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A SHORT GRAPH-THEORETIC PROOF OF THE 2×2 MATRIX ANTI-DIAGONALS RATIO INVARIANCE WITH EXPONENTIATION

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Abstract We state and prove the invariance, with respect to matrix power, of the anti-diagonals ratio of a general 2×2 matrix. The proof methodology—utilising basic properties of a so called (two vertex) quiver—is new, and adds to others found recently.

1 Introduction

Consider the general matrix

$$\mathbf{M} = \mathbf{M}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.1}$$

for which

$$\mathbf{M}^n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \tag{1.2}$$

say, where $A_1 = A, \ldots, D_1 = D$. The following result holds:

Theorem 1.1. *Unless otherwise indeterminate, the ratio B_n/C_n of the two anti-diagonal terms in $\mathbf{M}^n$ is the quantity B/C, being invariant with respect to integer power $n \geq 1$.*

This somewhat counter-intuitive result has been established formally, through a variety of approaches, in the papers [1, 4] (see also a proof [3] for the instance when $D = A$ and $\mathbf{M}$ becomes slightly specialised), while the articles [2, 5] extend it to address invariance of all anti-diagonal ratios possessed by a tri-diagonal matrix of arbitrary dimension (an n-square tri-diagonal matrix has $n - 1$ such invariants, of which the above result is merely the $n = 2$ special case); collectively, these detail background information on Theorem 1.1 (and the tri-diagonal matrix extension), together with the wider topic of matrix exponentiation with regard to methods of obtaining closed forms for entries of powers of matrices such as $\mathbf{M}(A, B, C, D)$ (and those of higher dimension) that are fully symbolic—there are, accordingly, a number of articles published in which explicit expressions for B_n and C_n in (1.2) yield Theorem 1.1 trivially by inspection, but the studies referenced above exhibit a desire to move beyond the phenomenon as a mere mathematical curiosity so as to understand more about it by formulating different first principles proof arguments.

A couple of remarks are in order here. Firstly, this result would appear to have slipped through the academic net, so to speak, as confirmed by the eminent Gilbert Strang (that first rate expositor in linear algebra) who was taken by surprise, described it in a private communication with P.J.L. as a "neat observation", and now mentions it regularly to many audiences on his invited lecture circuit. Secondly, some discussions with Fields Medalist Timothy Gowers has revealed that this simple anti-diagonals ratio is but one of an infinite number of matrix power invariants whose rational expressions are each first order—they form a Class 1 group, and others of greater Class Number exist (work ongoing); for now, we concentrate on the simple Class 1 ratio invariant B_n/C_n, our proof being a new one which emerges naturally from a familiar and convenient construct in graph theory.

2 The Proof

Proof. Suppose, in a 2-vertex graph, that the number of n-length walks (edges) from vertex 1 to vertex 1 is A_n (self-loops), that from vertex 1 to vertex 2 is B_n, from vertex 2 to vertex 1 is C_n, and from vertex 2 to itself is D_n. We note that this is what is known as a *quiver* (a directed graph where multiple loops around, and paths between, vertices are allowed), which we denote as $\mathcal{Q}$, and in particular we define the collections of these four types of paths to be the respective sets $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n, \mathfrak{D}_n$—their cardinalities are, accordingly, A_n, B_n, C_n, D_n, whose i, j position in the nth power of the matrix $\mathbf{M}$ aligns with the number of n-length paths from vertex i to vertex j.

Let $\mathfrak{S}_1$ be a set of paths on $\mathcal{Q}$ that all end at the same vertex, and $\mathfrak{S}_2$ be a set of paths that all begin at the common terminal vertex characterising $\mathfrak{S}_1$, defining $\mathfrak{S}_1 \times \mathfrak{S}_2$ to be the set of all paths formed by attaching some element of $\mathfrak{S}_2$ to some element of $\mathfrak{S}_1$; clearly, the cardinality relation $|\mathfrak{S}_1 \times \mathfrak{S}_2| = |\mathfrak{S}_1||\mathfrak{S}_2|$ holds in standard fashion.

Based on paths beginning with an edge A_1 or B_1, it is straightforward to see that (where $\uplus$ is disjoint set union)

$$\mathfrak{A}_{n+1} = (\mathfrak{A}_1 \times \mathfrak{A}_n) \uplus (\mathfrak{B}_1 \times \mathfrak{C}_n), \tag{P.1}$$

with

$$\begin{aligned} |\mathfrak{A}_{n+1}| &= |(\mathfrak{A}_1 \times \mathfrak{A}_n) \uplus (\mathfrak{B}_1 \times \mathfrak{C}_n)| \\ &= |\mathfrak{A}_1 \times \mathfrak{A}_n| + |\mathfrak{B}_1 \times \mathfrak{C}_n| \\ &= |\mathfrak{A}_1||\mathfrak{A}_n| + |\mathfrak{B}_1||\mathfrak{C}_n| \\ &= A_1 A_n + B_1 C_n \\ &= A A_n + B C_n. \end{aligned} \tag{P.2}$$

Based on paths ending with an edge A_1 or C_1, on the other hand, we can write down

$$\mathfrak{A}_{n+1} = (\mathfrak{A}_n \times \mathfrak{A}_1) \uplus (\mathfrak{B}_n \times \mathfrak{C}_1), \tag{P.3}$$

with

$$|\mathfrak{A}_{n+1}| = \cdots = A_n A + B_n C, \tag{P.4}$$

whence the result follows from equating (P.2) and (P.4) which give $B_n/C_n = B/C$. □

As an aside, expressing the path set $\mathfrak{D}_{n+1}$ as both $(\mathfrak{D}_1 \times \mathfrak{D}_n) \uplus (\mathfrak{C}_1 \times \mathfrak{B}_n)$ and $(\mathfrak{D}_n \times \mathfrak{D}_1) \uplus (\mathfrak{C}_n \times \mathfrak{B}_1)$ (in other words replacing (pairwise) $\mathfrak{A}$ with $\mathfrak{D}$, and $\mathfrak{B}$ with $\mathfrak{C}$, in each of (P.1) and (P.3)) yields the result similarly, something that is intuitively obvious from the symmetric nature of the quiver deployed.

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