Palestine Journal of Mathematics

Vol. 10(1)(2021) , 312–321

$h(x)-$ JACOBSTHAL and $h(x)-$ JACOBSTHAL- LUCAS REPRESENTATION POLYNOMIALS

Nayil Kilic

Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: Primary 05A15, 11B39; Secondary 11B37.

Keywords and phrases: Generating functions, Fibonacci and Lucas numbers and polynomials and generalizations, Recurrences.

Abstract In this paper, we introduce $h(x)-$ Jacobsthal type polynomials and give some properties of them and then using $h(x)-$ Jacobsthal type polynomials, we describe rising and decreasing diagonal functions and give some basic properties of them. Finally, we define augmented $h(x)-$ Jacobsthal- type representation polynomials, we compute generating functions, Binet formulas, summation formulas, Simson formulas and explicit combinatorial form of them.

1 Introduction

Let $h(x)$ be a polynomial with real coefficients. For $n > 0$, the $h(x)$-Jacobsthal polynomials $J_{h,n}(x)$, and the $h(x)$-Jacobsthal-Lucas polynomials $j_{h,n}(x)$, are defined by

$$J_{h,n+1}(x) = J_{h,n}(x) + h(x)J_{h,n-1}(x), \quad J_{h,0}(x) = 0, \quad J_{h,1}(x) = 1, \tag{1.1}$$

and

$$j_{h,n+1}(x) = j_{h,n}(x) + h(x)j_{h,n-1}(x), \quad j_{h,0}(x) = 2, \quad j_{h,1}(x) = 1, \tag{1.2}$$

respectively.
Note that, in particular case where $h(x) = 2x$, (1.1) reduces to the Jacobsthal polynomials and (1.2) reduces to the Jacobsthal-Lucas polynomials.

The solutions of the characteristic equation $z^2 - z - h(x) = 0$ associated to the recurrence relations (1.1) and (1.2) are $\lambda = \frac{1+\sqrt{1+4h(x)}}{2}$ and $\gamma = \frac{1-\sqrt{1+4h(x)}}{2}$.
Note that:

$$\lambda + \gamma = 1, \quad \lambda\gamma = -h(x), \quad \lambda - \gamma = \sqrt{1 + 4h(x)} \tag{1.3}$$

In the literature there are many studies about Jacobsthal-type polynomials. Some of them are as follows: In [5], Horadam studied Jacobsthal representation polynomials. He gave some properties of Jacobsthal representation numbers in [4], including generating functions, Binet formulas, Simson formulas and summation formulas. Also in [6], author introduced convolutions for Jacobsthal type polynomials and gave some important results of them. Also, in [1, 2, 3, 7, 8, 9], authors studied on Jacobsthal- type polynomials and Jacobsthal- type numbers.

2 The $h(x)-$ Jacobsthal- Type Polynomials

In the following Tables 1 and 2, we give the first few polynomials of (1.1) and (1.2) of these Jacobsthal-type sequences.

Table 1. $h(x)-$ Jacobsthal Polynomials $\{J_{h,n}(x)\}$: $0 \leq n \leq 10$

$J_{h,0}(x) = 0$	$J_{h,6}(x) = 1 + 4h(x) + 3h^2(x)$
$J_{h,1}(x) = 1$	$J_{h,7}(x) = 1 + 5h(x) + 6h^2(x) + h^3(x)$
$J_{h,2}(x) = 1$	$J_{h,8}(x) = 1 + 6h(x) + 10h^2(x) + 4h^3(x)$
$J_{h,3}(x) = 1 + h(x)$	$J_{h,9}(x) = 1 + 7h(x) + 15h^2(x) + 10h^3(x) + h^4(x)$
$J_{h,4}(x) = 1 + 2h(x)$	$J_{h,10}(x) = 1 + 8h(x) + 21h^2(x) + 20h^3(x) + 5h^4(x)$
$J_{h,5}(x) = 1 + 3h(x) + h^2(x)$	

Table 2. $h(x)-$ Jacobsthal-Lucas Polynomials $\{j_{h,n}(x)\}$: $0 \leq n \leq 10$

$J_{h,0}(x) = 2$	$J_{h,6}(x) = 1 + 6h(x) + 9h^2(x) + 2h^3(x)$
$J_{h,1}(x) = 1$	$J_{h,7}(x) = 1 + 7h(x) + 14h^2(x) + 7h^3(x)$
$J_{h,2}(x) = 1 + 2h(x)$	$J_{h,8}(x) = 1 + 8h(x) + 20h^2(x) + 16h^3(x) + 2h^4(x)$
$J_{h,3}(x) = 1 + 3h(x)$	$J_{h,9}(x) = 1 + 9h(x) + 27h^2(x) + 30h^3(x) + 9h^4(x)$
$J_{h,4}(x) = 1 + 4h(x) + 2h^2(x)$	$J_{h,10}(x) = 1 + 10h(x) + 35h^2(x) + 50h^3(x) + 25h^4(x) + 2h^5(x)$
$J_{h,5}(x) = 1 + 5h(x) + 5h^2(x)$	

3 Some Properties of the $h(x)-$ Jacobsthal- Type Polynomials

The Binet formulas for the $h(x)-$ Jacobsthal and $h(x)-$ Jacobsthal-Lucas polynomials are given by

$$J_{h,n}(x) = \frac{\lambda^n - \gamma^n}{\lambda - \gamma}, \qquad n \geq 0 \tag{3.1}$$

and

$$j_{h,n}(x) = \lambda^n + \gamma^n, \qquad n \geq 0 \tag{3.2}$$

respectively.

The generating functions for the $h(x)-$ Jacobsthal and $h(x)-$ Jacobsthal-Lucas polynomials are given as

$$\sum_{n=0}^{\infty} J_{h,n+1}(x) z^n = \frac{1}{1 - z - h(x)z^2} \tag{3.3}$$

and

$$\sum_{n=0}^{\infty} j_{h,n+1}(x) z^n = \frac{1 + 2h(x)z}{1 - z - h(x)z^2} \tag{3.4}$$

respectively.

Simson Formulas

From (3.1) and (3.2), we have the following Simson formulas, respectively:

$$J_{h,n+1}(x)J_{h,n-1}(x) - J_{h,n}^2(x) = (-1)^n (h(x))^{n-1}, \tag{3.5}$$

$$\begin{aligned} j_{h,n+1}(x)j_{h,n-1}(x) - j_{h,n}^2(x) &= [1 + 4h(x)](-h(x))^{n-1} \\ &= -[1 + 4h(x)][J_{h,n+1}(x)J_{h,n-1}(x) - J_{h,n}^2(x)]. \end{aligned} \tag{3.6}$$

Summation Formulas

Immediately, from (3.1) and (3.2), we have

$$\sum_{l=1}^{m} J_{h,l}(x) = \frac{J_{h,m+2}(x)-1}{h(x)}, \tag{3.7}$$

$$\sum_{l=0}^{m} j_{h,l}(x) = \frac{j_{h,m+2}(x)-1}{h(x)}. \tag{3.8}$$

Explicit Combinatorial Forms

In the next theorem, we give the explicit combinatorial forms of $J_{h,m}(x)$ and $j_{h,m}(x)$;

Theorem 3.1.

$$(i) \qquad J_{h,m}(x) = \sum_{l=0}^{\frac{m-1}{2}} \binom{m-l-1}{l} (h(x))^l, \tag{3.9}$$

$$(ii) \quad j_{h,m}(x) = \sum_{l=0}^{\frac{m}{2}} \frac{m}{m-l} \binom{m-l}{l} (h(x))^l. \tag{3.10}$$

Proof. (i) We use induction on m:

Verification of (3.9) for $m = 1, 2, 3$ is straightforward. Assume it is true for all $m \leq k$,

$$\begin{aligned}
&J_{h,k}(x) + h(x)J_{h,k-1}(x) \\
&= \sum_{l=0}^{\frac{k-1}{2}} \binom{k-l-1}{l} (h(x))^l + \sum_{l=0}^{\frac{k-2}{2}} \binom{k-l-2}{l} (h(x))^{l+1} \\
&= \binom{k-1}{0} + \binom{k-2}{1} h(x) + ... + \binom{\frac{k-1}{2}}{\frac{k-1}{2}} (h(x))^{\frac{k-1}{2}} \\
&+ \binom{k-2}{0} h(x) + \binom{k-3}{1} h^2(x) + ... + \binom{\frac{k-2}{2}}{\frac{k-2}{2}} (h(x))^{\frac{k}{2}} \\
&= \sum_{l=0}^{\frac{k}{2}} \binom{k-l}{l} (h(x))^l \quad \text{by} \quad \text{Pascal's} \quad \text{formula} \\
&= J_{h,k+1}(x).
\end{aligned}$$

(ii) The proof of (ii) is similar to the proof of (i), so it is omitted.□

Interrelationships

From (3.1) and (3.2), we obtain

$$j_{h,n}(x)J_{h,n}(x) = J_{h,2n}(x) \tag{3.11}$$

and

$$j_{h,n}(x) = J_{h,n+1}(x) + h(x)J_{h,n-1}(x). \tag{3.12}$$

Using (3.1), (3.2) and (1.3), we get

$$[1 + 4h(x)]J_{h,n}(x) = j_{h,n+1}(x) + h(x)j_{h,n-1}(x). \tag{3.13}$$

An immediate consequence of (3.1), (3.2), (1.3) and (1.1) is

$$J_{h,n}(x) + j_{h,n}(x) = 2J_{h,n+1}(x). \tag{3.14}$$

By (3.1), (3.2) and (3.12), we have

$$[1+4h(x)]J_{h,n}(x)+j_{h,n}(x)=2j_{h,n+1}(x). \tag{3.15}$$

From (3.1), (3.2) and (1.3), we arrive at

$$\sqrt{1+4h(x)}J_{h,n}(x)+j_{h,n}(x)=2\lambda^n \tag{3.16}$$

and

$$\sqrt{1+4h(x)}J_{h,n}(x)-j_{h,n}(x)=-2\gamma^n. \tag{3.17}$$

Using (3.1) and (3.2), we obtain

$$J_{h,m}(x)j_{h,n}(x)=J_{h,n}(x)j_{h,m}(x)=2J_{h,n+m}(x) \tag{3.18}$$

and

$$j_{h,m}(x)j_{h,n}(x)+(1+4h(x))J_{h,m}(x)J_{h,n}(x)=2j_{h,n+m}(x). \tag{3.19}$$

In particular, if we put $m=n$ in (3.18), we obtain (3.11). If we take m = n in (3.19), then we have

$$j^2_{h,m}(x)+(1+4h(x))J^2_{h,m}=2j_{h,2m}(x). \tag{3.20}$$

Using (3.3) and (3.4), we have

$$j_{h,n+1}(x)=J_{h,n+1}(x)+2h(x)J_{h,n}(x), \tag{3.21}$$

(3.21) is also obtained from (3.1) and (3.2).
In the following equations, we obtain the derivative of $h(x)-$ Jacobsthal and $h(x)-$ Jacobsthal-Lucas polynomials with respect to x:

$$\frac{dj_{h,n}(x)}{dx}=nh'(x)J_{h,n-1}(x) \tag{3.22}$$

and

$$[1+4h(x)]\frac{dJ_{h,n}(x)}{dx}=h'(x)nj_{h,n-1}(x)-2h'(x)J_{h,n}(x). \tag{3.23}$$

Suppose we describe the t^{th} associated sequences $\{J^{(t)}_{h,n}(x)\}$ and $\{j^{(t)}_{h,n}(x)\}$ of $\{J_{h,n}(x)\}$ and $\{j_{h,n}(x)\}$ to be, respectively ($t\geq 1$),

$$J^{(t)}_{h,n}(x)=J^{(t-1)}_{h,n+1}(x)+h(x)J^{(t-1)}_{h,n-1}(x) \tag{3.24}$$

and

$$j^{(t)}_{h,n}(x)=j^{(t-1)}_{h,n+1}(x)+h(x)j^{(t-1)}_{h,n-1}(x) \tag{3.25}$$

where $J^{(0)}_{h,n}(x)=J_{h,n}(x)$ and $j^{(0)}_{h,n}(x)=j_{h,n}(x)$. From (3.24) and (3.12), we obtain

$$J^{(1)}_{h,n}(x)=j_{h,n}(x). \tag{3.26}$$

By (3.25) and (3.13), we have that

$$j^{(1)}_{h,n}(x)=[1+4h(x)]J_{h,n}(x). \tag{3.27}$$

We generalize above formulas as follows:

$$J^{(2m)}_{h,n}(x)=j^{(2m-1)}_{h,n}(x)=\left(1+4h(x)\right)^m J_{h,n}(x), \tag{3.28}$$

$$J^{(2m+1)}_{h,n}(x)=j^{(2m)}_{h,n}(x)=\left(1+4h(x)\right)^m j_{h,n}(x). \tag{3.29}$$

When the structure of $\{R_{h,l}(x)\}$ and $\{r_{h,l}(x)\}$ is examined, it is seen that rising and descending diagonals will be obtained.

4 Rising Diagonal Functions

In Tables 1 and 2, imagine parallel upward-slanting lines, in Tables 1 and 2 there exist the rising diagonal functions $\{R_{h,l}(x)\}$ and $\{r_{h,l}(x)\}$, respectively. Some of the rising diagonal functions are , say,

$$R_{h,0}(x) = 0, \quad R_{h,1}(x) = R_{h,2}(x) = R_{h,3}(x) = 1, \quad R_{h,4}(x) = 1 + h(x), \tag{4.1}$$
$$R_{h,5}(x) = 1 + 2h(x), ..., \quad R_{h,10}(x) = 1 + 7h(x) + 10h^2(x) + h^3(x)$$

and

$$r_{h,0}(x) = 2, \quad r_{h,1}(x) = r_{h,2}(x) = 1, \quad r_{h,3}(x) = 1 + 2h(x), \tag{4.2}$$
$$r_{h,4}(x) = 1 + 3h(x), ..., \quad r_{h,10}(x) = 1 + 9h(x) + 20h^2(x) + 7h^3(x).$$

The generating functions for rising diagonal functions are

$$\sum_{l=1}^{\infty} R_{h,l}(x)t^{l-1} = \frac{1}{1 - t - h(x)t^3}, \tag{4.3}$$

$$\sum_{l=0}^{\infty} r_{h,l}(x)t^{l} = \frac{2 - t}{1 - t - h(x)t^3}. \tag{4.4}$$

From (4.3) and (4.4) , we write the following equation:

$$r_{h,n}(x) = 2R_{h,n+1}(x) - R_{h,n}(x). \tag{4.5}$$

For $n \geq 3$, by the aid of (4.3) and (4.4), the recurrence relations for rising diagonal functions are given by

$$R_{h,n}(x) = R_{h,n-1}(x) + h(x)R_{h,n-3}(x) \tag{4.6}$$

and

$$r_{h,n}(x) = r_{h,n-1}(x) + h(x)r_{h,n-3}(x). \tag{4.7}$$

We now give the explicit combinatorial form of rising diagonal functions:

Theorem 4.1.

$$(i) \qquad R_{h,m}(x) = \sum_{l=0}^{\frac{m-1}{3}} \binom{m-2l-1}{l} (h(x))^l, \tag{4.8}$$

$$(ii) \quad r_{h,m}(x) = 1 + \sum_{l=0}^{\frac{m}{3}} \frac{m-l}{l} \binom{m-2l-1}{l-1} (h(x))^l. \tag{4.9}$$

Proof. The proof can be done similar to the proof of Theorem 3.1. □

By (4.5) and (4.6), we have

$$r_{h,n}(x) = R_{h,n}(x) + 2h(x)R_{h,n-2}(x). \tag{4.10}$$

By using (4.5) and (4.10), we obtain the following result:

$$r^2_{h,n}(x) - R^2_{h,n}(x) = 4h(x)R_{h,n+1}(x)R_{h,n-2}(x). \tag{4.11}$$

Partially differential equations of the first order are readily described from (4.3) and (4.4). Let $R_h = R_h(x,t) = \sum_{l=1}^{\infty} R_{h,l}(x)t^{l-1}$ and $r_h = r_h(x,t) = \sum_{l=0}^{\infty} r_{h,l}(x)t^{l}$. These are

$$h'(x)t^3 \frac{\partial R_h}{\partial t} - (1 + 3t^2)\frac{\partial R_h}{\partial x} = 0 \tag{4.12}$$

and

$$h'(x)t^3 (\frac{\partial r_h}{\partial t} + R_h) - (1 + 3t^2)\frac{\partial r_h}{\partial x} = 0. \tag{4.13}$$

5 Descending Diagonal Functions

Imagine parallel downward-slanting lines in Tables 1 and 2 in which there exist the descending diagonal functions $\{D_{h,i}(x)\}$ and $\{d_{h,i}(x)\}$, respectively. Some of descending diagonal functions are, say,

$$D_{h,0}(x) = 0, \quad D_{h,1}(x) = 1, \quad D_{h,2}(x) = 1 + h(x), ..., \tag{5.1}$$
$$D_{h,5}(x) = 1 + 4h(x) + 6h^2(x) + 4h^3(x) + h^4(x)$$

and

$$d_{h,0}(x) = 2, \quad d_{h,1}(x) = 1 + 2h(x), \quad d_{h,2}(x) = 1 + 3h(x) + 2h^2(x), ..., \tag{5.2}$$
$$d_{h,5}(x) = 1 + 6h(x) + 14h^2(x) + 16h^3(x) + 9h^4(x) + 2h^5(x).$$

The generating functions for descending diagonal functions are

$$\sum_{l=1}^{\infty} D_{h,l}(x)t^l = \frac{1}{1-(1+h(x))t} \tag{5.3}$$

and

$$\sum_{l=1}^{\infty} d_{h,l}(x)t^{l-1} = \frac{1+2h(x)}{1-(1+h(x))t} \tag{5.4}$$

therefore $(l \geq 1)$

$$D_{h,l}(x) = (1+h(x))^{l-1} \tag{5.5}$$

and

$$d_{h,l}(x) = (1+2h(x))(1+h(x))^{l-1}. \tag{5.6}$$

From (5.5) and (5.6), we obtain

$$d_{h,l}(x) = (1+2h(x))D_{h,l}(x). \tag{5.7}$$

For $(l \geq 2)$

$$\frac{D_{h,l}(x)}{D_{h,l-1}(x)} = \frac{d_{h,l}(x)}{d_{h,l-1}(x)} = 1 + h(x). \tag{5.8}$$

By (5.8), we get

$$D_{h,l}(x)d_{h,l-1}(x) = D_{h,l-1}(x)d_{h,l}(x). \tag{5.9}$$

For $(l \geq 1)$

$$\frac{d_{h,l}(x)}{D_{h,l}(x)} = 1 + 2h(x). \tag{5.10}$$

$$d_{h,l}(x) = D_{h,l+1}(x) + h(x)D_{h,l}(x), \tag{5.11}$$

$$(1+2h(x))^2 D_{h,l}(x) = d_{h,l+1}(x) + h(x)d_{h,l}(x), \tag{5.12}$$

$$J_{h,5}(x)D_{h,l-1}(x) = D_{h,l+1}(x) + h(x)D_{h,l-1}(x), \tag{5.13}$$

$$J_{h,5}(x)d_{h,l-1}(x) = d_{h,l+1}(x) + h(x)d_{h,l-1}(x). \tag{5.14}$$

6 Differential Equations

Let

$$D_h \equiv D_h(x,t) = \sum_{l=1}^{\infty} D_{h,l}(x)t^{l-1} = \frac{1}{1-(1+h(x))t} \tag{6.1}$$

and

$$d_h \equiv d_h(x,t) = \sum_{l=1}^{\infty} d_{h,l}(x)t^{l-1} = \frac{1+2h(x)}{1-(1+h(x))t}. \tag{6.2}$$

It is easy to get from (5.5), (5.6), (6.1) and (6.2),

$$h'(x)t\frac{\partial D_h}{\partial t} - [1+h(x)]\frac{\partial D_h}{\partial x} = 0, \tag{6.3}$$

$$h'(x)t\frac{\partial d_h}{\partial t} - [1+h(x)][\frac{\partial d_h}{\partial x} - 2h'D_h] = 0, \tag{6.4}$$

$$[1+h(x)]\frac{dD_{h,n}(x)}{dx} = (n-1)h'D_{h,n}(x), \tag{6.5}$$

$$\frac{dd_{h,n}(x)}{dx} = h'(x)[D_{h,n-1}(x) + nd_{h,n-1}(x)], \tag{6.6}$$

$$\frac{d^{n-1}D_{h,n}(x)}{dx^{n-1}} = (n-1)!(h'(x))^{n-1}. \tag{6.7}$$

7 Augmented $h(x)$-Jacobsthal-Type Representation Polynomials

In [5], Horadam introduces the augmented Jacobsthal representation polynomial sequence $\{\mathfrak{T}_n(x)\}$ defined by

$$\mathfrak{T}_{n+2}(x) = \mathfrak{T}_{n+1}(x) + 2x\mathfrak{T}_n(x) + 3, \quad \mathfrak{T}_0(x) = 0, \quad \mathfrak{T}_1(x) = 1 \tag{7.1}$$

and the augmented Jacobsthal-Lucas representation polynomial sequence $\{\tau_n(x)\}$ defined by

$$\tau_{n+2}(x) = \tau_{n+1}(x) + 2x\tau_n(x) + 5, \quad \tau_0(x) = 0, \quad \tau_1(x) = 1. \tag{7.2}$$

We, now, introduce the augmented $h(x)$-Jacobsthal-type representation polynomial sequence $\{\mathfrak{T}_{h,n}(x)\}$ defined by

$$\mathfrak{T}_{h,n+2}(x) = \mathfrak{T}_{h,n+1}(x) + h(x)\mathfrak{T}_{h,n}(x) + 3, \quad \mathfrak{T}_{h,0}(x) = 0, \quad \mathfrak{T}_{h,1}(x) = 1 \tag{7.3}$$

and the augmented $h(x)$-Jacobsthal-Lucas representation polynomial sequence $\{\tau_{h,n}(x)\}$ defined by

$$\tau_{h,n+2}(x) = \tau_{h,n+1}(x) + h(x)\tau_{h,n}(x) + 5, \quad \tau_{h,0}(x) = 0, \quad \tau_{h,1}(x) = 1. \tag{7.4}$$

Example 7.1.

$$\mathfrak{T}_{h,0}(x) = 0, \quad \mathfrak{T}_{h,1}(x) = 1, ..., \mathfrak{T}_{h,8}(x) = 7h^3(x) + 40h^2(x) + 102h(x) + 22. \tag{7.5}$$

and

$$\tau_{h,0}(x) = 0, \quad \tau_{h,1}(x) = 1, ..., \tau_{h,8}(x) = 9h^3(x) + 60h^2(x) + 81h(x) + 36. \tag{7.6}$$

8 Some properties of $\{\mathfrak{T}_{h,n}(x)\}$ and $\{\tau_{h,n}(x)\}$

Generating Functions

$$\sum_{l=1}^{\infty} \mathfrak{T}_{h,l}(x)y^{l-1} = \frac{1+2y}{1-2y-(h(x)-1)y^2+h(x)y^3}, \tag{8.1}$$

$$\sum_{l=1}^{\infty} \tau_{h,l}(x)y^{l-1} = \frac{1+4y}{1-2y-(h(x)-1)y^2+h(x)y^3}. \tag{8.2}$$

Binet Formulas

Theorem 8.1. *For* $n \geq 0$

$$(i) \quad \mathfrak{T}_{h,n}(x) = \frac{J_{h,n+2}(x)+2J_{h,n+1}(x)-3}{h(x)}, \tag{8.3}$$

$$(ii) \quad \tau_{h,n}(x) = \frac{J_{h,n+2}(x)+4J_{h,n+1}(x)-5}{h(x)}. \tag{8.4}$$

Proof. (i) We use induction on n, verification of (8.3) for $n = 0, 1, 2, 3$ is straightforward. Assume (8.3) is true for $n = k$, so

$$\mathfrak{T}_{h,k}(x) = \frac{J_{h,k+2}(x)+2J_{h,k+1}(x)-3}{h(x)}$$

Now

$$\begin{aligned} \mathfrak{T}_{h,k+1}(x) &= \mathfrak{T}_{h,k}(x) + h(x)\mathfrak{T}_{h,k-1}(x) + 3 \\ &= \frac{J_{h,k+2}(x)+h(x)J_{h,k+1}(x)+2\big[J_{h,k+1}(x)+2J_{h,k}(x)\big]-3}{h(x)} \\ &= \frac{J_{h,k+3}(x)+2J_{h,k+2}(x)-3}{h(x)}. \end{aligned}$$

Thus, (8.3) is true for $n = k+1$. This completes induction.
(ii) The proof of (ii) is similar to (i), so it is omitted. □

From (8.3), (8.4) and (3.1), Binet formulas for $\mathfrak{T}_{h,n}(x)$ and $\tau_{h,n}(x)$ are derivable:

$$\mathfrak{T}_{h,n}(x) = \frac{\frac{\lambda^{n+2}-\gamma^{n+2}}{\lambda-\gamma} + 2(\frac{\lambda^{n+1}-\gamma^{n+1}}{\lambda-\gamma}) - 3}{h(x)}, \tag{8.5}$$

and

$$\tau_{h,n}(x) = \frac{\frac{\lambda^{n+2}-\gamma^{n+2}}{\lambda-\gamma} + 4(\frac{\lambda^{n+1}-\gamma^{n+1}}{\lambda-\gamma}) - 5}{h(x)}. \tag{8.6}$$

Simson Formulas

$$\begin{aligned} \mathfrak{T}_{h,n+1}(x)\mathfrak{T}_{h,n-1}(x) - \mathfrak{T}^2{}_{h,n}(x) =& (-h(x))^{n-2}[h(x)-6] \\ & - 3[J_{h,n-1}(x)+2J_{h,n-2}(x)], \end{aligned} \tag{8.7}$$

$$\begin{aligned} \tau_{h,n+1}(x)\tau_{h,n-1}(x) - \tau^2{}_{h,n}(x) =& (-h(x))^{n-2}[h(x)-20] \\ & - 5[J_{h,n-1}(x)+4J_{h,n-2}(x)]. \end{aligned} \tag{8.8}$$

Summation Formulas

$$\sum_{l=1}^{n} \mathfrak{T}_{h,l}(x) = \frac{\mathfrak{T}_{h,n+2}(x) - 3n - 4}{h(x)}, \tag{8.9}$$

$$\sum_{l=1}^{n} \tau_{h,l}(x) = \frac{\tau_{h,n+2}(x) - 5n - 6}{h(x)}. \tag{8.10}$$

Explicit Combinatorial Forms

Theorem 8.2.

$$(i) \quad \mathfrak{T}_{h,n}(x) = J_{h,n}(x) + 3\sum_{l=0}^{\frac{n-1}{2}} \binom{n-1-l}{l+1} h(x)^l, \tag{8.11}$$

$$(ii) \quad \tau_{h,n}(x) = J_{h,n}(x) + 5\sum_{l=0}^{\frac{n-1}{2}} \binom{n-1-l}{l+1} h(x)^l. \tag{8.12}$$

Proof.(i) We use induction on n, checking validates the case $n = 1, 2, 3$. Suppose (8.11) is true for $n = 1, 2, 3, ..., k-1, k$. Then, by using (1.1) and the hypothesis,

$$\begin{aligned}
&\mathfrak{T}_{h,k}(x) + h(x)\mathfrak{T}_{h,k-1}(x) + 3 \\
&= J_{h,k}(x) + h(x)J_{h,k-1}(x) \\
&+ 3\Big[\sum_{l=0}^{\frac{k-1}{2}} \binom{k-1-l}{l+1} h(x)^l + \sum_{l=0}^{\frac{k-2}{2}} \binom{k-2-l}{l+1} h(x)^{l+1} + 1\Big] \\
&= J_{h,k+1}(x) + 3\sum_{l=0}^{\frac{k}{2}} \binom{k-l}{l+1} h(x)^l \quad by \quad Pascal's \quad formula \\
&= \mathfrak{T}_{h,k+1}(x).
\end{aligned}$$

Thus, (8.11) is true for $n = k + 1$, and so for all n.
(ii) The proof of (ii) is similar to the proof of (i) and so it is omitted. □
Now, there is a connection between (8.3) and (8.4):

$$\tau_{h,n}(x) - \mathfrak{T}_{h,n}(x) = \frac{2(J_{h,n+1}(x) - 1)}{h(x)} \tag{8.13}$$

References

[1] P. Catarino and M. L. Morgado, On generalized Jacobsthal and Jacobsthal- Lucas polynomials, *An. Şt Univ. Ovidius Constanţa* **24(3)**, 61–78 (2016).

[2] Gospava B. Djordjević, Generalized Jacobsthal polynomials, *The Fibonacci Quarterly* **38(3)**, 239–243 (2000).

[3] V. E. Hoggatt, Jr. and M. Bicknell-Johnson, Convolution Arrays for Jacobsthal and Fibonacci Polynomials, *The Fibonacci Quarterly* **16(5)**, 385–402 (1978).

[4] A. F. Horadam, Jacobsthal Representation Numbers, *The Fibonacci Quarterly* **34(1)**, 40–54 (1996).

[5] A. F. Horadam, Jacobsthal Representation Polynomials, *The Fibonacci Quarterly* **35(2)**, 137–148 (1997).

[6] A. F. Horadam, Convolutions for Jacobsthal-Type Polynomials, *The Fibonacci Quarterly* **40(3)**, 212–222 (2002).

[7] N. Kilic, On split $k-$ Jacobsthal and $k-$ Jacobsthal Lucas quaternions, *Ars Combinatoria* **142**, 129–139 (2019).

[8] N. Kilic, On k-Jacobsthal and k-Jacobsthal-Lucas Octonions, *JP Journal of Algebra, Number Theory and Applications* **41(1)**, 1–17 (2019).

[9] Agnieszka Tereszkiewicz and Izabela Wawreniuk, Generalized Jacobsthal polynomials and special points for them, *Applied Mathematics and Computation* **268**, 806–814 (2015).

Author information

Nayil Kilic, Department of Mathematical Education, Hasan Ali Yucel Education Faculty, Istanbul University-Cerrahpasa, 34470, Fatih, Istanbul, Turkey.
E-mail: nayilkilic@gmail.com

Received: March 8, 2019.
Accepted: May 28, 2019.