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DIRICHLET SERIES AND CLOSED-FORM EXACT SOLUTIONS OF MHD CASSON FLUID FLOW OVER A PERMEABLE STRETCHING/SHRINKING SHEET

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Abstract. The paper presents analytical and semi-numerical solution of magneto-hydrodynamic (MHD) boundary layer flow of an electrically conducting incompressible Casson fluid over a permeable stretching/shrinking sheet. The governing equation admits a similarity solution, thereby reducing the model partial differential equations into nonlinear ordinary differential equations along with appropriate boundary conditions. The solution of the resulting third order boundary value problem in an infinite domain is obtained approximately using Dirichlet series method and also in exact analytical closed-form viz. method of stretching of variables. These methods have the advantage of obtaining the derived quantities accurately and require less computer memory space as compared with pure numerical methods.

1 Introduction

The study of viscous boundary layer flows induced by a moving boundary find many important applications in several engineering processes such as extrusion processes in plastic and metal industries [1, 2, 3]. The phenomena of velocities on the boundary towards a fixed point are known as shrinking phenomena, which often occur in the situations such as rising shrinking balloon. In certain situations, the shrinking sheet solutions do not exist, since the velocity cannot be confined in a boundary layer. These solutions may exist if either the magnetic field or the stagnation flow is taken into account. Sakiadis [4, 5] investigated the boundary layer flow on a continuously stretching surface with constant speed and carried out pioneering work in this area. Tsou et al. [6] experimentally verified the Sakiadis work. Many researchers [7, 8, 9, 10] investigated the following work of Sakiadis and they have generalized the boundary conditions on the surface. Liao [11] discussed a new branch solution for both impermeable and permeable stretching sheet which indicates multiple solutions for the stretching surfaces under certain conditions. Miklavcic and Wang [12] examined the flow over a shrinking sheet, in this flow configuration, fluid is stretched towards a slot and flow is entirely different from the stretching case. Crane [7] found a closed form solution for steady two-dimensional stretching where the velocity on the boundary is away and proportional to the distance from the fixed point. Wang [13] discussed the exact solutions which are based on the boundary layer assumption and are not exact solutions of Navier-Stokes (NS) equations except results by Crane [7]. Makinde [14] analysed the effects of convective cooling on nanofluids flow over an unsteady stretching sheet. Recently, Khan et al. [15] numerically investigated the non-aligned MHD stagnation point flow of variable viscosity nanofluids past a stretching sheet with radiative heat. Noor et al. [16][investigated simple non-perturbative solution for MHD viscous flow due to a shrinking sheet by series solution using Adomain decomposition method (ADM). Raftari and Yildirim [17], analysed MHD viscous flow due to a shrinking sheet by employing the homotopy perturbation method (HPM) and Pade' approximants.

Moreover, studies related to the flows of non-Newtonian fluid have attracted much attention of researchers in recent time due to their increasing industrial and technological applications such as in geothermal energy, cooling of nuclear reactor, underground disposal of nuclear wastes,

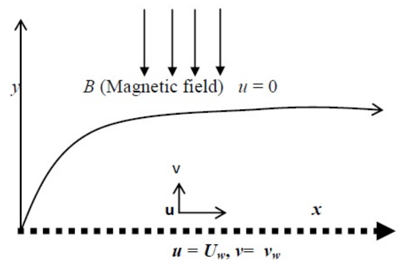

Figure 1. Problem geometry

petroleum reservoir operations, building insulation, irrigation systems, cooling of electronic component, etc. For instance, in a non-Newtonian fluid, the relation between the shear stress and the shear rate is different and can even be time-dependent. Therefore, the use of non−Newtonian fluids as the coolant or heat exchangers may reduce the required pumping power in some cases. There are several models in the literature to describe the flow behaviour of non-Newtonian fluid and one of them is Casson fluid model. Casson fluid can be defined as a shear thinning liquid which is assumed to have an infinite viscosity at zero rates of shear, a yield stress below which no flow occurs and a zero viscosity at an infinite rate of shear. Ibrahim and Makinde [18] numerically investigated the MHD stagnation point flow and heat transfer of Casson nanofluid past a stretching sheet with slip and convective boundary condition. Battacharyya et al. [19] found the analytic solution for MHD boundary layer flow of Casson fluid over a stretching /shrinking sheet with wall mass transfer. Meanwhile, Dirichlet series is particularly useful for obtaining solution of fluid flow problem and the derived quantities exactly. In a pioneering work, Kravnchenko and Yablonskii [20, 21] employed the Dirichlet series for solving third order nonlinear boundary value problem over an infinite domain. A general discussion of the convergence of the Dirichlet series may also be found in Riesz [22]. The accuracy as well as uniqueness of the solution can be confirmed using other powerful semi−numerical schemes. Sachdev et al. [23] have analyzed various problems from fluid dynamics of stretching sheet using this approach and found more accurate solution compared with earlier numerical findings. Awati et al. [24, 25] and Kudenatti et al. [26] have analyzed the problems from MHD boundary layer flow with nonlinear stretching sheet using these methods and found more accurate results compared with the classical numerical methods. Bhattacharyya [27] examined the effects of heat source/sink on the steady two dimensional MHD boundary layer flow and heat transfer over a shrinking sheet with wall mass suction using finite difference method.

In this present paper we employ the combination of Dirichlet series and the method of stretching of variables to tackle the nonlinear problem of hydro-magnetic boundary layer flow of Casson fluid over a permeable stretching /shrinking sheet. In section 2, a mathematical formulation of proposed problem with relevant boundary conditions is given. Section 3, is devoted to the solution of the problem using Dirichlet series. Section 4, gives the closed form exact analytical solution by means of method of stretching of variables. In Section 5, detailed results obtained are compared with the corresponding numerical schemes and Section 6, is about the conclusion.

2 Mathematical Formulation

The steady laminar two-dimensional incompressible viscous boundary layer flow of an electrically conducting Casson fluid over a continuously stretching or shrinking sheet is considered. The stretching or shrinking surface moves along x-axis in the direction and y-axis is perpendicular to it. The magnetic field strength of B is applied in the vertical direction and the induced magnetic field is neglected. The sheet stretching or shrinking velocity is $U_w = \pm U_0 x$ (note that positive sign indicates stretching while the negative sign indicate shrinking) and the wall suction/injection velocity is $\nu = \nu_w$ (see Figure 1). The rheological equation of state for an isotropic flow of a Casson fluid can be expressed as [19, 28, 29];

$$\tau_{ij} = \begin{cases} \{\mu_B + \frac{P_y}{\sqrt{2\pi}}\}2e_{ij}, & \pi > \pi_c \\ \{\mu_B + \frac{P_y}{\sqrt{2\pi_c}}\}2e_{ij}, & \pi < \pi_c \end{cases} \tag{2.1}$$

In the above equation $P_y = e_{ij}e_{ij}$ and e_{ij} denotes the $(i,j)^{th}$ component of the deformation rate, π be the product of the component of deformation rate with itself, π_c be a critical value of this product based on the non-Newtonian model, μ_B be the plastic dynamic viscosity of the non-Newtonian fluid and P_y be the yield stress of the fluid. From equation (2.1) we obtain (for $\pi < \pi_c$),

$$\tau_{ij} = \mu_B(1 + \frac{1}{\beta})2e_{ij}, \tag{2.2}$$

where $\beta = \mu\sqrt{2\pi_c}/P_y$ is the Casson fluid parameter. The governing conservation of mass and momentum boundary layer equations becomes [14, 19]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

$$u\frac{\partial u}{\partial x} + \nu\frac{\partial u}{\partial y} = \upsilon(1 + \frac{1}{\beta})(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \frac{\sigma B^2}{\rho}u, \tag{2.4}$$

The relevant boundary conditions for the present flow are

$$u = U_w = \pm U_0 x, \quad v = v_w \quad at \quad y = 0, \tag{2.5}$$

$$u = 0 \quad as \quad y \to \infty, \tag{2.6}$$

where u and v are the liquid velocity components in the x and y directions respectively, υ is the kinematic viscosity, ρ is the density of the fluid and σ is the electrical conductivity of the fluid. The velocity components u and v are related to the physical stream function ψ defined by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{2.7}$$

The stream function and similarity transformations can be defined in the form

$$\psi(x,y) = x\sqrt{\upsilon U_0}f(\eta), \quad \eta = y\sqrt{\frac{U_0}{\upsilon}}. \tag{2.8}$$

Using Eq.(2.8) the velocity components are expressed as $u = xU_0 f(\eta)$, $v = -\sqrt{U_0\upsilon}f(\eta)$ and the wall suction/injection velocity becomes $v_w = -\sqrt{U_0\upsilon}f(0)$, with these similarity variables Eqs. (2.3)-(2.6) reduces to the following form

$$(1 + \frac{1}{\beta})f''' + ff'' - f'^2 - M^2 f' = 0, \tag{2.9}$$

with

$$f(0) = f_w, \quad f'(0) = \pm 1, \quad f'(\infty) = 0. \tag{2.10}$$

where the prime symbol represents the derivative with respect to η, $M = \sqrt{\frac{\sigma B^2}{\rho U_0}}$ is the magnetic field parameter and $f_w = -\nu_w/\sqrt{U_0\upsilon}$ is the parameter such that $f_w > 0$ corresponds to suction and $f_w < 0$ corresponds to injection. Equations (2.9)-(2.10) can be conveniently written in a more general third order nonlinear differential equation as

$$f''' + Aff'' + Bf'^2 + Cf' = 0 \tag{2.11}$$

with the relevant boundary conditions

$$f(0) = \alpha_1, \quad f'(0) = \beta_1, \quad f'(\infty) = 0, \tag{2.12}$$

where A, B and C are constants. This third order nonlinear differential equation with infinite interval admits a Dirichlet series; necessary conditions for the existence and uniqueness of these solutions may be found in [20, 21]. We observe in equation (2.11) that the sign of A is always positive while that of B and C may be either positive, negative or zero. Other important quantity of interest is the local skin friction $C_f = \frac{\tau_w}{\rho U_w^2}$ given as

$$Re_x^{1/2} C_f = (1 + \frac{1}{\beta}) f''(0), \tag{2.13}$$

where $Re_x = U_w x/v$ is the local Reynolds number and the wall shear stress τ_w is define as

$$\tau_w = \mu(1 + \frac{1}{\beta}) \left\{ \frac{\partial u}{\partial y} \right\}_{y=0}. \tag{2.14}$$

3 Dirichlet Series Solution

We use Dirichlet series which is an elegant semi-numerical scheme to solve both categories of the problem. We seek Dirichlet series solution of Eq.(2.11) in the form [21]

$$f = \gamma_1 + \frac{6\gamma}{A} \sum_{i=1}^{\infty} b_i a^i e^{-i\gamma\eta} \tag{3.1}$$

where γ and a are parameters which are to be determined. Substituting Eq.(3.1) into Eq.(2.11), we get

$$\sum_{i=1}^{\infty} \{-\gamma^2 i^3 + A\gamma\gamma_1 i^2 - Ci\} b_i a^i e^{-i\gamma\eta} + \frac{6\gamma^2}{A} \sum_{i=2}^{\infty} \sum_{k=1}^{i-1} \{Ak^2 + Bk(i-k)\} b_k b_{i-k} a^i e^{-i\gamma\eta} = 0 \tag{3.2}$$

For i=1, we have

$$\gamma_1 = \frac{\gamma^2 + C}{A} \tag{3.3}$$

Substituting Eq.(3.3) into Eq. (3.2), the recurrence relation for obtaining coefficients is given by

$$b_i = \frac{6\gamma^2}{Ai(i-1)\{\gamma^2 i - C\}} \sum_{k=1}^{i-1} \{Ak^2 + Bk(i-k)\} b_k b_{i-k}, \quad i = 2, 3, 4, ... \tag{3.4}$$

Eq.(3.1) converges absolutely and uniformly when $\gamma > 0$ for some η_0, in the half plane Re $al(\eta) \geq$Re$al(\eta_0)$. It represents an analytic $2\pi/\gamma$ periodic function at $f = f(\eta_0)$ such that $f'(\infty) = 0$ [21]. Moreover, Eq. (3.1) contains two free parameters namely a and γ. These unknown parameters are determined from the remaining boundary conditions in Eq.(2.12) at $\eta = 0$.

$$f(0) = \frac{\gamma^2 + C}{A\gamma} + \frac{6\gamma}{A} \sum_{i=1}^{\infty} b_i a^i = \alpha_1 = f_w, \tag{3.5}$$

and

$$f'(0) = \frac{6\gamma^2}{A} \sum_{i=1}^{\infty} (-i) b_i a^i = \beta_1 \tag{3.6}$$

The solution of the above transcendental Eq. (3.5) and Eq. (3.6) yield constants a and γ. The solution of the above transcendental equations is equivalent to the unconstrained minimization of the functional

$$\left[\frac{\gamma^2 + C}{A\gamma} + \frac{6\gamma}{A} \sum_{i=1}^{\infty} b_i a^i - \alpha_1 \right]^2 + \left[\frac{6\gamma^2}{A} \sum_{i=1}^{\infty} (-i) b_i a^i - \beta_1 \right]^2 \tag{3.7}$$

We use Powell's method of conjugate directions [30] which is one of the most efficient techniques for solving unconstrained optimization problems. This helps in finding the unknown parameters a and γ uniquely for different values of the parameters $A, B, C, \alpha_1 = f_w, \beta_1 = 1$ for stretching sheet, $\beta_1 = -1$ for shrinking sheet. Alternatively, Newton's method is also used to determine the unknown parameters a and γ accurately. The local skin friction coefficient at the sheet surface is given by

$$f''(0) = \frac{6\gamma}{A}\sum_{i=1}^{\infty} b_i a^i (i\gamma)^2 \tag{3.8}$$

The velocity profiles of the problem is given by

$$f'(\eta) = \frac{6\gamma^2}{A}\sum_{i=1}^{\infty}(-i) b_i a^i e^{-i\gamma\eta} \tag{3.9}$$

4 Method of Stretching of Variables

Many nonlinear ODEs arising in MHD boundary layer problems which are not amenable for obtaining analytical solutions. In such situations, attempts have been made to develop an approximate analytical method for the solution of these problems. Method of stretching of variables is used here for the solution of such problems and obtained the closed form exact solution. In this method, we choose suitable derivative function H' such that the derivative boundary conditions are satisfied automatically and integration of H' will satisfy the remaining boundary condition. Substitution of this resulting function into the given equation gives the residual of the form $R(\xi, \alpha)$ called the defect function. The residual of the defect function can be minimized using least squares approximation method (for details see Ariel, [31]). Applying the transformation $f = f_w + F$ into Eq. (2.11), we get

$$F''' + A(f_w + F)F'' + BF'^2 + CF' = 0. \tag{4.1}$$

The boundary conditions in Eq.(2.12) then become

$$F(0) = 0, \quad F'(0) = \pm 1, \quad F'(\infty) = 0 \tag{4.2}$$

where $\beta_1 = 1$ represents stretching sheet and $\beta_1 = -1$ represents shrinking sheet. Introducing two variables $\xi = \alpha\eta$ and G in the form

$$G(\xi) = \alpha F(\eta) \tag{4.3}$$

where $\alpha > 0$ is an amplification factor. Substituting Eq.(4.3) into Eqs. (4.1)-(4.2), we obtain

$$\alpha^2 G''' + A(f_w\alpha + G)G'' + BG'^2 + CG' = 0, \tag{4.4}$$

with

$$G(0) = 0, \quad G'(0) = \pm 1, \quad G'(\infty) = 0. \tag{4.5}$$

For stretching sheet case:

We choose a trail velocity profile

$$G' = exp(-\xi) \tag{4.6}$$

which satisfies the derivative conditions in Eq.(4.5). Integrating Eq. (4.5) with respect to ξ from 0 to ξ using conditions (4.5), we get

$$G = 1 - exp(-\xi). \tag{4.7}$$

Substituting Eq.(4.7) into Eq. (4.4), we get the residual of defect function

$$R(\xi, \alpha) = (\alpha^2 - Af_w\alpha - A + C)exp(-\xi) + (A + B)exp(-2\xi). \tag{4.8}$$

Eq. (4.8) can be minimized using least squares approximation method as discussed in Ariel [31] for which

$$\frac{\partial}{\partial\alpha}\int_0^{\infty} R^2(\xi, \alpha)d\xi = 0 \tag{4.9}$$

Substituting (4.8) into Eq. (4.9) and solving cubic equation in α for a positive root, we get

$$\alpha = \frac{Af_w}{2}, \quad \alpha = \frac{1}{6}(3Af_w \pm \sqrt{3}\sqrt{4A - 8B - 12C + 3A^2 f_w^2}). \tag{4.10}$$

Once the amplification factor is calculated, then using Eq.(4.3), original function f can be written as

$$f = f_w + \frac{1}{\alpha}(1 - exp(-\alpha\eta)) \tag{4.11}$$

with α defined in Eq. (4.10). Thus Eq. (4.11) gives the solution of Eq. (2.11) for all A, B, C, f_w and β_1. It is striking that Eq.(2.11) also admits the following two analytical solutions. For $A = \frac{1}{D}, B = \frac{-1}{D}, C = \frac{-M}{D}$ and $\beta_1 = -1$, where $D = (1 + \frac{1}{\beta})$. Eq. (4.11) reduces to closed-form exact solution of [19]

$$f(\eta) = f_w + \frac{2(1+\frac{1}{\beta})}{f_w + \sqrt{f_w^2 + 4(1+\frac{1}{\beta})(1+M)}} - \frac{2(1+\frac{1}{\beta})}{f_w + \sqrt{f_w^2 + 4(1+\frac{1}{\beta})(1+M)}} e^{-\frac{f_w + \sqrt{f_w^2 + 4(1+\frac{1}{\beta})(1+M)}}{2(1+1/\beta)}\eta} \tag{4.12}$$

For shrinking sheet case:
We choose a trail velocity profile

$$G' = -exp(-\xi), \tag{4.13}$$

which satisfies the derivative conditions in Eq.(4.5). Integrating Eq. (4.12) with respect to ξ from 0 to ξ using conditions (4.5), we get

$$G = exp(-\xi) - 1. \tag{4.14}$$

Substituting Eq.(4.13) into Eq. (4.4), we get the residual of defect function

$$R(\xi, \alpha) = (-\alpha^2 + Af_w\alpha - A - C)exp(-\xi) + (A + B)exp(-2\xi). \tag{4.15}$$

Using the least squares method as discussed in Ariel [31] the equation (4.14) can be minimized for which

$$\frac{\partial}{\partial\alpha}\int_0^\infty R^2(\xi, \alpha)d\xi = 0 \tag{4.16}$$

Substituting (4.14) into equation (4.15) and solving cubic equation in α for a positive root, we get

$$\alpha = \frac{1}{6}(3Af_w \pm \sqrt{3}\sqrt{-4A + 8B - 12C + 3A^2 f_w^2}) \tag{4.17}$$

Once the amplification factor is calculated, then using Eq.(4.3), original function f can be written as

$$f = f_w + \frac{1}{\alpha}(exp(-\alpha\eta) - 1) \tag{4.18}$$

with α defined in Eq. (4.17). Thus Eq. (4.18) gives the solution of Eq. (2.11) for all A, B, C, f_w and β_1 . It is striking that Eq. (2.11) also admits the following two analytical solutions. For $A = 1, B = -1, C = -M^2$ and $\beta_1 = -1$. Eq. (4.18) reduces to closed-form exact solution of [32]

$$f(\eta) = f_w - \frac{2}{f_w \pm \sqrt{f_w^2 - (4 - 4M^2)}} + \frac{2}{f_w \pm \sqrt{f_w^2 - (4 - 4M^2)}} e^{\frac{f_w \pm \sqrt{f_w^2 - (4-4M^2)}}{2}\eta} \tag{4.19}$$

For$A = 1/D, B = -1/D, C = -M/D, \beta_1 = -1$ and $D = (1 + 1/\beta)$, Eq. (4.18) reduces to closed-form exact solution of [19]

$$f(\eta) = f_w - \frac{2(1+\frac{1}{\beta})}{f_w \pm \sqrt{f_w^2 - 4(1+\frac{1}{\beta})(1-M)}} + \frac{2(1+\frac{1}{\beta})}{f_w \pm \sqrt{f_w^2 - 4(1+\frac{1}{\beta})(1-M)}} e^{\frac{f_w \pm \sqrt{f_w^2 - 4(1+\frac{1}{\beta})(1-M)}}{2(1+1/\beta)}\eta} \tag{4.20}$$

It is interesting to note that the former exact solutions may also be obtained from the method of stretching of variables (cf. Eq. (4.18)).

5 Results and Discussion

The magneto-hydrodynamic (MHD) flow of Casson fluid due to a stretching or shrinking sheet caused by boundary layer of an incompressible viscous flow is analyzed by the more suggestive ways by using the Fortran programming and Mathematica. The third order nonlinear ordinary differential Eq.(2.11) subject to the infinite boundary conditions (2.12) has been solved semi-numerically using Dirichlet series method and method of stretching of variables. In Dirichlet series method it is important that the edge boundary layer $\eta \to \infty$ automatically satisfied. Also, the closed-form exact analytical solution is given by the method of stretching of variables. The numerical computations were performed for various values of the physical parameters viz.A, B, C, f_w and β_1. The present solution is also validated by comparing it with the previously published work of Battacharyya [27] as shown in Table 1. Also the closed form exact analytical solutions are compared with Battacharyya et al. [19], Fang and Zhang [32]. The graphs for the function $f'(\eta)$ i.e. velocity profiles which corresponds to the axial velocity component u are drawn against η for different values of parameters M, f_w and β as shown in Figs. 2-4. These figures match very well with that of earlier findings reported by Fang and Zhang [32]. The above computations were performed using Dirichlet series. It is observed that for both shrinking and stretching sheet, the dimensionless velocity profiles $f'(\eta)$ decreases with increasing values of magnetic parameter M, suction parameter $f_w > 0$ and the Casson parameter β. This inevitably leads to a decrease in the momentum boundary layer thickness. Moreover, it is well known that the presence of Lorentz force due to magnetic field tend to retard the fluid motion and consequently, the fluid velocity decreases. Meanwhile, an increase in the fluid injection i.e. $f_w < 0$ increases the momentum boundary layer thickness. Figs. 5-6 illustrate the local skin friction coefficient at the sheet surface. It is interesting to note that for both stretching and the shrinking sheet, the local skin friction coefficient increases with a rise in magnetic field intensity M and the fluid suction $f_w > 0$ but decreases with an increase in fluid injection $f_w < 0$ and Casson parameter β.

6 Conclusions

In this article, the analysis of a third order nonlinear ordinary differential equation modelling the MHD Casson fluid flow over a stretching /shrinking sheet is performed using Dirichlet series method and the method of stretching of variables for the closed form exact solution. The analytical method and semi-numerical scheme described here offer advantages over solutions obtained by any pure numerical methods. The pertinent results are presented graphically and discussed quantitatively. Finally, Dirichlet series solution is susceptible to the computer's memory limitations; it takes very less computer memory and can be employed to tackle a variety of nonlinear boundary value problems modeling real systems.

Table 1. The skin friction coefficient $f''(0)$ at $A = 1, B = 1, C = -2$ and $\beta_1 = -1$ (Eq.(11) and (12)) for various values of f_w by Dirichlet series solution and Method of stretching of variables

f_w	Dirichlet Series method			MSV	Numerical solution[27]
	a	γ	$f''(0)$	$f''(0)$	$f''(0)$
2	0.02859548	2.41421356	2.414214	2.4142136	2.414214
3	0.01527885	3.30277564	3.302776	3.3027756	3.302776
4	0.00928802	4.23606798	4.236067	4.2360679	4.236068

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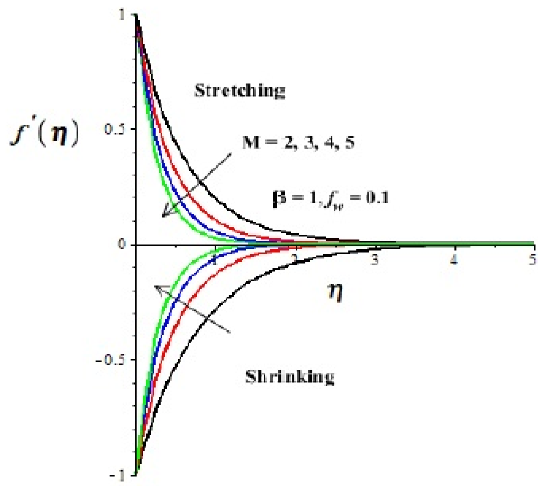

Figure 2. Velocity profiles with increasing M

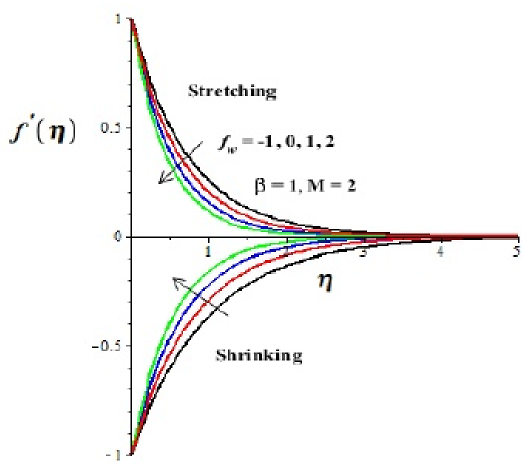

Figure 3. Velocity profiles with increasing f_w

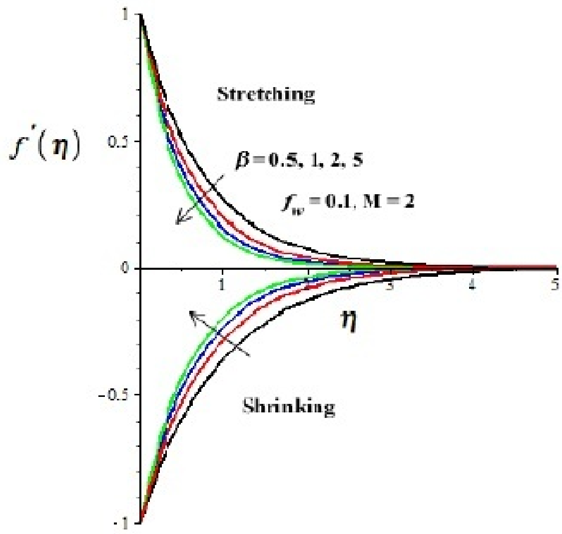

Figure 4. Velocity profiles with increasing β

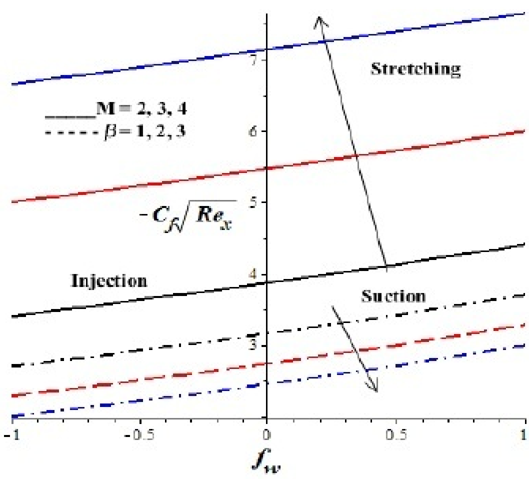

Figure 5. Skin friction coefficient for stretching sheet with increasing β,M and f_w

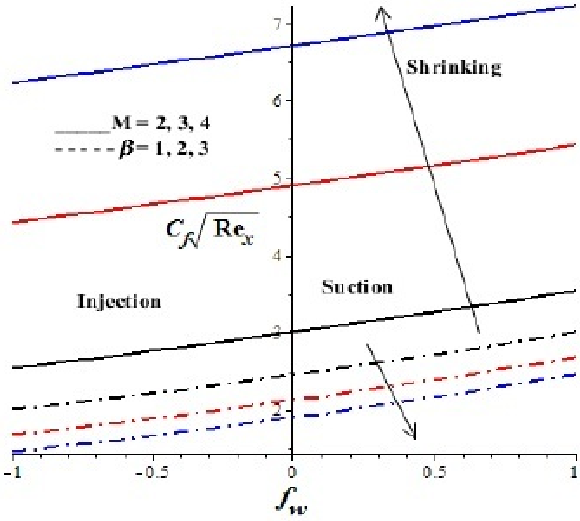

Figure 6. Skin friction coefficient for shrinking sheet with increasingβ,M and f_w.

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