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Application of $n-$tupled fixed points of contractive type operators for Ulam-Hyers stability

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Abstract. In this paper, we present existence, uniqueness and Ulam-Hyers stability results for the $n-$tupled fixed points of a pair of contractive type singlevalued and respectively multivalued operators on complete metric spaces. The approach is based on Perov type fixed point theorem for contractions in spaces endowed with vector- valued metrics.

1 Introduction and Preliminaries

Since the year 1922, Banach's contraction principle, due to its simplicity and applicability, has became a very popular tool in modern analysis, especially in nonlinear analysis including its applications to differential and integral equations, variational inequality theory, complementarity problems, equilibrium problems, minimization problems and many others. Also, many authors have improved, extended and generalized this contraction principle in several ways. Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [34] further studied by Nieto and Rodriguez -Lopez [33]. Samet and Vetro [43] introduced the notion of fixed point of N order in case of single-valued mappings. For some more work cited in [31, 32, 30].

In 1922, Banach [3] gives following definition of fixed point theorem,

Definition 1.1. An element $x \in X$ is called a fixed point of the mapping $T : X \to X$ if $Tx = x$.

In 2006, T.G. Bhaskar and V. Lakshmikantham [4] introduce the following definition of coupled fixed point

Definition 1.2. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

In 2011, Berinde and Borcut [5] introduce the concept of tripled fixed fixed point which is as follows,

Definition 1.3. An element $(x, y, z) \in X^3$ is called a tripled fixed point of $T : X^3 \to X$ if

$$T(x, y, z) = x, \;\; T(y, x, y) = y, \;\; and \;\; T(z, y, x) = z.$$

In 2011, Karapınar [40] give the following definition,

Definition 1.4. An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $T : X^4 \to X$ if

$$\begin{aligned} T(x, y, z, w) &= x, \;\; T(y, z, w, x) = y, \\ T(z, w, x, y) &= z, \;\; T(w, x, y, z) = w. \end{aligned} \tag{1.1}$$

Beside this, Ertürk and Karakaya [10] following concept of n-tupled fixed point in ordered metric space,

Definition 1.5. Let X be a nonempty set and $T : X^n \to X$ a given mapping. An element $(x_1, x_2, x_3, \ldots, x_n) \in X^n$ is called a $n-$tuplet fixed point of T if

$$\begin{aligned} T(x_1, x_2, x_3, \ldots, x_n) &= x_1, \\ T(x_2, x_3, \ldots, x_n, x_1) &= x_2, \\ &\vdots \\ T(x_n, x_1, x_2, x_3, \ldots, x_{n-1}) &= x_n. \end{aligned}$$

Remark 1.6. We observe following relations in above Definitions 1.1, 1.2, 1.3, 1.4, 1.5,

(i) If we take $x = y$ in Definition 1.2 then Definition 1.2 $\Longrightarrow$ Definition 1.1.

(ii) If we take $x = y = z$ in Definition 1.3 then Definition 1.3 $\Longrightarrow$ Definition 1.1.

(iii) If we take $x = y = z = w$ in Definition 1.4 then Definition 1.4 $\Longrightarrow$ Definition 1.1.

(iv) If we take $x_1 = x_2 = x_3 = \cdots = x_n$ in Definition 1.5 then Definition 1.5 $\Longrightarrow$ Definition 1.1.

It should be noted that through the Banach fixed point [3] technique we cannot solve a system with the following form,

$$\begin{cases} x^2 + 2y + 3 = 0, \\ y^2 + 2x + 3 = 0. \end{cases} \tag{1.2}$$

Above system 1.2 can be solve by using coupled fixed point [4] technique but not applicable to solve following system 1.3,

$$\begin{cases} x^3 + 2yz - 6x + 3 = 0, \\ y^3 + 2xz - 6y + 3 = 0, \\ z^3 + 2yx - 6z + 3 = 0. \end{cases} \tag{1.3}$$

System 1.3 can be solve by using tripled fixed point [5] technique but not applicable to solve following system 1.4,

$$\begin{cases} x^4 + 6yzw - 9x + 12 = & 0, \\ y^4 + 6xzw - 9y + 12 = & 0, \\ z^4 + 6yxw - 9z + 12 = & 0 \\ w^4 + 6yxz - 9w + 12 = & 0. \end{cases} \tag{1.4}$$

System 1.4 can can be solve by using quadrupled fixed point [40] but not applicable to solve following system 1.5,

$$\begin{cases} x_1^n + 2(n-1)\prod_{i=2}^{m} x_i - 3(n-1)x_1 + 4(n-1) = 0, \\ x_2^n + 2(n-1)\prod_{i=1, i\neq 2}^{m} x_i - 3(n-1)x_2 + 4(n-1) = 0, \\ x_3^n + 2(n-1)\prod_{i=1, i\neq 3}^{m} x_i - 3(n-1)x_3 + 4(n-1) = 0, \\ x_4^n + 2(n-1)\prod_{i=1, i\neq 4}^{m} x_i - 3(n-1)x_4 + 4(n-1) = 0, \\ \vdots \\ x_m^n + 2(n-1)\prod_{i=1}^{m-1} x_i - 3(n-1)x_m + 4(n-1) = 0. \end{cases} \tag{1.5}$$

System 1.5 can be solve by $m-$tupled fixed point [10] technique.

On the other way we can say that $n-$tupled fixed point [10] technique is more general than other fixed point theory like coupled [4], tripled [5] and quadrupled [40] in case that when its co-ordinates not equal.

Next we state following definitions and results which are used to prove of our main result.

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(a) $d(x, y) \geq 0$ for all $x, y \in X$,

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

If $x, y \in \mathbb{R}^m, x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, then, by definition: $x \leq y$ if and only if $x_i \leq y_i$ for $i \in \{1, 2, \ldots, m\}$.

A set endowed with a vector-valued metric d is called generalized metric space. The notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix.

Notice that we will make an identification between row and column vectors in $\mathbb{R}^m$.

For the proof of the main results we need the following theorems. A classical result in matrix analysis is the following theorem (see [1], [40], [45]).

Theorem 1.7. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent,*

(i) A is convergent towards zero,

(ii) $A^n \to 0$ as $n \to \infty$,

(iii) The eigenvalues of A are in the open unit disc, i.e $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with

$$det(A - \lambda I) = 0,$$

(iv) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + \cdots + A^n + \ldots. \tag{1.6}$$

(v) *The matrix* $(I - A)$ *is nonsingular and* $(I - A)^{-1}$ *has nonnegative elements*

(vi) $A^n q \to 0$ *and* $qA^n \to 0$ *as* $n \to \infty$, *for each* $q \in \mathbb{R}^m$.

We recall now Perov's fixed point theorem (see [36]).

Theorem 1.8. *Let* (X, d) *be a complete generalized metric space and the operator* $f : X \to X$ *with the property that there exists a matrix* $A \in M_{mm}(\mathbb{R})$ *such that* $d(f(x), f(y)) \leq Ad(x, y)$ *for all* $x, y \in X$. *If* A *is a matrix convergent towards zero, then:*

(i) $Fix(f) = \{x^*\}$ *(Here* $Fix(f)$ *denotes the set of fixed points of* f*),*

(ii) *the sequence of successive approximations* $(x_n)_{n\in\mathbb{N}}, x_n = f^n(x_0)$ *is convergent and has the limit* x^*, *for all* $x_0 \in X$,

(iii) *one has the following estimation*

$$d(x_n, x^*) \leq A^n(I - A)^{-1} d(x_0, x_1), \tag{1.7}$$

(iv) *if* $g : X \to X$ *is an operator such that there exist* $y^* \in Fix(g)$ *and* $\eta \in (\mathbb{R}^+_m)^*$ *with* $d(f(x), g(x)) \leq \eta$, *for each* $x \in X$, *then*

$$d(x^*, y^*) \leq (I - A)^{-1}\eta,$$

(v) *if* $g : X \to X$ *is an operator and there exists* $\eta \in (\mathbb{R}^m_+)^*$ *such that* $d(f(x), g(x)) \leq \eta$, *for all* $x \in X$, *then for the sequence* $y_n = g^n(x_0)$ *we have the following estimation*

$$d(y_n, x^*) \leq (I - A)^{-1}\eta + A^n(I - A)^{-1} d(x_0, x_1). \tag{1.8}$$

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial equations:

$$\begin{aligned} x_1 &= T_1(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ x_2 &= T_2(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ x_3 &= T_3(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ &\vdots \\ x_n &= T_n(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \end{aligned}$$

where $T_i : X^n \to X$ are given n operators where $i = 1, 2, 3, \dots, n$.

By definition, a solution $(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \in X^n$ of the above system is called a $n-$tupled fixed point for the multiple $(T_1, T_2, T_3, \dots, T_n)$. In a similar way, the case of an operatorial inclusion (using the symbol $\in$ instead of $=$) could be considered.

This paper deal with existence and uniqueness of $n-$tupled fixed point theorem the approach is based on Perov-type fixed point theorem for contractions in metric spaces endowed with vector-valued metrics. We are also studying Ulam-Hyers stability results for the $n-$tupled fixed points of a $n-$tupled of contractive type single-valued and respectively multi-valued operators on complete metric spaces. For related results to Perov's fixed point theorem and for some generalizations and applications of it we refer to [7], [11], [39].

2 Existence, uniqueness and stability results for multiple fixed points

Definition 2.1. Let (X, d) be a generalized metric space and $f : X \to X$ be an operator. Then, the fixed point equation

$$x = f(x) \tag{2.1}$$

is said to be generalized Ulam-Hyers stable if there exists an increasing function, $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$, continuous at 0 with $\psi(0) = 0$, such that, for any $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$ for $i \in \{1, \dots, m\}$ and any solution $y^* \in X$ of the in inequality

$$d(y, f(y)) \le \epsilon \tag{2.2}$$

there exists a solution x^* of (2.1) such that

$$d(x^*, y^*) \le \psi(\epsilon) \tag{2.3}$$

In particular, if $\psi(t) = Ct, \quad t \in \mathbb{R}^m_+, \ (where \quad C \in M_{mm}(\mathbb{R}_+))$, then the fixed point equation (2.1) is called Ulam-Hyers stable.

Our first abstract result is a direct consequence of Perov's fixed point theorem.

Theorem 2.2. *Let (X, d) be a generalized metric space and let $f : X \to X$ be an operator with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that A converges to zero and*

$$d(f(x), f(y)) \le Ad(x, y), \textit{ for all } \ x, y \in X.$$

Then the fixed point equation

$$x = f(x), \quad x \in X$$

is Ulam-Hyers stable.

Proof. From Perov's fixed point theorem we get that $Fix(f) = \{x^*\}$. Let $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$ for each $i \in \{1, \dots, m\}$ and let y^* be a solution of the in equation

$$d(y, f(y)) \le \epsilon.$$

Then we successively have that

$$\begin{aligned} d(x^*, y^*) &= d(f(x^*), y^*) \\ &\le d(f(x^*), f(y^*)) + d(f(y^*), y^*) \\ &\le Ad(x^*, y^*) + \epsilon. \end{aligned}$$

Thus, using Theorem 1.8, we get that

$$d(x^*, y^*) \le (I - A)^{-1}\epsilon.$$

□

Definition 2.3. Let (X, d) be a metric space and let $T_i : X^n \to X$ be n operators where $i = 1, 2, 3, \dots, n$. Then the system of operatorial equations

$$\left.\begin{aligned} x_1 &= T_1(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ x_2 &= T_2(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ x_3 &= T_3(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ &\vdots \\ x_{n-1} &= T_{n-1}(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \\ x_n &= T_n(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \end{aligned}\right\} \tag{2.4}$$

where $T_i : X^n \to X$ are n given operators where $i = 1, 2, 3, \dots, n$. is said to be Ulam-Hyers stable if there exist

$$c_{11}, c_{12}, c_{13}, \dots, c_{1n}, c_{21}, c_{22}, c_{23}, \dots, c_{2n}, c_{31}, c_{32}, c_{33}, \dots, c_{3n}, \dots, c_{n1}, c_{n2}, c_{n3}, \dots, c_{nn} > 0$$

such that for each

$$\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_n > 0$$

and each multiple $(u_1^*, u_2^*, u_3^*, \dots, u_n^*) \in X^n$ such that

$$\left.\begin{array}{r} d(u_1^*, T_1(u_1^*, u_2^*, u_3^*, \dots, u_n^*)) \le \epsilon_1 \\ d(u_2^*, T_2(u_1^*, u_2^*, u_3^*, \dots, u_n^*)) \le \epsilon_2 \\ d(u_3^*, T_3(u_1^*, u_2^*, u_3^*, \dots, u_n^*)) \le \epsilon_3 \\ \vdots \\ d(u_n^*, T_n(u_1^*, u_2^*, u_3^*, \dots, u_n^*)) \le \epsilon_n \end{array}\right\} \tag{2.5}$$

there exists a solution $(x_1^*, x_2^*, x_3^*, \dots, x_n^*) \in X^n$ of (2.5) such that

$$\left.\begin{array}{r} d(u_1^*, x_1^*) \le c_{11}\epsilon_1 + c_{12}\epsilon_2 + c_{13}\epsilon_3 + \cdots + c_{1n}\epsilon_n \\ d(u_2^*, x_2^*) \le c_{21}\epsilon_1 + c_{22}\epsilon_2 + c_{23}\epsilon_3 + \cdots + c_{2n}\epsilon_n \\ d(u_3^*, x_3^*) \le c_{31}\epsilon_1 + c_{32}\epsilon_2 + c_{33}\epsilon_3 + \cdots + c_{3n}\epsilon_n \\ \vdots \\ d(u_n^*, x_n^*) \le c_{n1}\epsilon_1 + c_{n2}\epsilon_2 + c_{n3}\epsilon_3 + \cdots + c_{nn}\epsilon_n \end{array}\right\}. \tag{2.6}$$

For examples and other considerations regarding Ulam-Hyers stability and generalized Ulam-Hyers stability of the operatorial equations and inclusions see I.A. Rus [41], Bota-Petruşel [6], Petru-Petruşel-Yao [37].

Our first main result is the following existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the $n-$tupled fixed point of single-valued operators $(T_1, T_2, T_3, \dots, T_n)$. The conclusions (i)-(ii) are originally proved by R. Precup [39], but for the sake of completeness we recall here the whole proof.

Theorem 2.4. *Let (X, d) be a complete metric space, $T_i : X^n \to X$ are n given operators where $i = 1, 2, 3, \dots, n$ such that*

$$\left.\begin{array}{r} d(T_1(x_1, x_2, x_3, \dots, x_n), T_1(u_1, u_2, u_3, \dots, u_n)) \le k_{11}d(x_1, u_1) + k_{12}d(x_2, u_2) + \cdots + k_{1n}d(x_n, u_n) \\ d(T_2(x_1, x_2, x_3, \dots, x_n), T_2(u_1, u_2, u_3, \dots, u_n)) \le k_{21}d(x_1, u_1) + k_{22}d(x_2, u_2) + \cdots + k_{2n}d(x_n, u_n) \\ d(T_3(x_1, x_2, x_3, \dots, x_n), T_3(u_1, u_2, u_3, \dots, u_n)) \le k_{31}d(x_1, u_1) + k_{32}d(x_2, u_2) + \cdots + k_{3n}d(x_n, u_n) \\ \vdots \\ d(T_n(x_1, x_2, x_3, \dots, x_n), T_n(u_1, u_2, u_3, \dots, u_n)) \le k_{n1}d(x_1, u_1) + k_{n2}d(x_2, u_2) + \cdots + k_{nn}d(x_n, u_n) \end{array}\right\} \tag{2.7}$$

for all $(x_1, x_2, x_3, \dots, x_n), (u_1, u_2, u_3, \dots, u_n) \in X^n$. We suppose that

$$A = \begin{pmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & k_{3n} \\ k_{41} & k_{42} & k_{43} & \dots & k_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn} \end{pmatrix}$$

converges to zero. Then,

(i) there exists a unique element $(x_1^*, x_2^*, x_3^*, \dots, x_n^*) \in X^n$ such that

$$\left.\begin{array}{r} x_1^* = T_1(x_1^*, x_2^*, x_3^*, \dots, x_n^*) \\ x_2^* = T_2(x_1^*, x_2^*, x_3^*, \dots, x_n^*) \\ x_3^* = T_3(x_1^*, x_2^*, x_3^*, \dots, x_n^*) \\ \vdots \\ x_n^* = T_n(x_1^*, x_2^*, x_3^*, \dots, x_n^*) \end{array}\right\} \tag{2.8}$$

(ii) *the sequence* $(T_1^p(x_1,x_2,x_3,\ldots,x_n),T_2^p(x_1,x_2,x_3,\ldots,x_n),\ldots,T_n^p(x_1,x_2,x_3,\ldots,x_n)), p \in N$ *converges to* $(x_1^*,x_2^*,x_3^*,\ldots,x_n^*)$ *as* $p \to \infty$*, where*

$$T_1^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) = T_1^p(T_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),T_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,T_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^*$$
$$T_2^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) = T_2^p(T_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),T_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,T_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^*$$
$$T_3^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) = T_3^p(T_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),T_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,T_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^*$$
$$T_n^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) = T_n^p(T_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),T_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,T_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^*$$

for all $p \in N$

(iii) *we have the following estimation:*

$$\begin{pmatrix} d(T_1^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0),x_1^*) \\ d(T_2^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0),x_2^*) \\ d(T_3^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0),x_3^*) \\ \vdots \\ d(T_n^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0),x_n^*) \end{pmatrix} \leq A^n(I-A)^{-1} \begin{pmatrix} d((x_1)_0,T_1((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \\ d((x_2)_0,T_2((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \\ d((x_3)_0,T_3((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \\ \\ d((x_n)_0,T_n((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \end{pmatrix}$$

(iv) *let* $F_i : X^n \to X$ *be* n *operators and there exist* $\eta_i > 0$ *such that* $i = 1,2,3,\ldots,n$ *with*

$$\begin{aligned} d(T_1(x_1,x_2,x_3,\ldots,x_n),F_1(x_1,x_2,x_3,\ldots,x_n)) &\leq \eta_1 \\ d(T_2(x_1,x_2,x_3,\ldots,x_n),F_2(x_1,x_2,x_3,\ldots,x_n)) &\leq \eta_2 \\ d(T_3(x_1,x_2,x_3,\ldots,x_n),F_3(x_1,x_2,x_3,\ldots,x_n)) &\leq \eta_3 \\ &\vdots \\ d(T_n(x_1,x_2,x_3,\ldots,x_n),F_n(x_1,x_2,x_3,\ldots,x_n)) &\leq \eta_n \end{aligned}$$

for all $(x_1,x_2,x_3,\ldots,x_n) \in X^n$*. If* $(a_1^*,a_2^*,a_3^*,\ldots,a_n^*) \in X^n$ *is such that*

$$\left.\begin{aligned} a_1^* &= F_1(a_1^*,a_2^*,a_3^*,\ldots,a_n^*) \\ a_2^* &= F_2(a_1^*,a_2^*,a_3^*,\ldots,a_n^*) \\ a_3^* &= F_3(a_1^*,a_2^*,a_3^*,\ldots,a_n^*) \\ &\vdots \\ a_n^* &= F_n(a_1^*,a_2^*,a_3^*,\ldots,a_n^*) \end{aligned}\right\} \tag{2.11}$$

then

$$\begin{pmatrix} d(a_1^*,x_1^*) \\ d(a_2^*,x_2^*) \\ d(a_3^*,x_3^*) \\ \vdots \\ d(a_n^*,x_n^*) \end{pmatrix} \leq (I-A)^{-1}\eta \tag{2.12}$$

where

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_n \end{pmatrix}$$

(v) let $F_i : X^n \to X$ be n operators and there exist $\eta_i > 0$ such that $i = 1, 2, 3, \ldots, n$ with

$$\left.\begin{aligned} d(T_1(x_1,x_2,x_3,\ldots,x_n), F_1(x_1,x_2,x_3,\ldots,x_n)) &\le \eta_1 \\ d(T_2(x_1,x_2,x_3,\ldots,x_n), F_2(x_1,x_2,x_3,\ldots,x_n)) &\le \eta_2 \\ d(T_3(x_1,x_2,x_3,\ldots,x_n), F_3(x_1,x_2,x_3,\ldots,x_n)) &\le \eta_3 \\ &\vdots \\ d(T_n(x_1,x_2,x_3,\ldots,x_n), F_n(x_1,x_2,x_3,\ldots,x_n)) &\le \eta_n \end{aligned}\right\} \tag{2.13}$$

for all $(x_1, x_2, x_3, \ldots, x_n) \in X^n$. If we consider the sequence

$$(F_1^p(x_1,x_2,x_3,\ldots,x_n), F_2^p(x_1,x_2,x_3,\ldots,x_n)), F_3^p(x_1,x_2,x_3,\ldots,x_n)), \quad p \in N,$$

given by

$$\begin{aligned} F_1^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) &= F_1^p(F_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*), F_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,F_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^* \\ F_2^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) &= F_2^p(F_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*), F_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,F_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^* \\ F_3^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) &= F_3^p(F_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*), F_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,F_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^* \\ \\ F_n^{p+1}(x_1^*,x_2^*,x_3^*,\ldots,x_n^*) &= F_n^p(F_1(x_1^*,x_2^*,x_3^*,\ldots,x_n^*), F_2(x_1^*,x_2^*,x_3^*,\ldots,x_n^*),\ldots,F_n(x_1^*,x_2^*,x_3^*,\ldots,x_n^* \end{aligned}$$

for all $p \in N$ and

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_n \end{pmatrix},$$

then

$$\begin{pmatrix} d(F_1^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0), x_1^*) \\ d(F_2^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0), x_2^*) \\ d(F_3^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0), x_3^*) \\ \vdots \\ d(F_n^p((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n)_0), x_n^*) \end{pmatrix} \le A^n(I-A)^{-1} \begin{pmatrix} d((x_1)_0, F_1((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \\ d((x_2)_0, F_2((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \\ d((x_3)_0, F_3((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \\ \\ d((x_n)_0, F_n((x_1)_0,(x_2)_0,(x_3)_0,\ldots,(x_n \end{pmatrix}$$

(vi) the system of operatorial equations

$$\left.\begin{array}{rl} x_1 = & T_1(x_1, x_2, x_3, \ldots, x_n) \\ x_2 = & T_2(x_1, x_2, x_3, \ldots, x_n) \\ x_3 = & T_3(x_1, x_2, x_3, \ldots, x_n) \\ \vdots & \\ x_n = & T_n(x_1, x_2, x_3, \ldots, x_n) \end{array}\right\} \tag{2.15}$$

is Ulam-Hyers stable.

Proof. For (i)-(ii) let us define $T_i : X^n \to X^n$ where $i = 1, 2, 3, \ldots, n$ by

$$\begin{aligned} T(x_1, x_2, x_3, \ldots, x_n) &= \begin{pmatrix} T_1(x_1, x_2, x_3, \ldots, x_n) \\ T_2(x_1, x_2, x_3, \ldots, x_n) \\ T_3(x_1, x_2, x_3, \ldots, x_n) \\ \vdots \\ T_n(x_1, x_2, x_3, \ldots, x_n) \end{pmatrix} \\ &= (T_1(x_1, x_2, x_3, \ldots, x_n), T_2(x_1, x_2, x_3, \ldots, x_n), \ldots, T_n(x_1, x_2, x_3, \ldots, x_n)). \end{aligned}$$

Denote $Z = X^n$ and consider $\tilde{d} : Z \times Z \to \mathbb{R}^n_+$,

$$\tilde{d}((x_1, x_2, x_3, \ldots, x_n), (u_1, u_2, u_3, \ldots, u_n)) = \begin{pmatrix} d(x_1, u_1) \\ d(x_2, u_2) \\ d(x_3, u_3) \\ \vdots \\ d(x_n, u_n) \end{pmatrix}.$$

Then we have

$$
\begin{aligned}
&\tilde{d}(T(x_1,x_2,x_3,\dots,x_n),T(u_1,u_2,u_3,\dots,u_n)) \\
&= \tilde{d}\left(\begin{pmatrix} T_1(x_1,x_2,x_3,\dots,x_n)\\ T_2(x_1,x_2,x_3,\dots,x_n)\\ T_3(x_1,x_2,x_3,\dots,x_n)\\ \vdots\\ T_n(x_1,x_2,x_3,\dots,x_n)\end{pmatrix}, \begin{pmatrix} T_1(u_1,u_2,u_3,\dots,u_n)\\ T_2(u_1,u_2,u_3,\dots,u_n)\\ T_3(u_1,u_2,u_3,\dots,u_n)\\ \vdots\\ T_n(u_1,u_2,u_3,\dots,u_n)\end{pmatrix}\right) \qquad (2.16)\\
&= \begin{pmatrix} d(T_1(x_1,x_2,x_3,\dots,x_n),T_1(u_1,u_2,u_3,\dots,u_n))\\ d(T_2(x_1,x_2,x_3,\dots,x_n),T_2(u_1,u_2,u_3,\dots,u_n))\\ d(T_3(x_1,x_2,x_3,\dots,x_n),T_3(u_1,u_2,u_3,\dots,u_n))\\ \vdots\\ d(T_n(x_1,x_2,x_3,\dots,x_n),T_n(u_1,u_2,u_3,\dots,u_n))\end{pmatrix}\\
&\le \begin{pmatrix} k_{11}d(x_1,u_1)+k_{12}d(x_2,u_2)+\cdots+k_{1n}d(x_n,u_n)\\ k_{21}d(x_1,u_1)+k_{22}d(x_2,u_2)+\cdots+k_{2n}d(x_n,u_n)\\ \vdots\\ k_{n1}d(x_1,u_1)+k_{n2}d(x_2,u_2)+\cdots+k_{nn}d(x_n,u_n)\end{pmatrix}\\
&= \begin{pmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n}\\ k_{21} & k_{22} & k_{23} & \dots & k_{2n}\\ k_{31} & k_{32} & k_{33} & \dots & k_{3n}\\ k_{41} & k_{42} & k_{43} & \dots & k_{4n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn}\end{pmatrix} \begin{pmatrix} d(x_1,u_1)\\ d(x_2,u_2)\\ d(x_3,u_3)\\ \vdots\\ d(x_n,u_n)\end{pmatrix} \qquad (2.17)\\
&= A\tilde{d}((x_1,x_2,x_3,\dots,x_n),(u_1,u_2,u_3,\dots,u_n)).
\end{aligned}
$$

If we denote $(x_1,x_2,x_3,\dots,x_n)=\alpha$, $(u_1,u_2,u_3,\dots,u_n)=\beta$, we get that

$$\tilde{d}(T(\alpha),T(\beta)) \le A\tilde{d}(\alpha,\beta).$$

Applying Perov's fixed point theorem 1.7 (i), we get that there exists a unique element $(x_1^*,x_2^*,x_3^*,\dots,x_n^*) \in X^n$ such that

$$(x_1^*,x_2^*,x_3^*,\dots,x_n^*) = T(x_1^*,x_2^*,x_3^*,\dots,x_n^*)$$

and is equivalent with

$$\left.\begin{aligned} x_1^* &= T_1(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\\ x_2^* &= T_2(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\\ x_3^* &= T_3(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\\ &\vdots\\ x_n^* &= T_n(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\end{aligned}\right\}.$$

Moreover, for each $\alpha \in X^n$, we have that $T(\alpha) \to \alpha^*$ as $p \to \infty$, where

$$
\begin{aligned}
T^0(\alpha) &= \alpha,\\
T^1(\alpha) &= T(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\\
&= (T_1(x_1^*,x_2^*,x_3^*,\dots,x_n^*),T_2(x_1^*,x_2^*,x_3^*,\dots,x_n^*),\dots,T_n(x_1^*,x_2^*,x_3^*,\dots,x_n^*))
\end{aligned}
$$

$$\begin{aligned} T^2(\alpha) &= T(T_1(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), T_2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \ldots, T_n(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ &= (T_1^2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), T_2^2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \ldots, T_n^2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \end{aligned}$$

and generally

$$\left.\begin{aligned} T_1^{p+1}(\alpha) &= T_1^p(T_1(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), T_2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \ldots, T_n(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ T_2^{p+1}(\alpha) &= T_2^p(T_1(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), T_2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \ldots, T_n(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ T_3^{p+1}(\alpha) &= T_3^p(T_1(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), T_2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \ldots, T_n(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ &\vdots \\ T_n^{p+1}(\alpha) &= T_n^p(T_1(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), T_2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \ldots, T_n(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \end{aligned}\right\} \tag{2.18}$$

We obtain that

$$T(\alpha) = (T_1(\alpha), T_2(\alpha), T_3(\alpha), \ldots, T_n(\alpha)) \to \alpha^* = (x_1^*, x_2^*, x_3^*, \ldots, x_n^*) \text{ as } p \to \infty,$$

for all $\alpha = (x_1, x_2, x_3, \ldots, x_n) \in X^n$. So, for all $(x_1, x_2, x_3, \ldots, x_n) \in X^n$, we have that

$$\left.\begin{aligned} T_1(x_1, x_2, x_3, \ldots, x_n) &\to x_1^* \text{ as } p \to \infty \\ T_2(x_1, x_2, x_3, \ldots, x_n) &\to x_2^* \text{ as } p \to \infty \\ T_3(x_1, x_2, x_3, \ldots, x_n) &\to x_3^* \text{ as } p \to \infty \\ &\vdots \\ T_n(x_1, x_2, x_3, \ldots, x_n) &\to x_n^* \text{ as } p \to \infty \end{aligned}\right\}. \tag{2.19}$$

(iii) By Perov's theorem (iii) we successively have

$$\begin{aligned} &\begin{pmatrix} d(T_1^p((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0), x_1^*) \\ d(T_2^p((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0), x_2^*) \\ d(T_3^p((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0), x_3^*) \\ \vdots \\ d(T_n^p((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0), x_n^*) \end{pmatrix} \\ &= \tilde{d}((T^p((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0)), (x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ &\leq A^p(I-A)^{-1} \begin{pmatrix} d((x_1)_0, T_1((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0)) \\ d((x_2)_0, T_2((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0)) \\ d((x_3)_0, T_3((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0)) \\ \vdots \\ d((x_n)_0, T_n((x_1)_0, (x_2)_0, (x_3)_0, \ldots, (x_n)_0)) \end{pmatrix}. \end{aligned}$$

(iv) If we consider $F : X^n \to X^n$ such that

$$F(x_1,x_2,x_3,\dots,x_n) = \begin{pmatrix} F_1(x_1,x_2,x_3,\dots,x_n) \\ F_2(x_1,x_2,x_3,\dots,x_n) \\ F_3(x_1,x_2,x_3,\dots,x_n) \\ \vdots \\ F_n(x_1,x_2,x_3,\dots,x_n) \end{pmatrix} \tag{2.20}$$

and

$$\begin{aligned} &\tilde{d}(T(x_1,x_2,x_3,\dots,x_n), F(x_1,x_2,x_3,\dots,x_n)) \\ = \quad &\tilde{d}\left(\begin{pmatrix} T_1(x_1,x_2,x_3,\dots,x_n) \\ T_2(x_1,x_2,x_3,\dots,x_n) \\ T_3(x_1,x_2,x_3,\dots,x_n) \\ \vdots \\ T_n(x_1,x_2,x_3,\dots,x_n) \end{pmatrix}, \begin{pmatrix} F_1(x_1,x_2,x_3,\dots,x_n) \\ F_2(x_1,x_2,x_3,\dots,x_n) \\ F_3(x_1,x_2,x_3,\dots,x_n) \\ \vdots \\ F_n(x_1,x_2,x_3,\dots,x_n) \end{pmatrix} \right) \\ = \quad &\begin{pmatrix} d(T_1(x_1,x_2,x_3,\dots,x_n), F_1(x_1,x_2,x_3,\dots,x_n)) \\ d(T_2(x_1,x_2,x_3,\dots,x_n), F_2(x_1,x_2,x_3,\dots,x_n)) \\ d(T_3(x_1,x_2,x_3,\dots,x_n), F_3(x_1,x_2,x_3,\dots,x_n)) \\ \vdots \\ d(T_n(x_1,x_2,x_3,\dots,x_n), F_n(x_1,x_2,x_3,\dots,x_n)) \end{pmatrix} \\ \leq \quad &\eta \end{aligned} \tag{2.21}$$

then, applying Perov‘s fixed point theorem 1.8 (iv) we get

$$\tilde{d}((x_1^*,x_2^*,x_3^*,\dots,x_n^*),(a_1^*,a_2^*,a_3^*,\dots,a_n^*)) \leq (I-A)^{-1}\eta. \tag{2.22}$$

(v) By (2.21) we get that

$$\tilde{d}(T(x_1,x_2,x_3,\dots,x_n), F(x_1,x_2,x_3,\dots,x_n)) \leq \eta.$$

Notice that

$$F^p(x_1,x_2,x_3,\dots,x_n) = F(F^{p-1}(x_1,x_2,x_3,\dots,x_n)),$$

for all $(x_1,x_2,x_3,\dots,x_n) \in X^n$.

Using the assertion (iii) of this theorem, we can successively write:

$$\begin{aligned} &\tilde{d}(F^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)),(x_1^*,x_2^*,x_3^*,\dots,x_n^*)) \\ \leq \quad &\tilde{d}(F^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)) \\ &+\tilde{d}(T^p(x_0,y_0,z_0),(x_1^*,x_2^*,x_3^*,\dots,x_n^*)) \\ \leq \quad &\tilde{d}(F^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)) \\ &+A^p(I-A)^{-1}\tilde{d}(T((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \tilde{d}(F^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)) \\
= & \tilde{d}(F(F^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T(T^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)) \\
\leq & \tilde{d}(F(F^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T(F^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)) \\
& +\tilde{d}(T(F^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T(T^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)) \\
\leq & \eta + A\tilde{d}((F^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T^{p-1}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)) \\
\leq & \eta + A[\eta + A.\tilde{d}((F^{p-2}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),T^{p-2}((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0))] \\
\leq & \cdots \leq \eta(I + A + A^2 + \cdots + A^p + \dots) \\
\leq & \eta(I-A)^{-1}.
\end{aligned} \tag{2.23}$$

Thus, we finally get the conclusion

$$\begin{aligned}
& \tilde{d}(F^p((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),(x_1^*,x_2^*,x_3^*,\dots,x_n^*)) \\
\leq & \eta(I-A)^{-1} + A^p(I-A)^{-1}\tilde{d}(T((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0),((x_1)_0,(x_2)_0,(x_3)_0,\dots,(x_n)_0)).
\end{aligned}$$

(vi) By (i) and (ii) there exists a unique element $(x_1^*,x_2^*,x_3^*,\dots,x_n^*) \in X^n$ such that $(x_1^*,x_2^*,x_3^*,\dots,x_n^*)$ is a solution for (2.15) and the sequence

$$(T_1^p(x_1,x_2,x_3,\dots,x_n),T_2^p(x_1,x_2,x_3,\dots,x_n),\dots,T_n^p(x_1,x_2,x_3,\dots,x_n)) \to (x_1^*,x_2^*,x_3^*,\dots,x_n^*) \quad as \quad n \to$$

Let $\epsilon_i > 0$ where $i = 1,2,3,\dots n$ and $(u_1^*,u_2^*,u_3^*,\dots,u_n^*) \in X^n$ such that

$$\left.\begin{aligned}
d(u_1^*,T_1(u_1^*,u_2^*,u_3^*,\dots,u_n^*)) \leq \epsilon_1 \\
d(u_2^*,T_2(u_1^*,u_2^*,u_3^*,\dots,u_n^*)) \leq \epsilon_2 \\
d(u_3^*,T_3(u_1^*,u_2^*,u_3^*,\dots,u_n^*)) \leq \epsilon_3 \\
\vdots \\
d(u_n^*,T_n(u_1^*,u_2^*,u_3^*,\dots,u_n^*)) \leq \epsilon_n
\end{aligned}\right\}. \tag{2.24}$$

Then we have

$$
\begin{aligned}
&\tilde{d}((u_1^*, u_2^*, u_3^*, \ldots, u_n^*), (x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\
\leq\ & \tilde{d}((u_1^*, u_2^*, u_3^*, \ldots, u_n^*), (T_1(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T_2(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), \ldots, T_n(u_1^*, u_2^*, u_3^*, \ldots, u_n^*))) \\
&+\tilde{d}((T_1(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T_2(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), \ldots, T_3(u_1^*, u_2^*, u_3^*, \ldots, u_n^*)), (x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\
=\ & \tilde{d}((u_1^*, u_2^*, u_3^*, \ldots, u_n^*), (T_1(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T_2(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), \ldots, T_n(u_1^*, u_2^*, u_3^*, \ldots, u_n^*))) \\
&+\tilde{d}\left(\begin{pmatrix} T_1(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), \\ T_2(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), \\ \vdots \\ T_n(u_1^*, u_2^*, u_3^*, \ldots, u_n^*) \end{pmatrix}, \begin{pmatrix} T_1(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \\ T_2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*), \\ \vdots \\ T_n(x_1^*, x_2^*, x_3^*, \ldots, x_n^*) \end{pmatrix}\right) \\
=\ & \begin{pmatrix} d(u_1^*, T_1(u_1^*, u_2^*, u_3^*, \ldots, u_n^*)) \\ d(u_2^*, T_2(u_1^*, u_2^*, u_3^*, \ldots, u_n^*)) \\ d(u_3^*, T_3(u_1^*, u_2^*, u_3^*, \ldots, u_n^*)) \\ \vdots \\ d(u_n^*, T_n(u_1^*, u_2^*, u_3^*, \ldots, u_n^*)) \end{pmatrix} + \begin{pmatrix} d(T_1(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T_1(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ d(T_2(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T_2(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ d(T_3(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T_3(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\ \vdots \\ d(T_n(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T_n(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \end{pmatrix} \\
\leq\ & \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_n \end{pmatrix} + \tilde{d}(T(u_1^*, u_2^*, u_3^*, \ldots, u_n^*), T(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \\
\leq\ & \epsilon + A\tilde{d}((u_1^*, u_2^*, u_3^*, \ldots, u_n^*), (x_1^*, x_2^*, x_3^*, \ldots, x_n^*)).
\end{aligned}
$$

Since $(I - A)$ is invertible and $(I - A)^{-1}$ has positive elements, we immediately obtain

$$
\tilde{d}((u_1^*, u_2^*, u_3^*, \ldots, u_n^*), (x_1^*, x_2^*, x_3^*, \ldots, x_n^*)) \leq (I - A)^{-1}\epsilon
$$

or equivalently

$$
\begin{pmatrix} d(u_1^*, x_1^*) \\ d(u_2^*, x_2^*) \\ d(u_3^*, x_3^*) \\ \vdots \\ d(u_n^*, x_n^*) \end{pmatrix} \leq (I - A)^{-1}\epsilon.
$$

If we denote

$$
(I - A)^{-1} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \ldots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \ldots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \ldots & c_{3n} \\ c_{41} & c_{42} & c_{43} & \ldots & c_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \ldots & c_{nn} \end{pmatrix},
$$

then we obtain

$$\left.\begin{array}{r}d(u_1^*, x_1^*) \leq c_{11}\epsilon_1 + c_{12}\epsilon_2 + c_{13}\epsilon_3 + \cdots + c_{1n}\epsilon_n \\ d(u_2^*, x_2^*) \leq c_{21}\epsilon_1 + c_{22}\epsilon_2 + c_{23}\epsilon_3 + \cdots + c_{2n}\epsilon_n \\ d(u_3^*, x_3^*) \leq c_{31}\epsilon_1 + c_{32}\epsilon_2 + c_{33}\epsilon_3 + \cdots + c_{3n}\epsilon_n \\ \vdots \\ d(u_n^*, x_n^*) \leq c_{n1}\epsilon_1 + c_{n2}\epsilon_2 + c_{n3}\epsilon_3 + \cdots + c_{nn}\epsilon_n\end{array}\right\} \tag{2.25}$$

proving that the operatorial system (2.15) is Ulam-Hyers stable. □

Definition 2.5. Let X be a nonempty set. An element $x = (x_1, x_2, \ldots, x_n) \in X^n$, $n \geq 0$, is said to be a fixed point of m-order of a mapping $F : X^n \to X$ if

$$\left.\begin{array}{r}T(x_1, x_2, x_3, x_4 \ldots, x_{n-2}, x_{n-1}, x_n) = x_1 \\ T(x_2, x_3, x_4, x_5 \ldots, x_{n-1}, x_n, x_1) = x_2 \\ T(x_3, x_4, x_5, x_6 \ldots, x_n, x_1, x_2) = x_3 \\ \vdots \\ T(x_n, x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}) = x_n\end{array}\right\} \tag{2.26}$$

Observe that 2.26 can be written as

$$T(t_i(x)) = x_i, \quad for \quad all \quad i \in \{1, 2, 3, \ldots, n\}, \tag{2.27}$$

where t_i is the $i-$th line of the circular matrix of x,

$$t(x) = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\ x_2 & x_3 & x_4 & \ldots & x_n & x_1 \\ x_3 & x_4 & x_5 & \ldots & x_1 & x_2 \\ x_4 & x_5 & x_6 & \ldots & x_2 & x_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \end{pmatrix}, \tag{2.28}$$

Remark 2.6. Notice that, if (X, d) is a metric space and $T : X^n \to X$ is an operator and we define

$$T_1(x_1, x_2, x_3, x_4 \ldots, x_{n-2}, x_{n-1}, x_n) = T(t_1(x)),$$

$$T_2(x_1, x_2, x_3, x_4 \ldots, x_{n-2}, x_{n-1}, x_n) = T(t_2(x)),$$

$$T_3(x_1, x_2, x_3, x_4 \ldots, x_{n-2}, x_{n-1}, x_n) = T(t_3(x)),$$

and so on,

$$T_n(x_1, x_2, x_3, x_4 \ldots, x_{n-2}, x_{n-1}, x_n) = T(t_n(x)),$$

then the above approach leads to some well-known n-tupled fixed point theorems.

We will consider now the case of multi-valued operators. We need first some notations. Let (X, d) be a generalized metric space with $d : X \times X \to \mathbb{R}_+^m$ given by

$$d(x, y) = \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_m(x, y) \end{pmatrix}.$$

Then, for $x \in X$ and $A \subseteq X$ we denote:

$$D_d(x,A) = \begin{pmatrix} D_{d_1}(x,A) \\ \vdots \\ D_{d_m}(x,A) \end{pmatrix} = \begin{pmatrix} \inf_{a \in A} d_1(x,a) \\ \vdots \\ \inf_{a \in A} d_m(x,a) \end{pmatrix}.$$

$$P(X) = \{Y \subseteq X | Y \text{is nonempty}\}$$
$$P_{cl}(X) = \{Y \subseteq P(X) | Y \text{closed}\}.$$

We also denote

$$D((x_1, x_2, x_3, x_4 \ldots, x_n), A_1 \times A_2 \times \cdots \times A_n) = \begin{pmatrix} D_d(x_1, A_1) \\ D_d(x_2, A_2) \\ D_d(x_3, A_3) \\ \vdots \\ D_d(x_n, A_n) \end{pmatrix}.$$

Our second main result is an existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the $n-$tupled fixed point of a triple of multi-valued operators (T_1, T_2, T_3). For the proof of our main result, we give the following theorem.

Theorem 2.7. *Let (X,d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multi-valued $A-$contraction, i.e. there exists $A \in M_{mm}(\mathbb{R}_+)$ which converges towards zero as $p \to \infty$ and for each $x, y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $d(u,v) \leq A.d(x,y)$. Then T is a $MWP-$operator, i.e. $Fix(T) \neq \phi$, and for each $(x,y) \in Graph(T)$ there exists a sequence $(x_n)_{n \in N}$ of succesive approximations for T starting from (x,y) which converges to a fixed point x^* of T. Moreover $d(x,x^*) \leq (I-A)^{-1} d(x,y)$, for all $(x,y) \in Graph(T)$.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Then by the $A-$contraction condition, there exists $x_2 \in T(x_1)$ such that $d(x_1,x_2) \leq Ad(x_0,x_1)$. Now, for $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that

$$d(x_2,x_3) \leq Ad(x_1,x_2) \leq A^2 d(x_0,x_1).$$

In this way, by an iterative construction, we get a sequence $(x_n)_{n \in N}$ such that

$$\left.\begin{aligned} x_0 &\in X \\ x_{n+1} &\in T(x_n) \\ d(x_n,x_{n+1}) &\leq A^n d(x_0,x_1) \end{aligned}\right\}.$$

for all $n \in N$.

Thus, by the above relation, we get

$$\begin{aligned} d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{p-1},x_{n+p}) \\ &\leq Ad(x_0,x_1) + A^2 d(x_0,x_1) + \cdots + A^{p-1} d(x_0,x_1) \\ &= A^n(I + A + \cdots + A^{n+p-1}) d(x_0,x_1) \end{aligned}$$

Letting $n \to \infty$ we get that the sequence $(x_n)_{n \in N}$ is Cauchy. Hence there exists $x^* \in X$ such that $x^* = \lim_{n \to \infty} x_n$.

We prove that $x^* \in T(x^*)$. Indeed, for $x_n \in T(x_{n-1})$ there exists $u_n \in T(x^*)$ such that

$$d(x_n,u_n) \leq Ad(x_{n-1},x^*),$$

for all $n \in N$.

On the other side

$$d(x^*, u_n) \leq d(x^*, x_n) + d(x_n, u_n) \leq d(x^*, x_n) + Ad(x_{n-1}, x^*) \to 0, \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty} u_n = x^*$. But $u_n \in T(x^*)$, for $n \in N$ and because $T(x^*)$ is closed, we have that $x^* \in T(x^*)$.

Moreover we can write

$$d(x_n, x_{n+p}) \leq A^n(I + A + \cdots + A^{p-1} + \dots).d(x_0, x_1) = A^n(I - A)-1d(x_0, x_1).$$

Letting $p \to \infty$ we get that

$$d(x_n, x^*) \leq A^n(I - A)^{-1}d(x_0, x_1).$$

for all $n \geq 1$. Thus

$$\begin{aligned} d(x_0, x^*) &\leq d(x_0, x_1) + d(x_1, x^*) \leq d(x_0, x_1) + A(I - A)^{-1}d(x_0, x_1) \\ &= (I + A(I - A)^{-1})d(x_0, x_1) = (I + A + A^2 + \dots)d(x_0, x_1) \\ &= (I - A)^{-1}d(x_0, x_1) \end{aligned}$$

□

Definition 2.8. Let (X, d) generalized metric space and $F : X \to P(X)$. The fixed point inclusion

$$x \in F(x), x \in X \tag{2.29}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ increasing, continuous at 0 with $\psi(0) = 0$ such that for each $\epsilon = (\epsilon_1, \dots, \epsilon_m) > 0$ and for each $\epsilon-$solution y^* of 2.29, i.e.

$$D_d(y^*, F(y^*)) \leq \epsilon$$

there exists a solution x^* of the fixed point inclusion (2.29) such that

$$d(y^*, x^*) \leq \psi(\epsilon).$$

In particular, if $\psi(t) = Ct$, for each $t \in \mathbb{R}^m_+$ (where $C \in M_{mm}(\mathbb{R}_+)$), then 2.29 is said to be Ulam-Hyers stable.

Definition 2.9. A subset U of a generalized metric space (X, d) is called proximinal if for each $x \in X$ there exists $u \in U$ such that $d(x, u) = D_d(x, U)$.

Theorem 2.10. *Let (X, d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multi-valued $A-$contraction with proximinal values. Then, the fixed point inclusion (2.29) is Ulam-Hyers stable.*

Proof. Let $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_1 > 0$, for each $i \in (1, 2, \dots, m)$ and let $y^* \in X$ an $\epsilon-$solution of (2.29), i.e.,

$$D_d(y^*, F(y^*)) \leq \epsilon.$$

By the second conclusion of Theorem 2.7 we have that for any $(x, y) \in Graph(T)$

$$d(x, x^*(x, y)) \leq (I - A)^{-1}d(x, y), \tag{2.30}$$

where $x^*(x, y)$ denotes the fixed point of F which is obtained by Theorem 2.7 by successive approximations starting from (x, y).

Since $T(y^*)$ is proximinal there exists $u \in T(y^*)$ such that

$$d(y^*, u^*) = D_d(y, T(y^*)).$$

Hence, by 2.30

$$d(y^*, x^*(y^*, u^*)) \leq (I - A)^{-1} d(y^*, u^*) \leq (I - A)^{-1}\epsilon.$$

□

Theorem 2.11. *Let (X, d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multi-valued $A-$contraction such that there exists $x^* \in X$ with $T(x^*) = \{x^*\}$. Then the fixed point inclusion (2.29) is Ulam-Hyers stable.*

Proof. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ with $\epsilon_i > 0$, for each $i \in (1, 2, \ldots, m)$ and let $y^* \in X$ an $\epsilon-$solution of (2.29), i.e.,

$$D_d(y^*, T(y^*)) \leq \epsilon.$$

By the $A-$contraction condition, for $x = y^*, y = x^*$ and $u \in T(y^*)$ we get that

$$d(u, x^*) \leq Ad(y^*, x^*).$$

Then, for any $u \in T(y^*)$ we have

$$d(y^*, x^*) \leq d(y^*, u) + d(u, x^*) \leq d(y^*, u) + A.d(y^*, x^*).$$

Hence

$$d(y^*, x^*) \leq (I - A)^{-1} d(y^*, u),$$

for any $u \in T(y^*)$. Thus

$$d(y^*, x^*) \leq (I - A)^{-1} D_d(y^*, T(y^*)) \leq (I - A)^{-1}\epsilon.$$

□

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial inclusions:

$$\left.\begin{aligned} x_1 &\in T_1(x_1, x_2, x_3, \ldots, x_n) \\ x_2 &\in T_2(x_1, x_2, x_3, \ldots, x_n) \\ x_3 &\in T_3(x_1, x_2, x_3, \ldots, x_n) \\ &\vdots \\ x_n &\in T_n(x_1, x_2, x_3, \ldots, x_n) \end{aligned}\right\} \tag{2.31}$$

where $T_i : X^n \to P(X)$ where $i = 1, 2, 3, \ldots, n$ are n given multi-valued operators. By definition, a solution $(x_1, x_2, x_3, \ldots, x_n) \in X^n$ of the above system is called a n fixed point for $(T_1, T_2, T_3, \ldots, T_n)$.

Definition 2.12. Let (X, d) be a metric space and let $T_i : X^n \to P(X)$ where $i = 1, 2, 3, \ldots, n$ are n multi-valued operators. Then the operatorial inclusions system (2.31) is said to be Ulam-Hyers stable if there exist

$$c_{11}, c_{12}, c_{13}, \ldots, c_{1n}, c_{21}, c_{22}, c_{23}, \ldots, c_{2n}, c_{31}, c_{32}, c_{33}, \ldots, c_{3n}, \ldots, c_{n1}, c_{n2}, c_{n3}, \ldots, c_{nn} > 0$$

such that for each

$$\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_n > 0$$

and each triple $(u_1^*, u_2^*, u_3^*, \ldots, u_n^*) \in X^n$ which satisfies the relations

$$\left.\begin{aligned} d(u_1^*, \alpha_1) &\leq \epsilon_1 \text{ for all } \alpha \in T_1(u_1^*, u_2^*, u_3^*, \ldots, u_n^*) \\ d(u_2^*, \alpha_2) &\leq \epsilon_2 \text{ for all } \alpha \in T_2(u_1^*, u_2^*, u_3^*, \ldots, u_n^*) \\ d(u_3^*, \alpha_3) &\leq \epsilon_3 \text{ for all } \alpha \in T_3(u_1^*, u_2^*, u_3^*, \ldots, u_n^*) \\ &\vdots \\ d(u_n^*, \alpha_n) &\leq \epsilon_n \text{ for all } \alpha \in T_n(u_1^*, u_2^*, u_3^*, \ldots, u_n^*) \end{aligned}\right\} \tag{2.32}$$

there exists a solution $(x^*, y^*, z^*) \in X \times X \times X$ of (2.31) such that

$$\left.\begin{array}{r} d(u_1^*, x_1^*) \leq c_{11}\epsilon_1 + c_{12}\epsilon_2 + c_{13}\epsilon_3 + \cdots + c_{1n}\epsilon_n \\ d(u_2^*, x_2^*) \leq c_{21}\epsilon_1 + c_{22}\epsilon_2 + c_{23}\epsilon_3 + \cdots + c_{2n}\epsilon_n \\ d(u_3^*, x_3^*) \leq c_{31}\epsilon_1 + c_{32}\epsilon_2 + c_{33}\epsilon_3 + \cdots + c_{3n}\epsilon_n \\ \vdots \\ d(u_n^*, x_n^*) \leq c_{n1}\epsilon_1 + c_{n2}\epsilon_2 + c_{n3}\epsilon_3 + \cdots + c_{nn}\epsilon_n \end{array}\right\} \tag{2.33}$$

Definition 2.13. Let (X, d) be a metric space, we say that $S : X^n \to P(X)$ has proximinal values with respect to the first variable if for any $x_1, x_2, x_3, \ldots, x_n \in X$ there exists $u_1 \in S(x_1, x_2, x_3, \ldots, x_n)$ such that

$$d(x_1, u_1) = D_d(x_1, S(x_1, x_2, x_3, \ldots, x_n)).$$

Definition 2.14. Let (X, d) be a metric space, we say that $S : X^n \to P(X)$ has proximinal values with respect to the second variable if for any $x_1, x_2, x_3, \ldots, x_n \in X$ there exists $u_2 \in S(x_1, x_2, x_3, \ldots, x_n)$ such that

$$d(x_2, u_2) = D_d(x_2, S(x_1, x_2, x_3, \ldots, x_n)).$$

Definition 2.15. Let (X, d) be a metric space, we say that $S : X^n \to P(X)$ has proximinal values with respect to the third variable if for any $x_1, x_2, x_3, \ldots, x_n \in X$ there exists $u_3 \in S(x_1, x_2, x_3, \ldots, x_n)$ such that

$$d(x_3, u_3) = D_d(x_3, S(x_1, x_2, x_3, \ldots, x_n)).$$

Similarly we can say that

Definition 2.16. Let (X, d) be a metric space, we say that $S : X^n \to P(X)$ has proximinal values with respect to the n^{th} variable if for any $x_1, x_2, x_3, \ldots, x_n \in X$ there exists $u_n \in S(x_1, x_2, x_3, \ldots, x_n)$ such that

$$d(x_n, u_n) = D_d(x_n, S(x_1, x_2, x_3, \ldots, x_n)).$$

Now we are in the position to give our next main results.

Theorem 2.17. *Let (X, d) be a complete metric space and let $T_i : X^n \to P_{cl}(X)$ where $i = 1, 2, 3, \ldots, n$ be n multi-valued operators. Suppose that T_1 has proximinal values with respect to the first variable, T_2 with respect to the second variable and similarly T_n with respect to the n variable. For each $(x_1, x_2, x_3, \ldots, x_n), (u_1, u_2, u_3, \ldots, u_n) \in X^n$ and each*

$$\alpha_1 \in T_1(x_1, x_2, x_3, \ldots, x_n), \quad \alpha_2 \in T_2(x_1, x_2, x_3, \ldots, x_n),$$

$$\alpha_3 \in T_3(x_1, x_2, x_3, \ldots, x_n), \quad \ldots \quad, \alpha_n \in T_n(x_1, x_2, x_3, \ldots, x_n)$$

there exist

$$\beta_1 \in T_1(u_1, u_2, u_3, \ldots, u_n), \quad \beta_2 \in T_2(u_1, u_2, u_3, \ldots, u_n),$$

$$\beta_3 \in T_3(u_1, u_2, u_3, \ldots, u_n), \quad \ldots \quad \beta_n \in T_n(u_1, u_2, u_3, \ldots, u_n),$$

satisfying

$$\left.\begin{array}{r} d(\alpha_1, \beta_1) \leq k_{11}d(x_1, u_1) + k_{12}d(x_2, u_2) + k_{13}d(x_3, u_3) + \cdots + k_{1n}d(x_n, u_n) \\ d(\alpha_2, \beta_2) \leq k_{21}d(x_1, u_1) + k_{22}d(x_2, u_2) + k_{23}d(x_3, u_3) + \cdots + k_{2n}d(x_n, u_n) \\ d(\alpha_3, \beta_3) \leq k_{31}d(x_1, u_1) + k_{32}d(x_2, u_2) + k_{33}d(x_3, u_3) + \cdots + k_{3n}d(x_n, u_n) \\ \vdots \\ d(\alpha_n, \beta_n) \leq k_{n1}d(x_1, u_1) + k_{n2}d(x_2, u_2) + k_{n3}d(x_3, u_3) + \cdots + k_{nn}d(x_n, u_n) \end{array}\right\}$$

We suppose that

$$A = \begin{pmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \dots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \dots & k_{3n} \\ k_{41} & k_{42} & k_{43} & \dots & k_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn} \end{pmatrix}$$

converges to zero. Then,

(i) there exists $(x_1^*, x_2^*, x_3^*, \dots, x_n^*) \in X^n$ *a solution for (2.31).*

(ii) the operatorial system (2.31) is Ulam-Hyers stable.

Proof. (i)-(ii) Let us define $T : X^n \to (P_{cl}(X))^n$ by

$T(x_1, x_2, x_3, \dots, x_n) = T_1(x_1, x_2, x_3, \dots, x_n) \times T_2(x_1, x_2, x_3, \dots, x_n) \times \dots \times T_n(x_1, x_2, x_3, \dots, x_n)$.

Denote $\Gamma = X^n$ and consider $\tilde{d} : \Gamma \times \Gamma \to \mathbb{R}_+^n$,

$$\tilde{d}((x_1, x_2, x_3, \dots, x_n), (u_1, u_2, u_3, \dots, u_n)) = \begin{pmatrix} d(x_1, u_1) \\ d(x_2, u_2) \\ d(x_3, u_3) \\ \vdots \\ d(x_n, u_n) \end{pmatrix}.$$

Then, from the hypotheses of the theorem, we get that for each $s = (x_1, x_2, x_3, \dots, x_n), t = (u_1, u_2, u_3, \dots, u_n) \in X^n$ and each $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n) \in T(x_1, x_2, x_3, \dots, x_n)$, there exists $\beta = (\beta_1, \beta_2, \beta_3, \dots, \beta_n) \in T(u_1, u_2, u_3, \dots, u_n)$ satisfying the relation

$$\tilde{d}(\alpha, \beta) \leq A\tilde{d}(s, t),$$

which proves that T is a multi-valued $A-$contraction. Since $T_1(x_1, x_2, x_3, \dots, x_n) \leq X$ is proximinal with respect to the first variable we have that, for any $(x_1, x_2, x_3, \dots, x_n) \in X$ there exists $u_1 \in T_1(x_1, x_2, x_3, \dots, x_n)$ such that

$$d(x_1, u_1) = D_d(x_1, T_1(x_1, x_2, x_3, \dots, x_n)).$$

Since $T_2(x_1, x_2, x_3, \dots, x_n) \subset X$ is proximinal with respect to the second variable we get that, for any $(x_1, x_2, x_3, \dots, x_n) \in X$ there exists $u_2 \in T_2(x_1, x_2, x_3, \dots, x_n)$ such that

$$d(x_2, u_2) = D_d(x_2, T_2(x_1, x_2, x_3, \dots, x_n)).$$

Since $T_3(x_1, x_2, x_3, \dots, x_n) \subset X$ is proximinal with respect to the third variable we get that, for any $(x_1, x_2, x_3, \dots, x_n) \in X$ there exists $u_3 \in T_3(x_1, x_2, x_3, \dots, x_n)$ such that

$$d(x_3, u_3) = D_d(x_3, T_3(x_1, x_2, x_3, \dots, x_n)).$$

Similarly, $T_n(x_1, x_2, x_3, \dots, x_n) \subset X$ is proximinal with respect to the n^{th} variable we get that, for any $(x_1, x_2, x_3, \dots, x_n) \in X$ there exists $u_n \in T_n(x_1, x_2, x_3, \dots, x_n)$ such that

$$d(x_n, u_n) = D_d(x_n, T_n(x_1, x_2, x_3, \dots, x_n)).$$

Then the set

$T(x_1, x_2, x_3, \dots, x_n) = T_1(x_1, x_2, x_3, \dots, x_n) \times T_2(x_1, x_2, x_3, \dots, x_n) \times \dots \times T_n(x_1, x_2, x_3, \dots, x_n)$

is proximinal, since for any $(x_1, x_2, x_3, \dots, x_n) \in X$ there exists $(u_1, u_2, u_3, \dots, u_n) \in T(x_1, x_2, x_3, \dots, x_n)$ such that

$$\tilde{d}((x_1, x_2, x_3, \dots, x_n), (u_1, u_2, u_3, \dots, u_n)) = D_{\tilde{d}}((x_1, x_2, x_3, \dots, x_n), T(x_1, x_2, x_3, \dots, x_n)).$$

The conclusions follow now from Theorem 2.7 and Theorem 2.10. □

Theorem 2.18. *Let (X,d) be a complete metric space and let $T_i : X^n \to P_{cl}(X)$ where $i = 1,2,3,\dots,n$ be n multi-valued operators. Suppose there exist $x_1,x_2,x_3,\dots,x_n \in X$ such that*

$$\left.\begin{aligned} T_1(x_1^*,x_2^*,x_3^*,\dots,x_n^*) &= \{x_1^*\},\\ T_2(x_1^*,x_2^*,x_3^*,\dots,x_n^*) &= \{x_2^*\},\\ T_3(x_1^*,x_2^*,x_3^*,\dots,x_n^*) &= \{x_3^*\}\\ &\vdots\\ T_n(x_1^*,x_2^*,x_3^*,\dots,x_n^*) &= \{x_n^*\}.\end{aligned}\right\} \tag{2.34}$$

For each $(x_1,x_2,x_3,\dots,x_n),(u_1,u_2,u_3,\dots,u_n) \in X^n$ and each

$$\alpha_1 \in T_1(x_1,x_2,x_3,\dots,x_n), \alpha_2 \in T_2(x_1,x_2,x_3,\dots,x_n),$$
$$\alpha_3 \in T_3(x_1,x_2,x_3,\dots,x_n), \quad \dots \quad ,\alpha_n \in T_n(x_1,x_2,x_3,\dots,x_n)$$

there exist

$$\beta_1 \in T_1(u_1,u_2,u_3,\dots,u_n), \beta_2 \in T_2(u_1,u_2,u_3,\dots,u_n),$$
$$\beta_3 \in T_3(u_1,u_2,u_3,\dots,u_n), \quad \dots \quad ,\beta_n \in T_n(u_1,u_2,u_3,\dots,u_n)$$

satisfying

$$\left.\begin{aligned} d(\alpha_1,\beta_1) &\le k_{11}d(x_1,u_1)+k_{12}d(x_2,u_2)+k_{13}d(x_3,u_3)+\dots+k_{1n}d(x_n,u_n)\\ d(\alpha_2,\beta_2) &\le k_{21}d(x_1,u_1)+k_{22}d(x_2,u_2)+k_{23}d(x_3,u_3)+\dots+k_{2n}d(x_n,u_n)\\ d(\alpha_3,\beta_3) &\le k_{31}d(x_1,u_1)+k_{32}d(x_2,u_2)+k_{33}d(x_3,u_3)+\dots+k_{3n}d(x_n,u_n)\\ &\vdots\\ d(\alpha_n,\beta_n) &\le k_{n1}d(x_1,u_1)+k_{n2}d(x_2,u_2)+k_{n3}d(x_3,u_3)+\dots+k_{nn}d(x_n,u_n)\end{aligned}\right\}$$

We suppose that

$$A = \begin{pmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1n}\\ k_{21} & k_{22} & k_{23} & \dots & k_{2n}\\ k_{31} & k_{32} & k_{33} & \dots & k_{3n}\\ k_{41} & k_{42} & k_{43} & \dots & k_{4n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ k_{n1} & k_{n2} & k_{n3} & \dots & k_{nn}\end{pmatrix}$$

converges to zero. Then:

(i) there exists $(x_1^*,x_2^*,x_3^*,\dots,x_n^*) \in X^n$ a solution for (2.31).

(ii) the operatorial system (2.31) is Ulam-Hyers stable.

Proof. For the prove of (i)-(ii) let us define $T : X^n \to (P_{cl}(X))^n$ by

$$T(x_1,x_2,x_3,\dots,x_n) = T_1(x_1,x_2,x_3,\dots,x_n)\times T_2(x_1,x_2,x_3,\dots,x_n)\times\dots\times T_n(x_1,x_2,x_3,\dots,x_n).$$

Then from the hypotheses of the theorem we get that

$$\begin{aligned} T(x_1^*,x_2^*,x_3^*,\dots,x_n^*) &= T_1(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\times T_2(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\times\dots\times T_n(x_1^*,x_2^*,x_3^*,\dots,x_n^*)\\ &= (x_1^*,x_2^*,x_3^*,\dots,x_n^*).\end{aligned}$$

So, T has at least one strict fixed point. We denote $\Gamma = X^n$ and consider $\tilde{d} : \Gamma\times\Gamma \to \mathbb{R}_+^n$

$$\tilde{d}((x_1,x_2,x_3,\dots,x_n),(u_1,u_2,u_3,\dots,u_n)) = \begin{pmatrix} d(x_1,u_1)\\ d(x_2,u_2)\\ d(x_3,u_3)\\ \vdots\\ d(x_n,u_n)\end{pmatrix}.$$

Then from the hypotheses of the theorem, we have that for each $s = (x_1, x_2, x_3, \ldots, x_n), t = (u_1, u_2, u_3, \ldots, u_n) \in X^n$ and each $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in T(x_1, x_2, x_3, \ldots, x_n)$, there exists $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T(u_1, u_2, u_3, \ldots, u_n)$ satisfying the relation

$$\tilde{d}(\alpha, \beta) \leq A\tilde{d}(s, t),$$

which proves that T is a multi-valued $A-$contraction. The conclusions follow now from Theorem 2.7 and Theorem 2.10. □

Remark 2.19. Notice again that, if (X, d) is a metric space and $T : X^n \to P(X)$ is a multi-valued operator and we define

$$\begin{aligned} T_1(x_1, x_2, x_3, \ldots, x_n) &= T(x_1, x_2, x_3, \ldots, x_n), \\ T_2(x_1, x_2, x_3, \ldots, x_n) &= T(x_2, x_3, \ldots, x_n, x_1) \\ &\vdots \\ T_n(x_1, x_2, x_3, \ldots, x_n) &= T(x_n, x_1, x_2, \ldots, x_{n-1}) \end{aligned}$$

then the above approach leads to some $n-$tupled fixed point theorems in the classical sense.

Remark 2.20. (i) When we take $n = 2$ in Theorem 2.7 then we get following result of Urs [44].

(ii) When we take $n = 3$ in Theorem 2.7 then we get following result of Gupta [15].

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