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SOLUTIONS OF LINEAR PARABOLIC EQUATIONS WITH HOMOTOPY PERTURBATION METHOD

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Abstract. In this paper, numerical solutions of linear partial differential equations of parabolic type are investigated. We first solve heat equation numerically with Crank-Nicolson, Homotopy Perturbation and Adomian Decomposition methods and then compare the obtained numerical results. As a result of this comparison, we obtain that the Homotopy Perturbation and Adomian Decomposition methods are more stable than the Crank-Nicolson method.

1 Introduction

Parabolic partial differential equations show up in a wide range of applications in a natural science, engineering, and technology; typically, they must be solved numerically and there are many finite-difference and finite-element schemes in order to do this. The biggest advantage of this method is that some problems that cannot acquire analytic solution can be solved numerically with these methods.

In this study, we use Crank-Nicolson, Homotopy Perturbation and Adomian Decomposition methods to numerically solve the linear partial differential equations of the parabolic type. We investigate the stability properties of these methods with appropriate initial and boundary conditions.

The Crank-Nicolson method was developed by Crank and Nicolson in 1947. They discussed a numerical method developed by the authors in which both derivatives were replaced by finite difference ratios and the solution proceeded by finite steps in time[6].

Adomian Decomposition method was developed by George Adomian in the 1980's[2, 3, 4]. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential, partial differential and integral equations as well. This technique is based on the representation of a solution to a functional equation as a series of function. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function.

The Homotopy Perturbation method is a new and effective method for solving various differential and integral equations. This method was introduced and developed by Ji-Huan He in [9, 10, 11] and combined a homotopy technique of topology and a perturbation technique. Choosing an appropriate initial approach and homotopy is important for solving the problem. Considerable research work has recently been conducted in applying this method to a class of linear and nonlinear equations. It can be said that this method is a universal one, and is able to solve various kinds of nonlinear functional equations[1, 5, 7, 8, 12, 13]. It continuously deforms the difficult equation under study into a simple equation, easy to solve. Approximate solutions solved by the perturbation methods are valid only for the small values of the parameters. But the proposed method, requiring no small parameters in the equations, can readily eliminate the limitations of the traditional perturbation techniques. The advantage of the method is that it doesn't need a small parameter in linear and nonlinear problems.

2 Numerical Methods

2.1 Crank-Nicolson Implicit Method

There are many implicit-finite difference methods that are used to find approximate solutions of the parabolic differential equation. Given the stability in the problem, since the calculations must be made for a large period of time, the number of time steps and hence the number of operations will increase. One of the differential approaches, which is not limited by the $\delta t = k$ time step that removes this negativity, is the Crank-Nicolson implicit solution approach. Crank and Nicolson proposed and used a method that reduces the total volume of calculation and is valid (convergent and stable) for all finite values of r. They replaced $\frac{\partial^2 u}{\partial x^2}$ by the mean of its finite-difference representations on the $(j+1)$th and jth time rows and approximated the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) \tag{2.2}$$

giving

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j} \tag{2.3}$$

where $r = \frac{k}{h^2}$. The left side of equation (3) contains three unknowns and the right side three knowns, pivotal values of u. If there are n international mesh points along each time row then for $j = 0$ and $i = 1, 2, ..., n$, equation (3) gives n simultaneous equations for the n unknown pivotal values along the first time row in terms of known initial and boundary values. Similarly $j = 1$ expresses n unknown values of u along the second time row in terms of the calculated values along the first, etc. A method such as this where the calculation of an unknown pivotal value necessitates the solution of a set of simultaneous equations is described as an implicit one[14].

2.2 Homotopy Perturbation Method

Considering the differential equation

$$A(u) - f(r) = 0\,, \quad r \in \Omega \tag{2.4}$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0\,, \quad r \in \Gamma \tag{2.5}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary and Γ is the boundary of the domain Ω. The operator A can be divided into two parts, namely, linear (L) and nonlinear (N), and equation (4) can be rewritten in the form of

$$L(u) + N(u) - f(r) = 0. \tag{2.6}$$

Using homotopy technique, one can construct a homotopy $\nu : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(\nu(r,p),p) = (1-p)[L(\nu) - L(u_0)] + p[A(\nu) - f(r)] = 0\,, \quad p \in [0,1] \tag{2.7}$$

or

$$H(\nu(r,p),p) = L(\nu) - L(u_0) + pL(u_0) + p[N(\nu) - f(r)] = 0 \tag{2.8}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of equation (4), which satisfies the boundary conditions. It is obvious from (7) and(8) that

$$H(\nu(r,0),0) = L(\nu) - L(u_0) = 0\,, H(\nu(r,1),1) = A(\nu) - f(r) = 0. \tag{2.9}$$

The process of changes in p from zero to unity is that of $\nu(r,p)$ changing from $u_0(r)$ to u(r). We assume that solution for (7) or (8) can be written as a power series of p:

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + ... \tag{2.10}$$

When $p \to 1$, it yields the approximate solution for (4) in the form

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + ... \tag{2.11}$$

2.3 Adomian Decomposition Method

At first we consider how to use the Adomian Decomposition Method (ADM) by the linear differential equation written in an operator form by

$$Lu + Ru = g \tag{2.12}$$

where L is mostly the lower order derivative which is assumed to be invertible, R is other linear differential operator, and g is a source term. The standard Adomian method defines the solution u by an infinite series of components given as bellow,

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \tag{2.13}$$

Rewriting the equation (12) as

$$Lu = g - Ru \tag{2.14}$$

and then applying L^{-1} to both sides of equation (14) and using the given condition, we get

$$u = f - L^{-1}(Ru) \tag{2.15}$$

where the function f represents the terms arising from integrating the source term g. Decomposed series solution function can be written as

$$u = \sum_{n=0}^{\infty} u_n \tag{2.16}$$

Considering (15) and (16), we obtain

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}(R(\sum_{n=0}^{\infty} u_n)). \tag{2.17}$$

For simplicity, equation (17) can be rewritten as

$$u_0 + u_1 + u_2 + ... = f - L^{-1}(R(u_0 + u_1 + u_2 + ...)). \tag{2.18}$$

The zeroth component u_0 is usually defined by the function f described above. And then we obtain the terms u_1 , u_2 , u_3 , ... by using the term u_0 . These terms can be illustrated by using recursive relation as below
$u_0 = f$,
$u_{k+1} = -L^{-1}(R(u_k))$, $k \geq 0$.
Finally to obtain solution u , these terms are plugged in the equation (16).

3 Numerical Results

In this section, we work out an example, one dimensional heat flow equation, in details. We solve it by using three methods that we described in previous sections.

3.1 Example

We consider one dimensional heat flow equation. The boundary and the initial conditions are follow as

$$\begin{cases} u_t - u_{xx} = 0, \ 0 < x < \pi, \ t > 0 \\ u(x, 0) = sin(x) + 3sin(2x), \ 0 \leq x \leq \pi \\ u(0, t) = u(\pi, t) = 0, \ t \geq 0 \end{cases} \tag{3.1}$$

with the exact solution $u(x, t) = e^{-t}sin(x) + 3e^{-4t}sin(2x)$.

Solution by ADM

Equation (19) can be rewritten in operator form as

$$L_t u = L_x u \tag{3.2}$$

where L_t symbolizes the easily invertible linear differential operator. These operators are defined as

$$L_t = \frac{\partial}{\partial t}\ , L_t^{-1}(.) = \int_0^t (.)dt\ , L_x = \frac{\partial^2}{\partial x^2} \tag{3.3}$$

Applying L_t^{-1} to both sides of equation (20), we have

$$L_t^{-1} L_t u = L_t^{-1} L_x u \tag{3.4}$$

or

$$u(x,t) - u(x,0) = L_t^{-1} L_x u. \tag{3.5}$$

Decomposed series solution function can be written as

$$u = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.6}$$

Substituting (24) in (23), we find

$$\sum_{n=0}^{\infty} u_n(x,t) = sin(x) + 3sin(2x) + L_t^{-1}(L_x(\sum_{n=0}^{\infty} u_n(x,t))). \tag{3.7}$$

The zeroth component is

$$u_0 = sin(x) + 3sin(2x)$$

Other components by using recursive relation as below

$$\begin{aligned}
u_{n+1}(x,t) &= L_t^{-1}(L_x(\sum_{n=0}^{\infty} u_n(x,t))) \\
u_1 &= L_t^{-1} L_x u_0 = -t(sin(x) + 12sin(2x)) \\
u_2 &= L_t^{-1} L_x u_1 = \frac{t^2}{2!}(sin(x) + 48sin(2x)) \\
u_3 &= L_t^{-1} L_x u_2 = -\frac{t^3}{3!}(sin(x) + 192sin(2x)) \\
&\vdots
\end{aligned}$$

Thus, the solution can be expressed as
$u = sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...) + 3sin(2x)(1 - 4t + \frac{(4t)^2}{2!} - \frac{(4t)^3}{3!} + ...)$
$u = e^{-t} sin(x) + 3e^{-4t} sin(2x)$

Solution by Homotopy Perturbation Method

From Equation (19), one can establish the following homotopy

$$H(\nu,p) = \frac{\partial \nu}{\partial t} - \frac{\partial u_0}{\partial t} - p(\frac{\partial^2 \nu}{\partial x^2} - \frac{\partial u_0}{\partial t}) = 0 \tag{3.8}$$

Now the homotopy parameter p is used to expand the solution $\nu(x,t)$ as follows

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + ... \tag{3.9}$$

Setting $p = 1$ leads to the approximate solution of the problem:

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + ... \qquad (3.10)$$

Considering the initial condition and comparing the coefficients of equal p powers, we obtain

$$\begin{aligned}
p^0 &: \frac{\partial \nu_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \\
p^1 &: \frac{\partial \nu_1}{\partial t} = \frac{\partial^2 \nu_0}{\partial x^2} - \frac{\partial u_0}{\partial t} \\
p^2 &: \frac{\partial \nu_2}{\partial t} = \frac{\partial^2 \nu_1}{\partial x^2} \\
p^3 &: \frac{\partial \nu_3}{\partial t} = \frac{\partial^2 \nu_2}{\partial x^2} \\
&\vdots
\end{aligned}$$

Then

$$\begin{aligned}
\nu_0(x,t) &= sin(x) + 3sin(2x) \\
\nu_1(x,t) &= -t(sin(x) + 4.3sin(2x)) \\
\nu_2(x,t) &= \frac{t^2}{2!}(sin(x) + 4^2.3sin(2x)) \\
\nu_3(x,t) &= -\frac{t^3}{3!}(sin(x) + 4^3.3sin(2x)) \\
&\vdots
\end{aligned}$$

Collecting the components together, we obtain
$u = sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...) + 3sin(2x)(1 - 4t + \frac{(4t)^2}{2!} - \frac{(4t)^3}{3!} + ...)$
$u = e^{-t}sin(x) + 3e^{-4t}sin(2x)$

Solution by Crank-Nicolson Implicit Method

Although this method is valid for all finite values of $r = \frac{k}{h^2}$,the $\frac{\partial u}{\partial t}$ approach leads to an incorrect approach for a large value of r . Take $h = \frac{\pi}{10}$ for instance. A suitable value is $r = 1$ and has the advantage of making the coefficient of $u_{i,j}$ zero in (3). Then $k = 0,098696$ and (3) reads as
$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$
Denote $u_{i,j+1}$ by u_i $(i = 1, 2, ..., 9)$. The values of u for the firs time step satisfy

$$\begin{aligned}
-0 + 4u_1 - u_2 &= 0 + 3,440952 \\
-u_1 + 4u_2 - u_3 &= 2,072370 + 3,662187 \\
-u_2 + 4u_3 - u_4 &= 3,440952 + 2,714412 \\
&\vdots \\
-u_7 + 4u_8 - u_9 &= -2,044150 - 1,454340 \\
-u_8 + 4u_9 - 0 &= -2,265384 + 0
\end{aligned}$$

These are solved by Gauss elimination method and the results are
$u_1 = 1,477999$, $u_2 = 2,471047$, $u_3 = 2,671634$, $u_4 = 2,060125$,

$u_5 = 0,906681$, $u_6 = -0,335515$, $u_7 = -1,204594$, $u_8 = -1,405181$,
$u_9 = -0,917641$
Then u values are calculated until $t = 0,888264$ by continuing in this way. At the time $t = 0,888264$, the analytical solution of the partial differential equation together with comparing finite-difference solution is given in Table 1.

x	Analytical Solution(Exact)	Crank-Nicolson Method(CNM)	Error
0	0	0	0
0,314159	0,177614	0,194688	0,017074
0,628318	0,323499	0,355773	0,032274
0,942477	0,414507	0,460275	0,045768
1,256637	0,441729	0,505842	0,064113
1,570796	0,411368	0,527900	0,116532
1,884955	0,340738	0,652625	0,311887
2,199114	0,251100	0,352960	0,10186
2,513274	0,160092	0,198292	0,0382
2,827433	0,076624	0,090646	0,014022
3,141592	0	0	0

Table 1. Comparing of the analytical solution at the time $t = 0,888264$ with the results obtained using the Crank-Nicolson implicit solution method

x	Analytical Solution (Exact)	CNM	HPM-ADM (N=12)	Error $\mid u_{exact} - u_{CNM} \mid$	Error $\mid u_{exact} - u_{HPM} \mid$
0	0	0	0	0	0
0,314159	0,177614	0,194688	0,165953	0,017074	0,011661
0,628318	0,323499	0,355773	0,304631	0,032274	0,018868
0,942477	0,414507	0,460275	0,395639	0,045768	0,018868
1,256637	0,441729	0,505842	0,430069	0,064113	0,01166
1,570796	0,411368	0,527900	0,411369	0,116532	0,000001
1,884955	0,340738	0,652625	0,352400	0,311887	0,011662
2,199114	0,251100	0,352960	0,269969	0,10186	0,018869
2,513274	0,160092	0,198292	0,178961	0,0382	0,018869
2,827433	0,076624	0,090646	0,088285	0,014022	0,011661
3,141592	0	0	0	0	0

Table 2. Comparing of the analytical solution at the time $t = 0,888264$ with the results obtained using the Crank-Nicolson implicit solution method(CNM) and Homotopy Perturbation method (HPM) for example 1

4 Conclusions

To sum up, this article investigates numerical solutions of linear partial differential equations of parabolic type. Crank-Nicolson, Homotopy Perturbation and Adomian Decomposition methods are used to solve these equations numerically with suitable initial and boundary conditions. The numerical results obtained are given in the table. As a result, Homotopy Perturbation and Adomian Decomposition methods are more stable and convergent than Crank-Nicolson method.

In addition, Homotopy Perturbation and Adomian Decomposition methods are found to be related in application in the sense that the computational size is reduced and they give close form solution.

In the forthcoming paper, we will extend the homotopy perturbation method to obtain the numerical solutions of nonlinear parabolic equations.

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