Palestine Journal of Mathematics

Vol. 10(1)(2021) , 104–108

NEW CLASSES OF GENERALISED LINEAR RECURRENCE HORADAM SEQUENCE TERM IDENTITIES

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11K31; Secondary 15A24.

Keywords and phrases: Horadam sequence, matrix algebra, generalised linear recurrence identity classes.

Abstract We derive, using matrices, new classes of linear recurrence identities linking the general term of the so called Horadam sequence with those of an associated 'cohort' sequence that differs only in its initial values.

1 Introduction

Denote by $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a,b;p,q)\}_0^{\infty}$, in standard format, the four-parameter Horadam sequence arising from the second order linear recursion

$$w_{n+2} = pw_{n+1} - qw_n, \qquad n \geq 0, \tag{1.1}$$

for which $w_0 = a$ and $w_1 = b$ are arbitrary initial values. For any fixed p, q, let $\mathcal{C}(p,q)$ be the collection, or *cohort*, of associated sequences of individual form $\{v_n\}_0^{\infty} = \{w_n(v_0, v_1; p, q)\}_0^{\infty}$ that each arise as a particular instance of (1.1) with variable start values v_0 and v_1; any pair v_0, v_1 describes a so called cohort sequence within $\mathcal{C}(p,q)$, and there are an infinite number of them.

2 Result and Proof

Defining

$$\alpha(a,b;p,q) = qa^2 + b^2 - pab \tag{2.1}$$

(assumed non-zero), and

$$\begin{aligned}
\beta_1(v_0, v_1, a, b; p, q) &= bv_1 - a(pv_1 - qv_0),\\
\beta_2(v_0, v_1, a, b; q) &= q(av_1 - bv_0),\\
\beta_3(v_0, v_1, a, b) &= bv_0 - av_1,\\
\beta_4(v_0, v_1, a, b; p, q) &= bv_1 - (pb - qa)v_0,
\end{aligned} \tag{2.2}$$

we state and prove a new result. For fixed recurrence parameters p, q of (1.1), the general Horadam term $w_n(a,b;p,q)$ is expressible as a linear combination of neighbouring terms from a cohort sequence $\{w_n(v_0, v_1; p, q)\}_0^{\infty}$ within $\mathcal{C}(p,q)$ according to an interesting and new linear recurrence relation that we illustrate accordingly.

Governing Identity. For $n \geq 1$,

$$(\beta_1\beta_4 - \beta_2\beta_3)w_n(a,b;p,q) = \alpha[\beta_4 w_n(v_0, v_1; p, q) - \beta_2 w_{n-1}(v_0, v_1; p, q)].$$

Evidently, the Governing Identity describes *classes* of identities because it holds for all cohort sequences characterised by p, q (which latter variables, as a pair, give rise to any single class).

Proof. Let

$$\mathbf{H} = \mathbf{H}(p,q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{P.1}$$

from which the recursion (1.1) readily delivers the matrix power relation

$$\begin{pmatrix} w_n(a,b;p,q) \\ w_{n-1}(a,b;p,q) \end{pmatrix} = \mathbf{H}^{n-1}(p,q) \begin{pmatrix} b \\ a \end{pmatrix} \tag{P.2}$$

that holds for $n \geq 1$. We identify, and make use of, a matrix (formed by the parameters $\alpha, \beta_1, \ldots, \beta_4$ of (2.1) and (2.2))

$$\mathbf{B} = \mathbf{B}(v_0, v_1, a, b; p, q) = \frac{1}{\alpha} \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \tag{P.3}$$

that has the following crucial algebraic properties (where T denotes transposition):

Property 1: $\mathbf{B}(b,a)^T = (v_1, v_0)^T$;

Property 2: $\mathbf{BHB}^{-1} = \mathbf{H}$.

Given these (which the reader is invited to check as an exercise, or else the Appendix may be referred to for details), and assuming $\beta_1\beta_4 \neq \beta_2\beta_3$ (so that $\mathbf{B}$ is non-singular), the proof is a straightforward one for we write, from (P.2),

$$\begin{aligned}
(w_n, w_{n-1})^T &= (w_n(a,b;p,q), w_{n-1}(a,b;p,q))^T \\
&= \mathbf{H}^{n-1}(b,a)^T \\
&= [\mathbf{B}^{-1}\mathbf{B}]\mathbf{H}^{n-1}[\mathbf{B}^{-1}\mathbf{B}](b,a)^T \\
&= \mathbf{B}^{-1}[\mathbf{B}\mathbf{H}^{n-1}\mathbf{B}^{-1}]\mathbf{B}(b,a)^T \\
&= \mathbf{B}^{-1}[\mathbf{BHB}^{-1}]^{n-1}\mathbf{B}(b,a)^T \\
&= \mathbf{B}^{-1}[\mathbf{H}]^{n-1}\mathbf{B}(b,a)^T \qquad \text{(by Property 2)} \\
&= \mathbf{B}^{-1}\mathbf{H}^{n-1}(v_1, v_0)^T \qquad \text{(by Property 1)} \\
&= \mathbf{B}^{-1}(w_n(v_0,v_1;p,q), w_{n-1}(v_0,v_1;p,q))^T \\
&= \mathbf{B}^{-1}(v_n, v_{n-1})^T
\end{aligned} \tag{P.4}$$

(having used (P.2) with $a = v_0$, $b = v_1$), whose components read

$$w_n = \frac{\alpha}{(\beta_1\beta_4 - \beta_2\beta_3)}(\beta_4 v_n - \beta_2 v_{n-1}) \tag{P.5}$$

and

$$w_n = \frac{\alpha}{(\beta_1\beta_4 - \beta_2\beta_3)}(-\beta_3 v_{n+1} + \beta_1 v_n). \tag{P.6}$$

Since, however, $q\beta_3 = -\beta_2$ and $\beta_1 - p\beta_3 = \beta_4$ (as seen in (A.3) of the Appendix), the r.h.s. expressions of (P.5),(P.6) are, deploying (1.1), seen to be identical, and so we have but one (independent) algebraic relation (the Governing Identity). This completes the proof, noting that the result holds trivially in the case $v_0 = a$, $v_1 = b$ (it is self-satisfying, with $\beta_2 = \beta_3 = 0$ and $\beta_1 = \beta_4 = \alpha(a,b;p,q)$; a Horadam sequence lies within its own set of cohort sequences), while setting $v_0 = 1$, $v_1 = p$ recovers Identity II of [2] (namely, $w_n(a,b;p,q) = aw_n(1,p;p,q) - (pa - b)w_{n-1}(1,p;p,q)$) that has previously been generated by a different approach altogether. □

3 Examples

We finish by demonstrating the validity of our result, choosing to involve well known sequences (that is, the Fibonacci and Lucas sequences). Consider the part specialised Horadam (or quasi-Fibonacci) sequence given by setting $p = 1$ and $q = -1$, for which

$$\{w_n(a,b;1,-1)\}_0^\infty = \{a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, \ldots\}, \tag{3.1}$$

with all cohort sequences in $\mathcal{C}(1,-1)$ likewise of the form

$$\{w_n(v_0,v_1;1,-1)\}_0^\infty = \{v_0, v_1, v_0+v_1, v_0+2v_1, 2v_0+3v_1, 3v_0+5v_1, 5v_0+8v_1, \ldots\} \quad (3.2)$$

for arbitrary v_0, v_1; we make the (easily proven) observation that, for $n \geq 1$,

$$w_n(v_0,v_1;1,-1) = F_{n-1}v_0 + F_n v_1 \quad (3.3)$$

(where $\{F_n\}_0^\infty = \{0,1,1,2,3,5,8,\ldots\} = \{w_n(0,1;1,-1)\}_0^\infty$ is the celebrated Fibonacci sequence), which is key to our cases; we feel these examples are instructive in seeing the structure of the Governing Identity.

Case (i): $a = 0$, $b = 1$

With $\alpha(0,1;1,-1) = 1$, together with $\beta_1(v_0,v_1,0,1;1,-1) = v_1$, $\beta_2(v_0,v_1,0,1;-1) = \beta_3(v_0, v_1, 0, 1) = v_0$, and $\beta_4(v_0,v_1,0,1;1,-1) = v_1 - v_0$, we confirm the Governing Identity for general n.

Firstly, we see that its l.h.s. is $(\beta_1\beta_4 - \beta_2\beta_3)w_n(0,1;1,-1) = (v_1^2 - v_0v_1 - v_0^2)F_n$. On the other hand, the r.h.s. is $(v_1 - v_0)w_n(v_0,v_1;1,-1) - v_0w_{n-1}(v_0,v_1;1,-1) = (v_1-v_0)(F_{n-1}v_0 + F_nv_1) - v_0(F_{n-2}v_0 + F_{n-1}v_1) = (v_1^2 - v_0v_1)F_n - v_0^2(F_{n-1} + F_{n-2}) = (v_1^2 - v_0v_1)F_n - v_0^2F_n = (v_1^2 - v_0v_1 - v_0^2)F_n =$ l.h.s.

Case (ii): $a = 2$, $b = 1$

Here $\{w_n(2,1;1,-1)\}_0^\infty$ is the familiar Lucas sequence $\{2,1,3,4,7,11,18,\ldots\} = \{L_n\}_0^\infty$, say. This time $\alpha(2,1;1,-1) = -5$, $\beta_1(v_0,v_1,2,1;1,-1) = -(v_1+2v_0)$, $\beta_2(v_0,v_1,2,1;-1) = \beta_3(v_0,v_1,2,1) = v_0 - 2v_1$, and $\beta_4(v_0,v_1,2,1;1,-1) = v_1 - 3v_0$; validation runs along similar lines to those of Case (i).

The l.h.s. is $(\beta_1\beta_4 - \beta_2\beta_3)w_n(2,1;1,-1) = [-(v_1+2v_0)(v_1-3v_0) - (v_0-2v_1)^2]L_n = -5(v_1^2 - v_0v_1 - v_0^2)L_n$, while the r.h.s. is $-5[(v_1-3v_0)w_n(v_0,v_1;1,-1) - (v_0-2v_1)w_{n-1}(v_0,v_1;1,-1)] = -5[(v_1-3v_0)(F_{n-1}v_0+F_nv_1) - (v_0-2v_1)(F_{n-2}v_0+F_{n-1}v_1)] = -5[A(F_n,F_{n-1})v_1^2 - B(F_n,F_{n-2})v_0v_1 - C(F_{n-1},F_{n-2})v_0^2]$, where $A(F_n,F_{n-1}) = F_n + 2F_{n-1}$, $B(F_n,F_{n-2}) = 3F_n - 2F_{n-2}$ and $C(F_{n-1},F_{n-2}) = 3F_{n-1} + F_{n-2}$; evidently, this last expression matches the l.h.s. if each of the linear functions A, B, C reduces to L_n, which is simple to check using the fundamental Lucas-Fibonacci relation $L_n = F_{n-1} + F_{n+1}$ $(n \geq 1)$.

4 Summary

This short paper has formulated and illustrated a new result that describes classes of recurrence identities connecting the general term of a Horadam sequence to successive terms in any accompanying cohort sequence. Note that it has been verified for a wide range of parameter values in both sets of sequence types, and moreover—via algebraic software—in complete generality using those long established sequence term closed forms for the two root cases (that is, degenerate and non-degenerate) of the characteristic equation associated with the recurrence equation (1.1)—the latter are omitted here, but are readily available in the considerable Horadam literature (see, for instance, the surveys [1, 3]).

Appendix

Here we establish Properties 1 and 2 underpinning the proof of the Governing Identity.

Property 1: Consider, from (P.3),

$$\begin{aligned}\mathbf{B}\begin{pmatrix} b \\ a \end{pmatrix} &= \frac{1}{\alpha}\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} \\ &= \frac{1}{\alpha}\begin{pmatrix} b\beta_1 + a\beta_2 \\ b\beta_3 + a\beta_4 \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ v_0 \end{pmatrix}, \end{aligned} \quad (A.1)$$

since $b\beta_1+a\beta_2=\alpha v_1$ and $b\beta_3+a\beta_4=\alpha v_0$ trivially using (2.1),(2.2); Property 2 is also straightforward to derive.

Property 2: Consider, from (P.1),(P.3),

$$\begin{aligned}
\mathbf{BHB}^{-1} &= \frac{1}{\alpha}\begin{pmatrix}\beta_1 & \beta_2\\ \beta_3 & \beta_4\end{pmatrix}\cdot\begin{pmatrix}p & -q\\ 1 & 0\end{pmatrix}\cdot\frac{\alpha}{(\beta_1\beta_4-\beta_2\beta_3)}\begin{pmatrix}\beta_4 & -\beta_2\\ -\beta_3 & \beta_1\end{pmatrix}\\
&= \frac{1}{(\beta_1\beta_4-\beta_2\beta_3)}\begin{pmatrix}\beta_1(p\beta_4+q\beta_3)+\beta_2\beta_4 & -[\beta_1(p\beta_2+q\beta_1)+\beta_2^2]\\ \beta_3(p\beta_4+q\beta_3)+\beta_4^2 & -[\beta_3(p\beta_2+q\beta_1)+\beta_2\beta_4]\end{pmatrix}\\
&= \frac{1}{(\beta_1\beta_4-\beta_2\beta_3)}\begin{pmatrix}T_1 & T_2\\ T_3 & T_4\end{pmatrix}, \qquad (A.2)
\end{aligned}$$

say, after a little algebra. The entries $T_1,\dots,T_4$ are readily simplified upon deploying the relations (immediate from (2.2))

$$\begin{aligned}
\beta_2+q\beta_3 &= 0,\\
\beta_1-p\beta_3 &= \beta_4, \qquad (A.3)
\end{aligned}$$

as needed, for we see that, in order,

$$\begin{aligned}
T_1 &= p\beta_1\beta_4+q\beta_1\beta_3+\beta_2\beta_4\\
&= p\beta_1\beta_4+\beta_1(-\beta_2)+\beta_2\beta_4\\
&= p\beta_1\beta_4-\beta_2(\beta_1-\beta_4)\\
&= p\beta_1\beta_4-\beta_2(p\beta_3)\\
&= p(\beta_1\beta_4-\beta_2\beta_3), \qquad (A.4)
\end{aligned}$$

$$\begin{aligned}
T_2 &= -[p\beta_1\beta_2+q\beta_1^2+\beta_2^2]\\
&= -[p\beta_1(-q\beta_3)+q\beta_1(p\beta_3+\beta_4)+\beta_2(-q\beta_3)]\\
&= -q(\beta_1\beta_4-\beta_2\beta_3), \qquad (A.5)
\end{aligned}$$

$$\begin{aligned}
T_3 &= p\beta_3\beta_4+q\beta_3^2+\beta_4^2\\
&= (p\beta_3+\beta_4)\beta_4+(-\beta_2)\beta_3\\
&= \beta_1\beta_4-\beta_2\beta_3, \qquad (A.6)
\end{aligned}$$

$$\begin{aligned}
T_4 &= -[p\beta_2\beta_3+q\beta_1\beta_3+\beta_2\beta_4]\\
&= -[p\beta_2\beta_3+q\beta_1\beta_3+\beta_2(\beta_1-p\beta_3)]\\
&= -\beta_1(q\beta_3+\beta_2)\\
&= -\beta_1(0)\\
&= 0; \qquad (A.7)
\end{aligned}$$

given equations (A.4)-(A.7), the r.h.s. of (A.2) reduces to $\mathbf{H}$ (P.1) instantly, and Property 2 is delivered.

Remark A.1. The reader may have noticed that Property 2 is a consequence of the commutativity of the matrices $\mathbf{B}$ and $\mathbf{H}$. It is, on investigation, a simple enough matter to see that this condition requires the equations of (A.3) to hold, together with $q\beta_1+p\beta_2=q\beta_4$ which is a trivial relation derivable therefrom.

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Received: July 29, 2019.
Accepted: October 22, 2019.