Palestine Journal of Mathematics

Vol. 10(1)(2021) , 322–331

ON GENERALIZED THIRD ORDER MOCK THETA FUNCTIONS

Pramod Kumar Rawat

Communicated by H. M. Srivastava

MSC 2010 Classifications: 33D15, 33D05.

Keywords and phrases: q-Hypergeometric Series, q-Integrals.

I am thankful to Prof. Bhaskar Srivastava for his guidance.

Abstract We have considered three independent variable generalization of the third order mock theta functions. We have given some identities for these functions. We have also given q-integral representation and multibasic expansion for these functions.

1 Introduction

In mathematical world, Ramanujan and mock theta functions are synonyms. Ramanujan in his last letter to Hardy [11] gave 17 functions and called them mock theta functions as they were not theta functions. In the list of these 17 functions, 4 were of third order, 10 of fifth order and 3 of seventh order. Though Ramanujan classified them in different "order" he did not explain what he meant by "order". Later Watson [15] added three mock theta functions in the list of third order mock theta functions. We have generalized the third order mock theta functions, by introducing three independent variables. The advantage of generalization is that by specializing them we get some known functions and some new identities. Moreover the results I have proved for generalized functions are new and hold for these known functions also. This is done in last section. We have expressed these functions as q-integrals and given multibasic expansions. The third order mock functions are ;

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2},\qquad \phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n},$$

$$\chi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n},\qquad \psi(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}}{(q;q^2)_{n+1}},$$

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2},\qquad \nu(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}},$$

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{((1+q+q^2)...(1+q^{2n+1}+q^{4n+1}))}.$$

The first four are of Ramanujan and last three of Watson [15].
The third order generalized mock theta functions of Andrews [1] are;

$$f(\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{n^2-n}\alpha^n}{(q;q)_n(\alpha;q)_n},\qquad \phi(\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\alpha q;q^2)_n},$$

$$\psi(\alpha;q)=\sum_{n=1}^{\infty}\frac{q^{n^2}}{(-\alpha;q^2)_n},\qquad \nu(\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(\alpha^2q^{-1};q^2)_{n+1}},$$

$$\omega(\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}\alpha^{2n}}{(-q;q^2)_{n+1}(-\alpha^2q^{-1};q^2)_{n+1}}.$$

For $\alpha=q$ these functions reduce to classical mock theta functions of third order

2 Generalized Mock Theta Function of Third Order

We define the two variable generalized third order mock theta functions as;

$$f(t,\alpha,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2},\qquad \phi(t,\alpha,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2}z^{2n}\alpha^n}{(-zq^2;q^2)_n},$$

$$\psi(t,\alpha,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{(n+1)^2}z^{2n}\alpha^n}{(zq;q^2)_{n+1}},\qquad \nu(t,\alpha,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2+n}z^{2n}\alpha^n}{(-zq;q^2)_{n+1}},$$

$$\omega(t,\alpha,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{2n^2+2n}z^{2n}\alpha^n}{(zq;q^2)_{n+1}^2}.$$

We have given generalization of only five of the third order mock theta functions and will be considering only these generalization.
For $t=0$, $\alpha=1$ and $z=1$ these functions reduce to classical mock theta functions of third order.

3 Some Identities for Generalized Mock Theta Functions

We prove the following identities for the third order generalized mock theta functions as;

Theorem 1.

$$(i)\quad f(0,\alpha,z;q)+\sum_{n=1}^{\infty}\frac{q^n(-1/z;q)_n}{z^n\alpha^n}=\frac{(-q/z\alpha;q)_\infty^2}{(\alpha z^2q,q/\alpha z^2,\frac{q}{\alpha};q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^2q^{2r})(-z\alpha;q)_r^2z^{4r}\alpha^rq^{2r^2}}{(1-\alpha z^2)(-zq;q)_r^2},$$

$$(ii)\quad \phi(0,\alpha,z;q)+\sum_{n=1}^{\infty}\frac{q^n(-1/z;q^2)_n}{z^n\alpha^n}=\frac{(-q^2/z^3\alpha^2;q^2)_\infty}{(\alpha z^2q,q/\alpha z^2,q/z\alpha;q)_\infty}\times\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^2q^{2r})(-z^3\alpha^2;q^2)_rz^{3r}\alpha^rq^{2r^2}}{(1-\alpha z^2)(-zq^2;q^2)_r},$$

$$(iii) \quad \nu(0,\alpha,z;q)+\frac{1}{(1+zq)}\sum_{n=1}^{\infty}\frac{q^n(-1/zq;q^2)_n}{z^n\alpha^n}=\frac{(-1/z^3\alpha^2;q^2)_\infty}{(\alpha z^2q^2,1/\alpha z^2,q/z\alpha;q)_\infty}\times\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^2q^{2r+1})(-z^3q\alpha^2;q^2)_r z^{3r}\alpha^r q^{2r^2+2r}}{(1-\alpha z^2q)(-zq^3;q^2)_r},$$

$$(iv) \quad \psi(0,\alpha,z;q)+\frac{q}{(1-zq)}\sum_{n=1}^{\infty}\frac{(-1)^n(1/zq;q^2)_n}{z^n\alpha^n}=\frac{(q/z^3\alpha^2;q^2)_\infty}{(\alpha z^2q^3,1/\alpha z^2q,-1/z\alpha;q)_\infty}\times\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^2q^{2r+2})(z^3\alpha^2q^3;q^2)_r(-1)^r z^{3r}\alpha^r q^{2r^2+3r}}{(1-\alpha z^2q^2)(zq^3;q^2)_r}$$

Proof of (i).

For proving the theorem we require following transformation formula [9]

$$\sum_{n=-\infty}^{\infty}\frac{z^nq^{n^2}}{(sq,tq;q)_n}=\frac{(sq/z,tq/z;q)_\infty}{(zq,q/z,stq/z;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(1-zq^{2r})(z/s,z/t;q)_r(zst)^rq^{2r^2}}{(1-z)(sq,tq;q)_r} \tag{3.1}$$

Replacing z by $z^2\alpha$ and putting $s=t=-z$ then, the left side is equal to

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}=\sum_{n=0}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}+\sum_{n=-1}^{-\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2} \tag{3.2}$$

Writing $-n$ for n in the second term on the right side, we have

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}=\sum_{n=0}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}+\sum_{n=1}^{\infty}\frac{q^{n^2}z^{-2n}\alpha^{-n}}{(-zq;q)_{-n}^2}, \tag{3.3}$$

using

$$(a;q)_{-n}=\frac{(-q/a)^nq^{n(n-1)/2}}{(q/a;q)_n},$$

to simplify the second term on the right side, we have

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}=f(\alpha,z,q)+\sum_{n=1}^{\infty}\frac{q^n(-1/z;q)_n}{z^n\alpha^n}. \tag{3.4}$$

The right side of (3.1) is

$$=\frac{(-q/z\alpha;q)_\infty^2}{(\alpha z^2q,q/\alpha z^2,q/\alpha;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^2q^{2r})(-z\alpha;q)_r^2z^{4r}\alpha^rq^{2r^2}}{(1-\alpha z^2)(-zq;q)_r^2}. \tag{3.5}$$

From (3.4) and (3.5), we have

$$f(0,\alpha,z;q)+\sum_{n=1}^{\infty}\frac{q^n(-1/z;q)_n}{z^n\alpha^n}=\frac{(-q/z\alpha;q)_\infty^2}{(\alpha z^2q,q/\alpha z^2,q/\alpha;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(1-\alpha z^2q^{2r})(-z\alpha;q)_r^2z^{4r}\alpha^rq^{2r^2}}{(1-\alpha z^2)(-zq;q)_r^2}. \tag{3.6}$$

Which proves (i).

Proof of (ii).
Put $z=z^2\alpha$, $s=i\sqrt{z}$ and $t=-i\sqrt{z}$ in the transformation formula (3.1).

Proof of (iii).
Put $z=z^2q\alpha$, $s=i\sqrt{zq}$ and $t=-i\sqrt{zq}$ in the transformation formula (3.1).

Proof of (iv).
Put $z=z^2\alpha q^2$, $s=\sqrt{zq}$ and $t=-\sqrt{zq}$ in the transformation formula (3.1).

4 q-Integral Representation for the Generalized Third Order Mock Theta Functions

Thomae and Jackson [6, p.19] defined q-integral.

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n,$$

using limiting case of q-beta integral, we have

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty d_qt, \tag{4.1}$$

We give the integral representation for the generalized third order mock theta functions in the following theorem.

Theorem 2.

$$(i) \quad f(q^t,\alpha,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty f(0,\alpha u,z;q)d_qu,$$

$$(ii) \quad \phi(q^t,\alpha,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty \phi(0,\alpha u,z;q)d_qu,$$

$$(iii) \quad \psi(q^t,\alpha,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty \psi(0,\alpha u,z;q)d_qu,$$

$$(iv) \quad \nu(q^t,\alpha,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty \nu(0,\alpha u,z;q)d_qu,$$

$$(v) \quad \omega(q^t,\alpha,z;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty \omega(0,\alpha u,z;q)d_qu.$$

A detailed proof for $f(q^t,\alpha,z;q)$ is given. The proofs of the other functions are similar , so omitted.

Proof.
By definition

$$f(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}$$

Replacing t by q^t we have

$$f(q^t,\alpha,z;q) = \frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(q^t)_n q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}$$

$$= \sum_{n=0}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2(q^{n+t};q)_\infty}$$

By equation (4.1), we have

$$= \sum_{n=0}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{n+t-1}(uq;q)_\infty d_qu$$

But

$$f(0,\alpha,z;q)=\sum_{n=0}^{\infty}\frac{q^{n^2}z^{2n}\alpha^n}{(-zq;q)_n^2}$$

So we have

$$f(q^t,\alpha,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty f(0,z,\alpha u;q)d_qu,$$

which proves (i) .
The proof of all the other fuctions is similar, so omitted.

5 Multibasic Expansion of Generalized Third Order Mock Theta Functions

The following bibasic expansion will be used to give multibasic expansion for the generalized functions.

Theorem 3.

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_kq^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k}\sum_{m=0}^{\infty}\alpha_{m+k}$$
$$=\sum_{m=0}^{\infty}\frac{(ap,bp;p)_m(cq,aq/bc;q)_mq^m}{(q,aq/b;q)_m(ap/c,bcp;p)_m}\alpha_m \tag{5.1}$$

Using the summation formula [6, (3.6.7), p.71] and [10, Lemma 10, p.57], we have Theorem 3. We will consider the following cases of the Theorem 3.

Case I
Letting $q\to q^3$ and making $c\to\infty$, in Theorem 3, we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{3k})(1-bp^kq^{-3k})(a,b;p)_kq^{\frac{3k^2+3k}{2}}}{(1-a)(1-b)(q^3,aq^3/b;q^3)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k}$$
$$=\sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{\frac{3m^2+3m}{2}}}{(q^3,aq^3/b;q^3)_mb^mp^{\frac{m^2+m}{2}}}\alpha_m. \tag{5.2}$$

Case II.
Letting $q\to q^5$ and $c\to\infty$ in Theorem 3, we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{5k})(1-bp^kq^{-5k})(a,b;p)_kq^{\frac{5k^2+5k}{2}}}{(1-a)(1-b)(q^5,aq^5/b;q^5)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k}$$
$$=\sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{\frac{5m^2+5m}{2}}}{(q^5,aq^5/b;q^5)_mb^mp^{\frac{m^2+m}{2}}}\alpha_m. \tag{5.3}$$

We now give multibasic expansion of generalized third order mock theta functions.

Theorem 4.

$$(i)\quad f(t,\alpha,z;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}z^{2k}q^{k^2}\alpha^k}{(1-q^{k+2})(-zq;q)_k^2}$$
$$\times\,\phi\left[{q;0;tq^{3k};q^{3k+3}\atop q^{k+3},-zq^{k+1},-zq^{k+1};0:0};q,q^2,q^3;z^2q\alpha\right]$$

$$(ii) \quad \phi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}z^{2k}q^{k^2}\alpha^k}{(1-q^{k+2})(-zq^2;q^2)_k}$$
$$\times\,\phi\left[{q;0;tq^{3k},q^{3k+3} \atop q^{k+3};-zq^{2k+2};0};q,q^2,q^3;z^2q\alpha\right]$$

$$(iii) \quad \psi(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}z^{2k}q^{(k+1)^2}\alpha^k}{(1-q^{k+2})(zq;q^2)_{k+1}}$$
$$\times\,\phi\left[{q;0;tq^{3k},q^{3k+3} \atop q^{k+3};zq^{2k+3};0};q,q^2,q^3;z^2q^3\alpha\right]$$

$$(iv) \quad \nu(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}q^{k^2+k}z^{2k}\alpha^k}{(1-q^{k+2})(-zq;q^2)_{k+1}}$$
$$\times\,\phi\left[{q;0;q^{3k+3},tq^{3k} \atop q^{k+3};-zq^{2k+3};0};q,q^2,q^3;q^2z^2\alpha\right]$$

$$(v) \quad \omega(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+4})(t;q)_{k-1}q^{2k^2+2k}z^{2k}\alpha^k}{(1-q^{k+4})(zq;q^2)_{k+1}^2}$$
$$\times\,\phi\left[{q;0;q^{5k+5},tq^{5k} \atop q^{k+5};zq^{2k+3},zq^{2k+3};0};q,q^2,q^5;z^2q^4\alpha\right]$$

We shall give the proof of (i) in detail, for others we will state the value of the parameters.

Proof of (i).
Taking

$$a=t/q,\ b=q^2,\ p=q \text{ and } \alpha_m = \frac{(q^3;q^3)_m(t;q^3)_mq^mz^{2m}\alpha^m}{(q^3;q)_m(-zq;q)_m^2} \text{ in (5.2)}$$

we have

$$\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q;q)_k(q^2;q)_kq^{\frac{3k^2+3k}{2}}}{(1-t/q)(1-q^2)(q^3,t;q^3)_kq^{2k}q^{\frac{k^2+k}{2}}}$$
$$\times\sum_{m=0}^{\infty}\frac{(q^3;q^3)_{m+k}(t;q^3)_{m+k}q^{m+k}z^{2m+2k}\alpha^{m+k}}{(q^3;q)_{m+k}(-zq;q)_{m+k}^2}$$

$$=\sum_{m=0}^{\infty}\frac{(t,q^3;q)_mq^{\frac{3m^2+3m}{2}}}{(q^3,t;q^3)_mq^{2m}q^{\frac{m^2+m}{2}}}\frac{(q^3;q^3)_mq^m(t;q^3)_mz^{2m}\alpha^m}{(q^3;q)_m(-zq;q)_m^2} \tag{5.4}$$

The right hand side is equal to

$$=\sum_{m=0}^{\infty}\frac{(t;q)_mq^{m^2}z^{2m}\alpha^m}{(-zq;q)_m^2}$$

$$=(t)_\infty f(t,z,\alpha;q).$$

The left hand side of (5.4) is equal to

$$\sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q;q)_k(q^2;q)_k q^{k^2+k}}{(1-t/q)(1-q^2)(q^3,t;q^3)_k q^{2k}}$$
$$\times \sum_{m=0}^{\infty} \frac{(q^3;q^3)_k(q^{3k+3};q^3)_m(t;q^3)_k(tq^{3k};q^3)_m q^{m+k} z^{2m+2k}\alpha^{m+k}}{(q^3;q)_k(q^{k+3};q)_m(-zq;q)_k^2(-zq^{k+1};q)_m^2}$$

$$= \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1} q^{k^2} z^{2k}\alpha^k}{(1-q^{k+2})(-zq;q)_k^2}$$
$$\times \sum_{m=0}^{\infty} \frac{(q^{3k+3};q^3)_m(tq^{3k};q^3)_m q^m z^{2m}\alpha^m}{(q^{k+3};q)_m(-zq^{k+1};q)_m^2}$$

$$= \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1} q^{k^2} z^{2k}\alpha^k}{(1-q^{k+2})(-zq;q)_k^2}$$
$$\times \phi\left[\begin{matrix} q;0;tq^{3k};q^{3k+3} \\ q^{k+3},-zq^{k+1};-zq^{k+1};0 \end{matrix}; q,q^2,q^3;z^2q\alpha\right].$$

which proves (i).

Proof of (ii).
Take

$$a=t/q,\ b=q^2,\ p=q \text{ and } \alpha_m = \frac{(q^3;q^3)_m(t;q^3)_m q^m z^{2m}\alpha^m}{(q^3;q)_m(-zq^2;q^2)_m} \text{ in (5.2).}$$

Proof of (iii).
Take

$$a=t/q,\ b=q^2,\ p=q \text{ and } \alpha_m = \frac{(q^3;q^3)_m(t;q^3)_m q^{3m+1} z^{2m}\alpha^m}{(q^3;q)_m(zq;q^2)_{m+1}} \text{ in (5.2).}$$

Proof of (iv).
Take

$$a=t/q,\ b=q^2,\ p=q \text{ and } \alpha_m = \frac{(q^3;q^3)_m(t;q^3)_m q^{2m} z^{2m}\alpha^m}{(q^3;q)_m(-zq;q^2)_{m+1}} \text{ in (5.2).}$$

Proof of (v).
Take

$$a=t/q,\ b=q^4,\ p=q \text{ and } \alpha_m = \frac{(q^5;q^5)_m(t;q^5)_m q^m z^{2m}\alpha^m}{(q^5;q)_m(zq;q^2)_{m+1}^2} \text{ in (5.3).}$$

6 Special Cases

We have following special cases of genrealized third order mock theta functions;

$$(i) \qquad f(0,1,-1;q) = \frac{1}{(q)_\infty},$$

$$(ii) \qquad \phi(0,-1,-1;q) = \frac{f(-q,-q)}{(q)_\infty},$$

$$(iii) \qquad \phi(0,-1,-1;q) = \frac{f(-q,-q^5)}{\psi(q)},$$

$$(iv) \qquad \phi(0,q,-1;q) = \frac{f(-q,-q^3)}{(q)_\infty},$$

$$(v) \qquad \nu(0,-q^{-1},-1;q) = 1.$$

Case (i).

By definition

$$f(t,\alpha,z,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2} z^{2n}\alpha^n}{(-zq;q)_n^2} \tag{6.1}$$

For $t=0$, $\alpha=1$ and $z=-1$, we have

$$f(0,1,-1;q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2} \tag{6.2}$$

From [13, eqn.(A.6), p.170]

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2} \tag{6.3}$$

By eqn.(6.2) and eqn.(6.3), we have

$$f(0,1,-1;q) = \frac{1}{(q)_\infty} \tag{6.4}$$

Case (ii).

By definition

$$\phi(t,\alpha,z,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2} z^{2n}\alpha^n}{(-zq^2;q^2)_n} \tag{6.5}$$

For $t=0$, $\alpha=-1$ and $z=-1$, we have

$$\phi(0,-1,-1;q) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} \tag{6.6}$$

From [13, eqn.(A.12), p.171]

$$\frac{f(-q,-q)}{(q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} \tag{6.7}$$

By eqn.(6.6) and eqn.(6.7), we have

$$\phi(0,-1,-1;q) = \frac{f(-q,-q)}{(q)_\infty} \tag{6.8}$$

Case (iii).

For $t = 0$, $\alpha = -1$ and $z = -1$, we have

$$\phi(0,-1,-1;q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} \tag{6.9}$$

From [13, eqn.(A.54), p.175]

$$\frac{f(-q,-q^5)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} \tag{6.10}$$

By eqn.(6.9) and eqn.(6.10), we have

$$\phi(0,-1,-1;q) = \frac{f(-q,-q^5)}{\psi(q)} \tag{6.11}$$

Case (iv).

For $t = 0$, $\alpha = q$ and $z = -1$, we have

$$\phi(0,q,-1;q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \tag{6.12}$$

From [13, eqn.(A.22), p.172]

$$\frac{f(-q,-q^3)}{(q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \tag{6.13}$$

By eqn.(6.12) and eqn.(6.13), we have

$$\phi(0,q,-1;q) = \frac{f(-q,-q^3)}{(q)_\infty} \tag{6.14}$$

Case (v).

By definition

$$\nu(t,\alpha,z,q) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n q^{n^2+n} z^{2n} \alpha^n}{(-zq;q^2)_{n+1}} \tag{6.15}$$

For $t = 0$, $\alpha = -q^{-1}$ and $z = -1$, we have

$$\nu(0,-q^{-1},-1;q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_{n+1}} \tag{6.16}$$

From [13, eqn.(A.207), p.191]

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_{n+1}} \tag{6.17}$$

By eqn.(6.16) and eqn.(6.17), we have

$$\nu(0, -q^{-1}, -1; q) = 1 \tag{6.18}$$

Conclusion: I have made comprehensive study of generalized mock theta functions of order fifth, sixth, eighth and tenth, and the papers have been communicated for publication.

References

[1] G. E. Andrews, *On basic hypergeometric series, mock theta functions and partitions* (I), Quart. J. Math. (Oxford) (2), **17**(1966), 64-80.

[2] G. E. Andrews, *The fifth and seventh order mock theta functions*, Trans. Amer. Math. Soc. **293**(1986), 113-134.

[3] G. E. Andrews, *Mock Theta Functions, Proc. Sympos*. Pure Math. **49**(1989), Part 2, 283-298.

[4] G. E. Andrews, F. G. Garvan, *Ramanujan's "Lost" Notebook VI: The Mock Theta Conjectures*, Advances in mathematics. **73**(1989), 242-255.

[5] Y.-S. Choi, *The basic bilateral hypergeometric series and the mock theta functions*, Ramanujan J. **24**(2011), 345-386.

[6] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge (1990).

[7] K. Hikami, *Transformation formula of the second order mock theta function*, Letter. Math. Phys. **75(1)**(2006), 93-98.

[8] L. J. *Slater, Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **54**(1952), 147-167.

[9] J. M. Laughlin, *Mock Theta Function Identities Deriving from Bilateral Basic Hypergeometric Series*, Analytic Number Theory , Modular Forms and q-Hypergeometric Series, 503-531, Springer Proc. Math .Stat. 221, Springer, Cham, 2017.

[10] E. D. Rainville, Special Function, Chelsea Publishing Company, Bronx, New York (1960).

[11] S. Ramanujan, Collected Paper, Cambridge University Press 1927, reprinted by Chelsea New York, (1962).

[12] B. Srivastava, *Certain bilateral basic hypergeometric transformations and mock theta functions,* Hiroshima Math. J. **29**(1999), 19-26.

[13] A. V. Sills, An Invitation to the Rogers-Ramanujan identities, Taylor & Francis (2017).

[14] C. Truesdell, An essay toward a unified theory of special functions, Princeton University Press, Princenton (1948).

[15] G. N. Watson, *The final problem: An account of the mock theta functions*, J. London Math. Soc. **11**(1936), 55-80.

[16] G. N. Watson, *The mock theta functions (2)*, Proc. London Math. Soc. **(2)42**(1937), 274-304.

Author information

Pramod Kumar Rawat, Department of Mathematics and Astronomy
University of Lucknow, Lucknow.
226007, India.
E-mail: `pramodkrawat@yahoo.com`

Received: May 26, 2019.
Accepted: August 28, 2019