold and (N, g') a Riemannian manifold. Let $\pi : (M, g, F) \to (N, g')$ be a v-semi-slant submersion. Then for $U, V \in \Gamma(\ker\pi_*)$, we have*

$$\hat{\nabla}_U \phi V + \mathcal{T}_U \omega V = \phi \hat{\nabla}_U V + B\mathcal{T}_U V$$
$$\mathcal{T}_U \phi V + \mathcal{H}\nabla_U \omega V = \omega \hat{\nabla}_U V + C\mathcal{T}_U V.$$

For $X, Y \in \Gamma\left((\ker\pi_*)^\perp\right)$ we get

$$\mathcal{V}\nabla_X BY + \mathcal{A}_X CY = \phi\mathcal{A}_X Y + B\mathcal{H}\nabla_X Y$$
$$\mathcal{A}_X BY + \mathcal{H}\nabla_X CY = \omega\mathcal{A}_X Y + C\mathcal{H}\nabla_X Y.$$

Also, for $U \in \Gamma(\ker\pi_*)$ and $X \in \Gamma\left((\ker\pi_*)^\perp\right)$ we have

$$\hat{\nabla}_U BX + \mathcal{T}_U CX = \phi\mathcal{T}_U X + B\mathcal{H}\nabla_U X$$
$$\mathcal{T}_U BX + \mathcal{H}\nabla_U CX = \omega\mathcal{T}_U X + C\mathcal{H}\nabla_U X.$$

Example 3.3. We define an almost product structure F on R^6 as follows:

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = (x_2, x_1, x_5, x_6, x_3, x_4).$$

Given a map $\pi : \mathbb{R}^6 \to \mathbb{R}^4$ by

$$\pi(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_3\sin\alpha - x_4\cos\alpha, x_2, x_5\cos\beta - x_6\sin\beta).$$

Then we have

$$\ker\pi_* = \left\{V_1 = \cos\alpha\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_4}, V_2 = \sin\beta\frac{\partial}{\partial x_5} + \cos\beta\frac{\partial}{\partial x_6}\right\},$$
$$(\ker\pi_*)^\perp = \{H_1 = \frac{\partial}{\partial x_1}, H_2 = \frac{\partial}{\partial x_2}, H_3 = \sin\alpha\frac{\partial}{\partial x_3} - \cos\alpha\frac{\partial}{\partial x_4},$$
$$H_4 = \cos\beta\frac{\partial}{\partial x_5} - \sin\beta\frac{\partial}{\partial x_6}\}.$$

Thus the map π is a v-semi-slant submersion such that

$$D_1 = \left\{H_1 = \frac{\partial}{\partial x_1}, H_2 = \frac{\partial}{\partial x_2}\right\} \text{ and}$$
$$D_2 = \left\{H_3 = \sin\alpha\frac{\partial}{\partial x_3} - \cos\alpha\frac{\partial}{\partial x_4}, H_4 = \cos\beta\frac{\partial}{\partial x_5} - \sin\beta\frac{\partial}{\partial x_6}\right\}$$

with the v-semi-slant angle $\cos\theta = \sin(\alpha + \beta)$.

Example 3.4. We determine an almost product structure F on R^8 as follows:

$$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_3, x_4, x_1, x_2, x_7, x_8, x_5, x_6).$$

Define a map $\pi : \mathbb{R}^8 \to \mathbb{R}^4$ by

$$\pi(x_1, x_2, ..., x_8) = \left(\frac{x_2 + x_8}{\sqrt{2}}, \cos\alpha x_3 - \sin\alpha x_5, \frac{x_4 + x_6}{\sqrt{2}}, x_1\right).$$

Then we obtain

$$\begin{aligned}\ker\pi_* =&\{V_1 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_8}, V_2 = \sin\alpha\frac{\partial}{\partial x_3} + \cos\alpha\frac{\partial}{\partial x_5},\\ &V_3 = \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_6}, V_4 = \frac{\partial}{\partial x_7}\}\\ (\ker\pi_*)^\perp =&\{H_1 = \frac{\partial}{\partial x_1}, H_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_8}, H_3 = \cos\alpha\frac{\partial}{\partial x_3} - \sin\alpha\frac{\partial}{\partial x_5},\\ &H_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6}\}.\end{aligned}$$

Therefore the map π is a v-semi-slant submersion such that

$$\begin{aligned}D_1 &= \left\{H_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_8}, H_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6}\right\} \text{ and}\\ D_2 &= \left\{H_1 = \frac{\partial}{\partial x_1}, H_3 = \cos\alpha\frac{\partial}{\partial x_3} - \sin\alpha\frac{\partial}{\partial x_5}\right\}\end{aligned}$$

with the v-semi-slant angle $\theta = \alpha$.

Theorem 3.5. *Let π be a Riemannian submersion from an almost product Reimannian manifold (M, g, F) onto a Riemannian manifold (N, g'). Then π is a v-semi-slant submersion if and only if*

$$C^2X = \lambda X$$

for $X \in \Gamma(D_2)$, where $\lambda = \cos^2\theta$ and θ is the v-semi-slant angle of D_2.

Proof. Since $cos\theta = \frac{\|CX\|}{\|FX\|}$, we can write

$$\begin{aligned}\cos\theta &= \frac{g(FX, CX)}{\|CX\|\|FX\|}\\ \Rightarrow \cos^2\theta &= \frac{g(X, C^2X)}{\|X\|^2}\end{aligned}$$

for $X \in \Gamma(D_2)$. Therefore, for $X \in \Gamma(D_2)$, we arrive

$$C^2X = \cos^2\theta X$$

which prove the theorem. □

Theorem 3.6. *Let π be a v-semi-slant submersion from a locally product Reimannian manifold (M, g, F) onto a Riemannian manifold (N, g'). Then the distribution D_1 is integrable if and only if*

$$g(\mathcal{A}_X FY - \mathcal{A}_Y FX, BZ) = g(\mathcal{H}\nabla_Y FX - \mathcal{H}\nabla_X FY, CZ)$$

for $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. Using the equations (2.2), (2.3), (2.9) and (3.2), we get

$$\begin{aligned}g([X, Y], Z) =&g(\nabla_X FY, FZ) - g(\nabla_Y FX, FZ)\\ =&g(\mathcal{A}_X FY, BZ) + g(\mathcal{H}\nabla_X FY, CZ)\\ &- g(\mathcal{A}_Y FX, BZ) - g(\mathcal{H}\nabla_Y FX, CZ)\end{aligned}$$

for $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. Then we arrive

$$\begin{aligned}g([X, Y], Z) =&g(\mathcal{A}_X FY - \mathcal{A}_Y FX, BZ)\\ &+ g(\mathcal{H}\nabla_X FY - \mathcal{H}\nabla_Y FX, CZ).\end{aligned}$$

Thus we have the result. □

Theorem 3.7. *Let π be a v-semi-slant submersion from a locally product Reimannian manifold (M, g, F) onto a Riemannian manifold (N, g'). Then the distribution D_2 is integrable if and only if*

$$g\left(\mathcal{A}_Z BW - \mathcal{A}_W BZ, FX\right) = g\left(\mathcal{A}_W BCZ - \mathcal{A}_Z BCW, X\right)$$

for $Z, W \in \Gamma(D_2)$ and $X \in \Gamma(D_1)$.

Proof. From (2.2), (2.3) and (3.2) we have

$$\begin{aligned} g\left([Z,W], X\right) &= g\left(\nabla_Z FW, FX\right) - g\left(\nabla_W FZ, FX\right) \\ &= g\left(\nabla_Z BW, FX\right) + g\left(\nabla_Z CW, FX\right) \\ &- g\left(\nabla_W BZ, FW\right) - g\left(\nabla_W CZ, FX\right) \end{aligned}$$

for $Z, W \in \Gamma(D_2)$ and $X \in \Gamma(D_1)$. By using (2.8) and Theorem 3.5, we can write

$$\begin{aligned} g\left([Z,W], X\right) =& g\left(\mathcal{A}_Z BW - \mathcal{A}_W BZ, FX\right) + g\left(\nabla_Z BCW - \nabla_W BCZ, X\right) \\ &+ \cos^2\theta g\left(\nabla_Z W - \nabla_W Z, X\right) \\ \Rightarrow \sin^2 g\left([Z,W], X\right) =& g\left(\mathcal{A}_Z BW - \mathcal{A}_W BZ, FX\right) + g\left(\mathcal{A}_Z BCW - \mathcal{A}_W BCZ, X\right). \end{aligned}$$

Therefore, the proof is completed. □

Theorem 3.8. *Let π be a v-semi-slant submersion from a locally product Reimannian manifold (M, g, F) onto a Riemannian manifold (N, g'). Then the distribution D_1 defines a totally geodesic foliation on M if and only if*

$$g\left(\mathcal{A}_X FY, BZ\right) = -g\left(\mathcal{A}_X Y, BCZ\right)$$

and

$$g\left(\mathcal{A}_X FY, \phi U\right) = -g\left(\mathcal{H}\nabla_X FY, \omega U\right)$$

for $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $U \in \Gamma(\ker \pi_*)$.

Proof. By using (2.2), (2.3), (3.2) and Theorem 3.5, we obtain

$$\begin{aligned} g\left(\nabla_X Y, Z\right) &= g\left(\nabla_X FY, BZ\right) + g\left(F\nabla_X Y, CZ\right) \\ &= g\left(\nabla_X FY, BZ\right) + \cos^2\theta g\left(\nabla_X Y, Z\right) + g\left(\nabla_X Y, BCZ\right) \end{aligned}$$

for $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. From (2.9), we get

$$\sin^2\theta g\left(\nabla_X Y, Z\right) = g\left(\mathcal{A}_X FY, BZ\right) + g\left(\mathcal{A}_X Y, BCZ\right).$$

Thus we obtain the first equation.
On the other hand, for $U \in \Gamma(\ker \pi_*)$ we have

$$\begin{aligned} g\left(\nabla_X Y, U\right) =& g\left(\nabla_X FY, FU\right) \\ =& g\left(\mathcal{A}_X FY + \mathcal{H}\nabla_X FY, FU\right). \end{aligned}$$

Then using (3.1) we write

$$g\left(\nabla_X Y, U\right) = g\left(\mathcal{A}_X FY, \phi U\right) + g\left(\mathcal{H}\nabla_X FY, \omega U\right).$$

Thus the proof is completed. □

Theorem 3.9. *Let π be a v-semi-slant submersion from a locally product Reimannian manifold (M, g, F) onto a Riemannian manifold (N, g'). Then the distribution D_2 defines a totally geodesic foliation on M if and only if*

$$g\left(\mathcal{A}_Z BW, FX\right) = -g\left(\mathcal{A}_Z BCW, X\right)$$

and

$$-g\left(\mathcal{A}_Z BW, \omega U\right) = g\left(\mathcal{V}\nabla_Z BW, \phi U\right) + g\left(\mathcal{V}\nabla_Z BCW, U\right)$$

for $Z, W \in \Gamma(D_2)$, $X \in \Gamma(D_1)$ and $U \in \Gamma(\ker \pi_*)$.

Proof. Using the equations (2.2), (2.3) and (3.2), we get

$$g(\nabla_Z W, X) = g(\nabla_Z BW, FX) + g(F\nabla_Z CW, X)$$

for $Z, W \in \Gamma(D_2), X \in \Gamma(D_1)$. From (2.8) and Theorem 3.5, we obtain

$$g(\nabla_Z W, X) = g(\nabla_Z BW, FX) + g(\nabla_Z BCW, X) + \cos^2\theta g(\nabla_Z W, X)$$

or

$$\Rightarrow \sin^2\theta g(\nabla_Z W, X) = g(\mathcal{A}_Z BW, FX) + g(\mathcal{A}_Z BCW, X).$$

Similarly, for $Z, W \in \Gamma(D_2)$ and $U \in \Gamma(\ker \pi_*)$

$$\begin{aligned} g(\nabla_Z W, U) =& g(\nabla_Z FW, FU) \\ =& g(\nabla_Z BW, FU) + g(\nabla_Z BCW, U) + \cos^2\theta g(\nabla_Z W, U). \end{aligned}$$

Thus we arrive

$$\begin{aligned} \sin^2\theta g(\nabla_Z W, U) =& g(\mathcal{A}_Z BW, \omega U) + g(\mathcal{V}\nabla_Z BW, \phi U) \\ &+ g(\mathcal{V}\nabla_Z BCW, U). \end{aligned}$$

Then the proof is completed. □

Theorem 3.10. *Let π be a v-semi-slant submersion from a locally product Reimannian manifold (M, g, F) onto a Riemannian manifold (N, g'). Then π is a totally geodesic map if and only if*

$$\begin{aligned} \omega\left(\mathcal{T}_U \omega V + \hat{\nabla}_U \phi V\right) + C(\mathcal{T}_U \phi V + \mathcal{H}\nabla_U \omega V) = 0 \\ C\mathcal{T}_U BZ + \omega\hat{\nabla}_U BZ + \mathcal{T}_U BCZ + \cos^2\theta \mathcal{H}\nabla_U Z = 0 \\ \omega\mathcal{T}_U FX + C\mathcal{H}\nabla_U FX = 0 \end{aligned}$$

for $U, V \in \Gamma(\ker \pi_*)$, $X \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$.

Proof. Since π is a Riemannian submersion, for $X_1, X_2 \in \Gamma\left((\ker \pi_*)^\perp\right)$

$$(\nabla\pi_*)(X_1, X_2) = 0.$$

From (2.12) we have

$$(\nabla\pi_*)(U, V) = -\pi_*(F\nabla_U FV)$$

for $U, V \in \Gamma(\ker \pi_*)$. By using (2.6) and (2.7) we obtain

$$(\nabla\pi_*)(U, V) = -\pi_*\left(F\left(\mathcal{T}_U \phi V + \hat{\nabla}_U \phi V\right) + F(\mathcal{T}_U \omega V + \mathcal{H}\nabla_U \omega V)\right).$$

Thus, using (3.1) and (3.2), we get

$$(\nabla\pi_*)(U, V) = -\pi_*\left(\omega\left(\mathcal{T}_U \omega V + \hat{\nabla}_U \phi V\right) + C(\mathcal{T}_U \phi V + \mathcal{H}\nabla_U \omega V)\right) = 0.$$

If π is a totally geodesic map, we arrive

$$\omega\left(\mathcal{T}_U \omega V + \hat{\nabla}_U \phi V\right) + C(\mathcal{T}_U \phi V + \mathcal{H}\nabla_U \omega V) = 0.$$

Similarly, we can write

$$(\nabla\pi_*)(U, X) = -\pi_*(F\nabla_U FX)$$

for $U \in \Gamma(\ker \pi_*)$ and $X \in \Gamma(D_1)$. Using (2.7) we get

$$(\nabla\pi_*)(U, X) = -\pi_*(F(\mathcal{T}_U FX + \mathcal{H}\nabla_U FX)).$$

Then we have

$$\omega \mathcal{T}_U FX + C\mathcal{H}\nabla_U FX = 0.$$

Also we obtain

$$(\nabla \pi_*)(U, Z) = -\pi_* (F\nabla_U FZ)$$

for $U \in \Gamma(\ker \pi_*)$ and $Z \in \Gamma(D_2)$. From Theorem 3.5 we get

$$\begin{aligned}(\nabla \pi_*)(U, Z) = &- \pi_* \left(F\left(\mathcal{T}_U BZ + \hat{\nabla}_U BZ\right) + \nabla_U BCZ + \nabla_U C^2 Z\right)\\ = &- \pi_*(B\mathcal{T}_U BZ + C\mathcal{T}_U BZ + \phi\hat{\nabla}_U BZ + \omega\hat{\nabla}_U BZ\\ &+ \mathcal{T}_U BCZ + \hat{\nabla}_U BCZ + \cos^2\theta\,(\mathcal{T}_U Z + \mathcal{H}\nabla_U Z)).\end{aligned}$$

Therefore we have

$$C\mathcal{T}_U BZ + \omega\hat{\nabla}_U BZ + \mathcal{T}_U BCZ + \cos^2\theta\mathcal{H}\nabla_U Z = 0$$

which prove the theorem. □

We recall a fiber of a Riemannian submersion $\pi : (M, g) \to (N, g')$ is called totally umbilical if

$$\mathcal{T}_U V = g(U, V)H$$

for $U \in \Gamma(\ker \pi_*)$, where H is the mean curvature vector field of the fiber.
Then we give the following theorem

Theorem 3.11. *Let π be a v-semi-slant submersion with totally umbilical fibers from a locally product Reimannian manifold (M, g, F) onto a Riemannian manifold (N, g'). Then we obtain*

$$H \perp \Gamma(D_1)$$

Proof. From the equations (2.3) and (2.6) we have

$$T_U FV + \hat{\nabla}_U FV = \nabla_U FV = F\nabla_U V$$

for $U, V \in \Gamma(\mu)$. For $X \in \Gamma(D_1)$, we get

$$g(U, FV)\, g(H, X) = g(U, V)\, g(H, FX)$$

If U and V are replaced, we find

$$g(V, FU)\, g(H, X) = g(V, U)\, g(H, FX)$$

Then we obtain

$$g(U, V)\, g(H, FX) = 0.$$

Therefore the proof is completed. □

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Author information

Sezin Aykurt Sepet, Department of Mathematics, University of Kırşehir Ahi Evran, Kırşehir, Turkey.
E-mail: `sezinaykurt@hotmail.com`

Hülya Gün Bozok, Department of Mathematics, University of Osmaniye Korkut Ata, Osmaniye, Turkey.
E-mail: `hulyagun@osmaniye.edu.tr`

Received: April 6, 2019.
Accepted: May 29, 2019.