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Some different type parameterized inequalities via generalized integral operators and their applications

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Abstract The author discover an identity for a generalized integral operator with parameters via differentiable function. By using this integral equation, we derive some new bounds on Hermite–Hadamard and Ostrowski type integral inequalities. By taking the special parameter values for various suitable choices of function, some interesting results are obtained. At the end, some applications of presented results to special means and new error estimates for the trapezium and midpoint formula have been analyzed. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

1 Introduction

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $a_1, a_2 \in I$ with $a_1 < a_2$. Then the following inequality holds:*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1.1}$$

This inequality (1.1) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.1) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1]-[25].
The aim of this paper is to establish trapezium and Ostrowski type generalized integral inequalities for preinvex functions. Interestingly, the special cases of presented results, are fractional integral inequalities. Therefore, it is important to summarize the study of fractional integrals. Let us recall some special functions and evoke some basic definitions as follows:

Definition 1.2. For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k} - 1}}{(x)_{n,k}}. \tag{1.2}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha - 1} e^{-\frac{t^k}{k}} dt. \tag{1.3}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha). \tag{1.4}$$

For $k = 1$, eq. (1.3) gives integral representation of gamma function.

Definition 1.3. [17] Let $f \in L[a_1, a_2]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a_1 \geq 0$ are defined by

$$I_{a_1^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{x} (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a_1$$

and

$$I_{a_2^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{a_2} (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad a_2 > x. \tag{1.5}$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals. For $\alpha = k = 1$, k-fractional integrals give classical integrals.
Also, let define a function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < +\infty, \tag{1.6}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\varphi(r)} \leq A \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.7}$$

$$\frac{\varphi(r)}{r^2} \leq B\frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{1.8}$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq C|r-s|\frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.9}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies (1.6)-(1.9), see [21]. Therefore, the left-sided and right-sided generalized integral operators are defined as follows:

$$_{a_1^+}I_\varphi f(x) = \int_{a_1}^{x} \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a_1, \tag{1.10}$$

$$_{a_2^-}I_\varphi f(x) = \int_{x}^{a_2} \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < a_2. \tag{1.11}$$

The most important feature of generalized integrals is that; they produce Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc., see [20].

Motivated by the above literatures, the main objective of this paper is to discover in Section 2, an identity for a generalized integral operator with parameters via differentiable function. By using the established identity as an auxiliary result, some new estimates on Hermite–Hadamard and Ostrowski type integral inequalities are obtained. It is pointed out that some new fractional integral inequalities have been deduced by taking special parameter values for various suitable choices of function. In Section 3, some applications to special means and new error estimates for the trapezium and midpoint formula are given. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

2 Main results

Throughout this study, let $P = [ma_1, a_2]$ with $a_1 < a_2$, $m \in (0, 1]$ be an invex subset with respect to $\eta : P \times P \longrightarrow \mathbb{R}$ and $n \in \mathbb{N}^*$. Also for all $t \in [0, 1]$, for brevity, we define

$$\Lambda_{m,n}(t) = \int_0^t \frac{\varphi\left(\frac{\eta(x,ma_1)}{n+1}u\right)}{u} du < \infty, \quad \eta(x, ma_1) > 0 \tag{2.1}$$

and

$$\Delta_{m,n}(t) = \int_0^t \frac{\varphi\left(\frac{\eta(a_2,mx)}{n+1}u\right)}{u} du < \infty, \quad \eta(a_2, mx) > 0. \tag{2.2}$$

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 2.1. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2). If $f' \in L(P)$ and $\lambda \in (0,1]$, then the following identity for generalized fractional integrals hold:*

$$\frac{f(mx)+f\left(ma_1+\frac{\lambda}{n+1}\eta(x,ma_1)\right)}{2}-\frac{1}{2\Delta_{m,n}(\lambda)}\times {}_{(mx)^+}I_{\varphi}f\left(mx+\frac{\lambda}{n+1}\eta(a_2,mx)\right)$$
$$-\frac{1}{2\Lambda_{m,n}(\lambda)}\times {}_{\left(ma_1+\frac{\lambda}{n+1}\eta(x,ma_1)\right)^-}I_{\varphi}f\left(ma_1\right)$$
$$=\frac{\lambda\eta(x,ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)}\times\int_0^1\Lambda_{m,n}(\lambda t)f'\left(ma_1+\frac{\lambda t}{n+1}\eta(x,ma_1)\right)dt \tag{2.3}$$
$$-\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)}\times\int_0^1\Delta_{m,n}(\lambda(1-t))f'\left(mx+\frac{\lambda t}{n+1}\eta(a_2,mx)\right)dt.$$

We denote

$$T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)=\frac{\lambda\eta(x,ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)} \tag{2.4}$$
$$\times\int_0^1\Lambda_{m,n}(\lambda t)f'\left(ma_1+\frac{\lambda t}{n+1}\eta(x,ma_1)\right)dt$$
$$-\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)}\times\int_0^1\Delta_{m,n}(\lambda(1-t))f'\left(mx+\frac{\lambda t}{n+1}\eta(a_2,mx)\right)dt.$$

Proof. Integrating by parts eq. (2.4) and changing the variables of integration, we have

$$T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)=\frac{\lambda\eta(x,ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)}$$
$$\times\left\{\frac{\Lambda_{m,n}(\lambda t)(n+1)f\left(ma_1+\frac{\lambda t}{n+1}\eta(x,ma_1)\right)}{\lambda\eta(x,ma_1)}\right|_0^1$$
$$\left.-\frac{(n+1)}{\eta(x,ma_1)}\times\int_0^1\frac{\varphi\left(\frac{\eta(x,ma_1)}{n+1}\lambda t\right)}{\lambda t}f\left(ma_1+\frac{\lambda t}{n+1}\eta(x,ma_1)\right)dt\right\}$$
$$-\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)}\times\left\{\frac{\Delta_{m,n}(\lambda(1-t))(n+1)f\left(mx+\frac{\lambda t}{n+1}\eta(a_2,mx)\right)}{\lambda\eta(a_2,mx)}\right|_0^1$$
$$\left.+\frac{(n+1)}{\eta(a_2,mx)}\times\int_0^1\frac{\varphi\left(\frac{\eta(a_2,mx)}{n+1}\lambda(1-t)\right)}{\lambda(1-t)}f\left(mx+\frac{\lambda t}{n+1}\eta(a_2,mx)\right)dt\right\}$$
$$=\frac{\lambda\eta(x,ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)}$$
$$\times\left\{\frac{\Lambda_{m,n}(\lambda)(n+1)f\left(ma_1+\frac{\lambda}{n+1}\eta(x,ma_1)\right)}{\lambda\eta(x,ma_1)}-\frac{(n+1)}{\lambda\eta(x,ma_1)}\times {}_{\left(ma_1+\frac{\lambda}{n+1}\eta(x,ma_1)\right)^-}I_{\varphi}f\left(ma_1\right)\right\}$$
$$-\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)}$$
$$\times\left\{-\frac{\Delta_{m,n}(\lambda)(n+1)f(mx)}{\lambda\eta(a_2,mx)}+\frac{(n+1)}{\lambda\eta(a_2,mx)}\times {}_{(mx)^+}I_{\varphi}f\left(mx+\frac{\lambda}{n+1}\eta(a_2,mx)\right)\right\}$$
$$=\frac{f(mx)+f\left(ma_1+\frac{\lambda}{n+1}\eta(x,ma_1)\right)}{2}-\frac{1}{2\Delta_{m,n}(\lambda)}\times {}_{(mx)^+}I_{\varphi}f\left(mx+\frac{\lambda}{n+1}\eta(a_2,mx)\right)$$
$$-\frac{1}{2\Lambda_{m,n}(\lambda)}\times {}_{\left(ma_1+\frac{\lambda}{n+1}\eta(x,ma_1)\right)^-}I_{\varphi}f\left(ma_1\right).$$

The proof of Lemma 2.1 is completed. □

Remark 2.2. Taking $\lambda = \frac{1}{2}$, $m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$ and $\varphi(t) = t$ in Lemma 2.1, we get the following new trapezium type integral identity:

$$T_{f,\Lambda_{1,0},\Delta_{1,0}}\left(\frac{a_1+a_2}{2}; \frac{1}{2}, a_1, a_2\right) = \frac{f\left(\frac{a_1+a_2}{2}\right) + f\left(\frac{3a_1+a_2}{4}\right)}{2} \tag{2.5}$$

$$-\frac{2}{a_2 - a_1}\left[\int_{a_1}^{\frac{3a_1+a_2}{4}} f(t)dt + \int_{\frac{a_1+a_2}{2}}^{\frac{a_1+3a_2}{4}} f(t)dt\right].$$

Remark 2.3. Taking $\lambda = m = 1$, $n = 0$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$ and $\varphi(t) = t$ in Lemma 2.1, we get the following Ostrowski type integral identity:

$$T_{f,\Lambda_{1,0},\Delta_{1,0}}(x; 1, a_1, a_2) = f(x) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(t)dt. \tag{2.6}$$

Theorem 2.4. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on (ma_1, a_2), where $\lambda \in (0, 1]$. If $|f'|^q$ is preinvex on P for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality for generalized fractional integrals hold:*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x; \lambda, a_1, a_2)\right| \leq \frac{\lambda\eta(x, ma_1)}{2(n+1)\sqrt[q]{2(n+1)}\Lambda_{m,n}(\lambda)}\sqrt[p]{B_{\Lambda_{m,n}}(\lambda; p)} \tag{2.7}$$

$$\times\sqrt[q]{(2(n+1) - \lambda)\,|f'(ma_1)|^q + \lambda|f'(x)|^q}$$

$$+\frac{\lambda\eta(a_2, mx)}{2(n+1)\sqrt[q]{2(n+1)}\Delta_{m,n}(\lambda)}\sqrt[p]{C_{\Delta_{m,n}}(\lambda; p)} \times \sqrt[q]{(2(n+1) - \lambda)\,|f'(mx)|^q + \lambda|f'(a_2)|^q},$$

where

$$B_{\Lambda_{m,n}}(\lambda; p) := \int_0^1 \left[\Lambda_{m,n}(\lambda t)\right]^p dt, \quad C_{\Delta_{m,n}}(\lambda; p) := \int_0^1 \left[\Delta_{m,n}(\lambda t)\right]^p dt. \tag{2.8}$$

Proof. From Lemma 2.1, preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x; \lambda, a_1, a_2)\right| \leq \frac{\lambda\eta(x, ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)}$$

$$\times\int_0^1 \Lambda_{m,n}(\lambda t)\left|f'\left(ma_1 + \frac{\lambda t}{n+1}\eta(x, ma_1)\right)\right|dt$$

$$+\frac{\lambda\eta(a_2, mx)}{2(n+1)\Delta_{m,n}(\lambda)} \times \int_0^1 \Delta_{m,n}(\lambda(1-t))\left|f'\left(mx + \frac{\lambda t}{n+1}\eta(a_2, mx)\right)\right|dt$$

$$\leq \frac{\lambda\eta(x, ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)}$$

$$\times\left(\int_0^1 \left[\Lambda_{m,n}(\lambda t)\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(ma_1 + \frac{\lambda t}{n+1}\eta(x, ma_1)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\lambda\eta(a_2, mx)}{2(n+1)\Delta_{m,n}(\lambda)}$$

$$\times\left(\int_0^1 \left[\Delta_{m,n}(\lambda(1-t))\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(mx + \frac{\lambda t}{n+1}\eta(a_2, mx)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\lambda\eta(x, ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)}\sqrt[p]{B_{\Lambda_{m,n}}(\lambda; p)}$$

$$\times\left(\int_0^1\left[\left(1-\frac{\lambda t}{n+1}\right)|f'(ma_1)|^q+\frac{\lambda t}{n+1}|f'(x)|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)}\sqrt[p]{C_{\Delta_{m,n}}(\lambda;p)}$$
$$\times\left(\int_0^1\left[\left(1-\frac{\lambda t}{n+1}\right)|f'(mx)|^q+\frac{\lambda t}{n+1}|f'(a_2)|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{2(n+1)}\Lambda_{m,n}(\lambda)}\sqrt[p]{B_{\Lambda_{m,n}}(\lambda;p)}\times\sqrt[q]{(2(n+1)-\lambda)|f'(ma_1)|^q+\lambda|f'(x)|^q}$$
$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{2(n+1)}\Delta_{m,n}(\lambda)}\sqrt[p]{C_{\Delta_{m,n}}(\lambda;p)}\times\sqrt[q]{(2(n+1)-\lambda)|f'(mx)|^q+\lambda|f'(a_2)|^q}.$$

The proof of Theorem 2.4 is completed. □

We point out some special cases of Theorem 2.4.

Corollary 2.5. *Taking* $\lambda=\frac{1}{2}$, $m=1$, $n=0$, $x=\frac{ma_1+a_2}{2}$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$ *and* $\varphi(t)=t$ *in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{1,0},\Delta_{1,0}}\left(\frac{a_1+a_2}{2};\frac{1}{2},a_1,a_2\right)\right|\le\frac{(a_2-a_1)}{2^{\frac{3q+2}{q}}\sqrt[p]{p+1}} \tag{2.9}$$
$$\times\left\{\sqrt[q]{3|f'(a_1)|^q+|f'(x)|^q}+\sqrt[q]{3|f'(x)|^q+|f'(a_2)|^q}\right\}.$$

Corollary 2.6. *Taking* $\lambda=1$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$ *and* $\varphi(t)=t$ *in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{1,0},\Delta_{1,0}}(x;1,a_1,a_2)\right|\le\frac{1}{2\sqrt[q]{2}\sqrt[p]{p+1}} \tag{2.10}$$
$$\times\left\{(x-a_1)\sqrt[q]{|f'(a_1)|^q+|f'(x)|^q}+(a_2-x)\sqrt[q]{|f'(x)|^q+|f'(a_2)|^q}\right\}.$$

Corollary 2.7. *Taking* $p=q=2$ *in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right|\le\frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt{2(n+1)}\Lambda_{m,n}(\lambda)}\sqrt{B_{\Lambda_{m,n}}(\lambda;2)} \tag{2.11}$$
$$\times\sqrt{(2(n+1)-\lambda)|f'(ma_1)|^2+\lambda|f'(x)|^2}$$
$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt{2(n+1)}\Delta_{m,n}(\lambda)}\sqrt{C_{\Delta_{m,n}}(\lambda;2)}\times\sqrt{(2(n+1)-\lambda)|f'(mx)|^2+\lambda|f'(a_2)|^2}.$$

Corollary 2.8. *Taking* $\|f'\|_\infty\le K$ *in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right|\le\frac{K\lambda}{2(n+1)} \tag{2.12}$$
$$\times\left\{\frac{\eta(x,ma_1)}{\Lambda_{m,n}(\lambda)}\sqrt[p]{B_{\Lambda_{m,n}}(\lambda;p)}+\frac{\eta(a_2,mx)}{\Delta_{m,n}(\lambda)}\sqrt[p]{C_{\Delta_{m,n}}(\lambda;p)}\right\}.$$

Corollary 2.9. *Taking* $\varphi(t)=t$ *in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right|\le\frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{2(n+1)}\sqrt[p]{p+1}} \tag{2.13}$$
$$\times\sqrt[q]{(2(n+1)-\lambda)|f'(ma_1)|^q+\lambda|f'(x)|^q}$$
$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{2(n+1)}\sqrt[p]{p+1}}\times\sqrt[q]{(2(n+1)-\lambda)|f'(mx)|^q+\lambda|f'(a_2)|^q}.$$

Corollary 2.10. *Taking $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{2(n+1)}\sqrt[p]{p\alpha+1}} \tag{2.14}$$
$$\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(ma_1)\right|^q+\lambda|f'(x)|^q}$$
$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{2(n+1)}\sqrt[p]{p\alpha+1}}\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(mx)\right|^q+\lambda|f'(a_2)|^q}.$$

Corollary 2.11. *Taking $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{2(n+1)}\sqrt[p]{\frac{p\alpha}{k}+1}} \tag{2.15}$$
$$\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(ma_1)\right|^q+\lambda|f'(x)|^q}$$
$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{2(n+1)}\sqrt[p]{\frac{p\alpha}{k}+1}}\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(mx)\right|^q+\lambda|f'(a_2)|^q}.$$

Corollary 2.12. *Taking $\varphi(t)=t(a_2-t)^{\alpha-1}$ for $\alpha\in(0,1)$ in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda\alpha\eta(x,ma_1)}{2(n+1)\sqrt[q]{2(n+1)}\left[a_2^{\alpha}-\left(a_2-\frac{\lambda\eta(x,ma_1)}{n+1}\right)^{\alpha}\right]} \tag{2.16}$$
$$\times\sqrt[p]{B^{\star}_{\Lambda_{m,n}}(\lambda;p)}\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(ma_1)\right|^q+\lambda|f'(x)|^q}$$
$$+\frac{\lambda\alpha\eta(a_2,mx)}{2(n+1)\sqrt[q]{2(n+1)}\left[a_2^{\alpha}-\left(a_2-\frac{\lambda\eta(a_2,mx)}{n+1}\right)^{\alpha}\right]}\sqrt[p]{C^{\star}_{\Delta_{m,n}}(\lambda;p)}$$
$$\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(mx)\right|^q+\lambda|f'(a_2)|^q},$$

where

$$B^{\star}_{\Lambda_{m,n}}(\lambda;p)=\frac{n+1}{\lambda\alpha^p\eta(x,ma_1)}\int_{a_2-\frac{\lambda\eta(x,ma_1)}{n+1}}^{a_2}\left(a_2^{\alpha}-t^{\alpha}\right)^p dt \tag{2.17}$$

and

$$C^{\star}_{\Delta_{m,n}}(\lambda;p)=\frac{n+1}{\lambda\alpha^p\eta(a_2,mx)}\int_{a_2-\frac{\lambda\eta(a_2,mx)}{n+1}}^{a_2}\left(a_2^{\alpha}-t^{\alpha}\right)^p dt. \tag{2.18}$$

Corollary 2.13. *Taking $\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha\in(0,1)$ in Theorem 2.4, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda(\alpha-1)\eta(x,ma_1)}{2(n+1)\sqrt[q]{2(n+1)}\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(x,ma_1)\lambda}{n+1}\right]-1\right\}} \tag{2.19}$$
$$\times\sqrt[p]{B^{\diamond}_{\Lambda_{m,n}}(\lambda;p)}\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(ma_1)\right|^q+\lambda|f'(x)|^q}$$
$$+\frac{\lambda(\alpha-1)\eta(a_2,mx)}{2(n+1)\sqrt[q]{2(n+1)}\left\{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(a_2,mx)\lambda}{n+1}\right]-1\right\}}\sqrt[p]{C^{\diamond}_{\Delta_{m,n}}(\lambda;p)}$$
$$\times\sqrt[q]{(2(n+1)-\lambda)\left|f'(mx)\right|^q+\lambda|f'(a_2)|^q},$$

where

$$B^{\diamond}_{\Lambda_{m,n}}(\lambda;p)=\frac{\alpha(n+1)}{\lambda(\alpha-1)^{p+1}\eta(x,ma_1)}\int_0^{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(x,ma_1)\lambda}{n+1}\right]-1}\frac{t^p}{t+1}dt \tag{2.20}$$

and

$$C^{\diamond}_{\Delta_{m,n}}(\lambda;p)=\frac{\alpha(n+1)}{\lambda(\alpha-1)^{p+1}\eta(a_2,mx)}\int_0^{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(a_2,mx)\lambda}{n+1}\right]-1}\frac{t^p}{t+1}dt. \tag{2.21}$$

Theorem 2.14. *Let $f:P\longrightarrow\mathbb{R}$ be a differentiable mapping on (ma_1,a_2), where $\lambda\in(0,1]$. If $|f'|^q$ is preinvex on P for $q\geq 1$, then the following inequality for generalized fractional integrals hold:*

$$\begin{aligned}\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| &\leq \frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{n+1}\Lambda_{m,n}(\lambda)}\left[B_{\Lambda_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}} \\ &\times\sqrt[q]{\left((n+1)B_{\Lambda_{m,n}}(\lambda;1)-\lambda D_{\Lambda_{m,n}}(\lambda)\right)|f'(ma_1)|^q+\lambda D_{\Lambda_{m,n}}(\lambda)|f'(x)|^q} \\ &+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{n+1}\Delta_{m,n}(\lambda)}\left[C_{\Delta_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}} \\ &\times\sqrt[q]{\left((n+1)C_{\Delta_{m,n}}(\lambda;1)-\lambda E_{\Delta_{m,n}}(\lambda)\right)|f'(mx)|^q+\lambda E_{\Delta_{m,n}}(\lambda)|f'(a_2)|^q},\end{aligned} \tag{2.22}$$

where

$$D_{\Lambda_{m,n}}(\lambda)=\int_0^1 t\Big[\Lambda_{m,n}(\lambda t)\Big]dt,\quad E_{\Delta_{m,n}}(\lambda)=\int_0^1 t\Big[\Delta_{m,n}(\lambda(1-t))\Big]dt \tag{2.23}$$

and $B_{\Lambda_{m,n}}(\lambda;1)$, $C_{\Delta_{m,n}}(\lambda;1)$ are defined as in Theorem 2.4.

Proof. From Lemma 2.1, preinvexity of $|f'|^q$, power mean inequality and properties of the modulus, we have

$$\begin{aligned}\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| &\leq \frac{\lambda\eta(x,ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)} \\ &\times\int_0^1\Lambda_{m,n}(\lambda t)\left|f'\left(ma_1+\frac{\lambda t}{n+1}\eta(x,ma_1)\right)\right|dt \\ &+\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)}\times\int_0^1\Delta_{m,n}(\lambda(1-t))\left|f'\left(mx+\frac{\lambda t}{n+1}\eta(a_2,mx)\right)\right|dt \\ &\leq\frac{\lambda\eta(x,ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)} \\ &\times\left(\int_0^1\Lambda_{m,n}(\lambda t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1\Lambda_{m,n}(\lambda t)\left|f'\left(ma_1+\frac{\lambda t}{n+1}\eta(x,ma_1)\right)\right|^q dt\right)^{\frac{1}{q}} \\ &+\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)} \\ &\times\left(\int_0^1\Delta_{m,n}(\lambda(1-t))dt\right)^{1-\frac{1}{q}}\left(\int_0^1\Delta_{m,n}(\lambda(1-t))\left|f'\left(mx+\frac{\lambda t}{n+1}\eta(a_2,mx)\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\leq\frac{\lambda\eta(x,ma_1)}{2(n+1)\Lambda_{m,n}(\lambda)}\left[B_{\Lambda_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}}\end{aligned}$$

$$\times\left(\int_0^1 \Lambda_{m,n}(\lambda t)\left[\left(1-\frac{\lambda t}{n+1}\right)|f'(ma_1)|^q+\frac{\lambda t}{n+1}|f'(x)|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\Delta_{m,n}(\lambda)}\left[C_{\Delta_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \Delta_{m,n}(\lambda(1-t))\left[\left(1-\frac{\lambda t}{n+1}\right)|f'(mx)|^q+\frac{\lambda t}{n+1}|f'(a_2)|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{n+1}\Lambda_{m,n}(\lambda)}\left[B_{\Lambda_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left((n+1)B_{\Lambda_{m,n}}(\lambda;1)-\lambda D_{\Lambda_{m,n}}(\lambda)\right)|f'(ma_1)|^q+\lambda D_{\Lambda_{m,n}}(\lambda)|f'(x)|^q}$$

$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{n+1}\Delta_{m,n}(\lambda)}\left[C_{\Delta_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left((n+1)C_{\Delta_{m,n}}(\lambda;1)-\lambda E_{\Delta_{m,n}}(\lambda)\right)|f'(mx)|^q+\lambda E_{\Delta_{m,n}}(\lambda)|f'(a_2)|^q}.$$

The proof of Theorem 2.14 is completed. □

We point out some special cases of Theorem 2.14.

Corollary 2.15. *Taking* $\lambda=\frac{1}{2}$, $m=1$, $n=0$, $x=\frac{ma_1+a_2}{2}$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$ *and* $\varphi(t)=t$ *in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{1,0},\Delta_{1,0}}\left(\frac{a_1+a_2}{2};\frac{1}{2},a_1,a_2\right)\right|\leq\frac{(a_2-a_1)}{16}\sqrt[q]{\frac{a_2-a_1}{6}} \tag{2.24}$$

$$\times\left\{\sqrt[q]{4|f'(a_1)|^q+2|f'(x)|^q}+\sqrt[q]{5|f'(x)|^q+|f'(a_2)|^q}\right\}.$$

Corollary 2.16. *Taking* $\lambda=1$, $m=1$, $n=0$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$ *and* $\varphi(t)=t$ *in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{1,0},\Delta_{1,0}}(x;1,a_1,a_2)\right|\leq\frac{(x-a_1)}{4}\sqrt[q]{\frac{x-a_1}{3}}\times\sqrt[q]{|f'(a_1)|^q+2|f'(x)|^q} \tag{2.25}$$

$$+\frac{(a_2-x)}{4}\sqrt[q]{\frac{a_2-x}{3}}\times\sqrt[q]{2|f'(x)|^q+|f'(a_2)|^q}.$$

Corollary 2.17. *Taking* $q=1$ *in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right|\leq\frac{\lambda\eta(x,ma_1)}{2(n+1)^2\Lambda_{m,n}(\lambda)} \tag{2.26}$$

$$\times\left\{\left((n+1)B_{\Lambda_{m,n}}(\lambda;1)-\lambda D_{\Lambda_{m,n}}(\lambda)\right)|f'(ma_1)|+\lambda D_{\Lambda_{m,n}}(\lambda)|f'(x)|\right\}$$

$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)^2\Delta_{m,n}(\lambda)}$$

$$\times\left\{\left((n+1)C_{\Delta_{m,n}}(\lambda;1)-\lambda E_{\Delta_{m,n}}(\lambda)\right)|f'(mx)|+\lambda E_{\Delta_{m,n}}(\lambda)|f'(a_2)|\right\}.$$

Corollary 2.18. *Taking* $\|f'\|_\infty\leq K$ *in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right|\leq\frac{K\lambda}{2(n+1)} \tag{2.27}$$

$$\times\left\{\frac{\eta(x,ma_1)}{\Lambda_{m,n}(\lambda)}B_{\Lambda_{m,n}}(\lambda;1)+\frac{\eta(a_2,mx)}{\Delta_{m,n}(\lambda)}C_{\Delta_{m,n}}(\lambda;1)\right\}.$$

Corollary 2.19. *Taking $\varphi(t)=t$ in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda}{4(n+1)\sqrt[q]{3(n+1)}} \tag{2.28}$$

$$\times\Big\{\eta(x,ma_1)\sqrt[q]{(3(n+1)-2\lambda)|f'(ma_1)|^q+2\lambda|f'(x)|^q}$$

$$+\eta(a_2,mx)\sqrt[q]{(3(n+1)-2\lambda)|f'(mx)|^q+2\lambda|f'(a_2)|^q}\Big\}.$$

Corollary 2.20. *Taking $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda}{2(\alpha+1)(n+1)\sqrt[q]{(\alpha+2)(n+1)}} \tag{2.29}$$

$$\times\Big\{\eta(x,ma_1)\sqrt[q]{((\alpha+2)(n+1)-(\alpha+1)\lambda)|f'(ma_1)|^q+(\alpha+1)\lambda|f'(x)|^q}$$

$$+\eta(a_2,mx)\sqrt[q]{((\alpha+2)(n+1)-(\alpha+1)\lambda)|f'(mx)|^q+(\alpha+1)\lambda|f'(a_2)|^q}\Big\}.$$

Corollary 2.21. *Taking $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda}{2\left(\frac{\alpha}{k}+1\right)(n+1)\sqrt[q]{\left(\frac{\alpha}{k}+2\right)(n+1)}} \tag{2.30}$$

$$\times\left\{\eta(x,ma_1)\sqrt[q]{\left(\left(\frac{\alpha}{k}+2\right)(n+1)-\left(\frac{\alpha}{k}+1\right)\lambda\right)|f'(ma_1)|^q+\left(\frac{\alpha}{k}+1\right)\lambda|f'(x)|^q}\right.$$

$$\left.+\eta(a_2,mx)\sqrt[q]{\left(\left(\frac{\alpha}{k}+2\right)(n+1)-\left(\frac{\alpha}{k}+1\right)\lambda\right)|f'(mx)|^q+\left(\frac{\alpha}{k}+1\right)\lambda|f'(a_2)|^q}\right\}.$$

Corollary 2.22. *Taking $\varphi(t)=t(a_2-t)^{\alpha-1}$ for $\alpha\in(0,1)$ in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right| \leq \frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{n+1}\Lambda^*_{m,n}(\lambda)}\left[B^*_{\Lambda_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}} \tag{2.31}$$

$$\times\sqrt[q]{\left((n+1)B^*_{\Lambda_{m,n}}(\lambda;1)-\lambda D^*_{\Lambda_{m,n}}(\lambda)\right)|f'(ma_1)|^q+\lambda D^*_{\Lambda_{m,n}}(\lambda)|f'(x)|^q}$$

$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{n+1}\Delta^*_{m,n}(\lambda)}\left[C^*_{\Delta_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left((n+1)C^*_{\Delta_{m,n}}(\lambda;1)-\lambda E^*_{\Delta_{m,n}}(\lambda)\right)|f'(mx)|^q+\lambda E^*_{\Delta_{m,n}}(\lambda)|f'(a_2)|^q},$$

where

$$\Lambda^*_{m,n}(\lambda t)=\frac{a_2^\alpha-\left(a_2-\frac{\eta(x,ma_1)(\lambda t)}{n+1}\right)^\alpha}{\alpha},\quad \Delta^*_{m,n}(\lambda t)=\frac{a_2^\alpha-\left(a_2-\frac{\eta(a_2,mx)(\lambda t)}{n+1}\right)^\alpha}{\alpha}, \tag{2.32}$$

$$B^*_{\Lambda_{m,n}}(\lambda;1)=\frac{n+1}{\lambda\alpha\eta(x,ma_1)}\left\{a_2^\alpha\left(a_2-\frac{\eta(x,ma_1)\lambda}{n+1}\right)-\frac{a_2^{\alpha+1}-\left(a_2-\frac{\eta(x,ma_1)\lambda}{n+1}\right)^{\alpha+1}}{\alpha+1}\right\}, \tag{2.33}$$

$$C^{*}_{\Delta_{m,n}}(\lambda;1)=\frac{n+1}{\lambda\alpha\eta(a_2,mx)}\left\{a_2^{\alpha}\left(a_2-\frac{\eta(a_2,mx)\lambda}{n+1}\right)-\frac{a_2^{\alpha+1}-\left(a_2-\frac{\eta(a_2,mx)\lambda}{n+1}\right)^{\alpha+1}}{\alpha+1}\right\} \tag{2.34}$$

and

$$D^{*}_{\Lambda_{m,n}}(\lambda)=\int_0^1 t\Big[\Lambda^{*}_{m,n}(\lambda t)\Big]dt,\quad E^{*}_{\Delta_{m,n}}(\lambda)=\int_0^1 t\Big[\Delta^{*}_{m,n}(\lambda(1-t))\Big]dt. \tag{2.35}$$

Corollary 2.23. *Taking* $\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ *for* $\alpha\in(0,1)$ *in Theorem 2.14, we get*

$$\left|T_{f,\Lambda_{m,n},\Delta_{m,n}}(x;\lambda,a_1,a_2)\right|\le\frac{\lambda\eta(x,ma_1)}{2(n+1)\sqrt[q]{n+1}\Lambda^{\diamond}_{m,n}(\lambda)}\left[B^{\diamond}_{\Lambda_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}} \tag{2.36}$$
$$\times\sqrt[q]{\left((n+1)B^{\diamond}_{\Lambda_{m,n}}(\lambda;1)-\lambda D^{\diamond}_{\Lambda_{m,n}}(\lambda)\right)|f'(ma_1)|^q+\lambda D^{\diamond}_{\Lambda_{m,n}}(\lambda)|f'(x)|^q}$$
$$+\frac{\lambda\eta(a_2,mx)}{2(n+1)\sqrt[q]{n+1}\Delta^{\diamond}_{m,n}(\lambda)}\left[C^{\diamond}_{\Delta_{m,n}}(\lambda;1)\right]^{1-\frac{1}{q}}$$
$$\times\sqrt[q]{\left((n+1)C^{\diamond}_{\Delta_{m,n}}(\lambda;1)-\lambda E^{\diamond}_{\Delta_{m,n}}(\lambda)\right)|f'(mx)|^q+\lambda E^{\diamond}_{\Delta_{m,n}}(\lambda)|f'(a_2)|^q},$$

where

$$\Lambda^{\diamond}_{m,n}(\lambda t)=\frac{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(x,ma_1)(\lambda t)}{n+1}\right]-1}{\alpha-1}, \tag{2.37}$$

$$\Delta^{\diamond}_{m,n}(\lambda t)=\frac{\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)\frac{\eta(a_2,mx)(\lambda t)}{n+1}\right]-1}{\alpha-1} \tag{2.38}$$

and

$$D^{\diamond}_{\Lambda_{m,n}}(\lambda)=\int_0^1 t\Big[\Lambda^{\diamond}_{m,n}(\lambda t)\Big]dt,\quad E^{\diamond}_{\Delta_{m,n}}(\lambda)=\int_0^1 t\Big[\Delta^{\diamond}_{m,n}(\lambda(1-t))\Big]dt \tag{2.39}$$

and $B^{\diamond}_{\Lambda_{m,n}}(\lambda;1)$, $C^{\diamond}_{\Delta_{m,n}}(\lambda;1)$ *are defined by eqs. (2.20) and (2.21) for* $p=1$.

3 Applications to special means and some new error estimates

Consider the following special means for different real numbers α,β and $\alpha\beta\neq 0$, as follows:

(i) The arithmetic mean:
$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},$$

(ii) The harmonic mean:
$$H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},$$

(iii) The logarithmic mean:
$$L(\alpha,\beta)=\frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},$$

(iv) The generalized log-mean:
$$L_r(\alpha,\beta)=\left[\frac{\beta^{r+1}-\alpha^{r+1}}{(r+1)(\beta-\alpha)}\right]^{\frac{1}{r}};\quad r\in\mathbb{Z}\setminus\{-1,0\}.$$

Using the theory results in Section 2, we give some applications to special means for different real numbers.

Proposition 3.1. *Let* $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, *where* $a_1 < a_2$. *Then for* $r \geq 2$ *and* $r \in \mathbb{N}$, *where* $q > 1$ *and* $p^{-1} + q^{-1} = 1$, *the following inequality hold:*

$$\left| A\left(A^r(a_1,a_2), \frac{A^r(3a_1,a_2)}{2^r}\right) - \frac{1}{2}\left[L_r^r\left(a_1, \frac{3a_1+a_2}{4}\right) + L_r^r\left(\frac{a_1+a_2}{2}, \frac{a_1+3a_2}{4}\right)\right]\right| \leq \frac{r(a_2-a_1)}{8\sqrt[q]{8}\sqrt[p]{p+1}} \tag{3.1}$$

$$\times\left\{\sqrt[q]{A\left(3|a_1|^{q(r-1)}, \left|\frac{a_1+a_2}{2}\right|^{q(r-1)}\right)} + \sqrt[q]{A\left(3\left|\frac{a_1+a_2}{2}\right|^{q(r-1)}, |a_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = \frac{1}{2}$, $m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.4, one can obtain the result immediately. □

Proposition 3.2. *Let* $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, *where* $a_1 < a_2$. *Then for* $r \geq 2$ *and* $r \in \mathbb{N}$, *where* $q > 1$ *and* $p^{-1} + q^{-1} = 1$, *the following inequality hold:*

$$\left|A^r(a_1,a_2) - L_r^r(a_1,a_2)\right| \leq \frac{r(a_2-a_1)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{3.2}$$

$$\times\left\{\sqrt[q]{A\left(|a_1|^{q(r-1)}, \left|\frac{a_1+a_2}{2}\right|^{q(r-1)}\right)} + \sqrt[q]{A\left(\left|\frac{a_1+a_2}{2}\right|^{q(r-1)}, |a_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.4, one can obtain the result immediately. □

Proposition 3.3. *Let* $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, *where* $a_1 < a_2$. *Then for* $q > 1$ *and* $p^{-1} + q^{-1} = 1$, *the following inequality hold:*

$$\left| A\left(\frac{1}{A(a_1,a_2)}, \frac{2}{A(3a_1,a_2)}\right) - \frac{1}{2}\left[\frac{1}{L\left(a_1, \frac{3a_1+a_2}{4}\right)} + \frac{1}{L\left(\frac{a_1+a_2}{2}, \frac{a_1+3a_2}{4}\right)}\right]\right| \leq \sqrt[q]{\frac{3}{2}}\frac{(a_2-a_1)}{8\sqrt[p]{p+1}} \tag{3.3}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(|a_1|^{2q}, 3\left|\frac{a_1+a_2}{2}\right|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(\left|\frac{a_1+a_2}{2}\right|^{2q}, 3|a_2|^{2q}\right)}}\right\}.$$

Proof. Taking $\lambda = \frac{1}{2}$, $m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.4, one can obtain the result immediately. □

Proposition 3.4. *Let* $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, *where* $a_1 < a_2$. *Then for* $q > 1$ *and* $p^{-1} + q^{-1} = 1$, *the following inequality hold:*

$$\left|\frac{1}{A(a_1,a_2)} - \frac{1}{L(a_1,a_2)}\right| \leq \frac{(a_2-a_1)}{8\sqrt[p]{p+1}} \tag{3.4}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(|a_1|^{2q}, \left|\frac{a_1+a_2}{2}\right|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(\left|\frac{a_1+a_2}{2}\right|^{2q}, |a_2|^{2q}\right)}}\right\}.$$

Proof. Taking $\lambda = m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.4, one can obtain the result immediately. □

Proposition 3.5. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $r \geq 2$ and $r \in \mathbb{N}$, where $q \geq 1$, the following inequality hold:*

$$\left| A\left(A^r(a_1, a_2), \frac{A^r(3a_1, a_2)}{2^r}\right) - \frac{1}{2}\left[L_r^r\left(a_1, \frac{3a_1 + a_2}{4}\right) + L_r^r\left(\frac{a_1 + a_2}{2}, \frac{a_1 + 3a_2}{4}\right)\right]\right|$$

$$\leq \frac{r(a_2 - a_1)}{16}\sqrt[q]{\frac{a_2 - a_1}{3}} \tag{3.5}$$

$$\times\left\{\sqrt[q]{A\left(4|a_1|^{q(r-1)}, 2\left|\frac{a_1 + a_2}{2}\right|^{q(r-1)}\right)} + \sqrt[q]{A\left(5\left|\frac{a_1 + a_2}{2}\right|^{q(r-1)}, |a_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = \frac{1}{2}$, $m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.14, one can obtain the result immediately. □

Proposition 3.6. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $r \geq 2$ and $r \in \mathbb{N}$, where $q \geq 1$, the following inequality hold:*

$$\left|A^r(a_1, a_2) - L_r^r(a_1, a_2)\right| \leq \frac{r(a_2 - a_1)}{8}\sqrt[q]{\frac{a_2 - a_1}{3}} \tag{3.6}$$

$$\times\left\{\sqrt[q]{A\left(|a_1|^{q(r-1)}, 2\left|\frac{a_1 + a_2}{2}\right|^{q(r-1)}\right)} + \sqrt[q]{A\left(2\left|\frac{a_1 + a_2}{2}\right|^{q(r-1)}, |a_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.14, one can obtain the result immediately. □

Proposition 3.7. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $q \geq 1$, the following inequality hold:*

$$\left| A\left(\frac{1}{A(a_1, a_2)}, \frac{2}{A(3a_1, a_2)}\right) - \frac{1}{2}\left[\frac{1}{L\left(a_1, \frac{3a_1+a_2}{4}\right)} + \frac{1}{L\left(\frac{a_1+a_2}{2}, \frac{a_1+3a_2}{4}\right)}\right]\right|$$

$$\leq \frac{(a_2 - a_1)}{16}\sqrt[q]{\frac{a_2 - a_1}{3}} \tag{3.7}$$

$$\times\left\{\sqrt[q]{\frac{4}{H\left(|a_1|^{2q}, 2\left|\frac{a_1+a_2}{2}\right|^{2q}\right)}} + \sqrt[q]{\frac{5}{H\left(\left|\frac{a_1+a_2}{2}\right|^{2q}, 5|a_2|^{2q}\right)}}\right\}.$$

Proof. Taking $\lambda = \frac{1}{2}$, $m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.14, one can obtain the result immediately. □

Proposition 3.8. *Let $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$. Then for $q \geq 1$, the following inequality hold:*

$$\left|\frac{1}{A(a_1, a_2)} - \frac{1}{L(a_1, a_2)}\right| \leq \frac{(a_2 - a_1)}{8}\sqrt[q]{\frac{2(a_2 - a_1)}{3}} \tag{3.8}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(2|a_1|^{2q}, \left|\frac{a_1+a_2}{2}\right|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(\left|\frac{a_1+a_2}{2}\right|^{2q}, 2|a_2|^{2q}\right)}}\right\}.$$

Proof. Taking $\lambda = m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.14, one can obtain the result immediately. □

Remark 3.9. Applying our Theorems 2.4 and 2.14 for special parameter values λ and various suitable choices of function $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi(t) = t(a_2-t)^{\alpha-1}$ and $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$, such that $|f'|^q$ to be convex, we can deduce some new general fractional integral inequalities using above special means. We omit the proof here and the details are left to the interested reader.

Next, we provide some new error estimates for the trapezium and midpoint formula. Let Q be the partition of the points $a_1 = x_0 < x_1 < \ldots < x_k = a_2$ of the interval $[a_1, a_2]$. Let consider the following quadrature formula:

$$\int_{a_1}^{\frac{3a_1+a_2}{4}} f(x)dx + \int_{\frac{a_1+a_2}{2}}^{\frac{a_1+3a_2}{4}} f(x)dx = T(f,Q) + E(f,Q),$$

where

$$T(f,Q) = \sum_{i=0}^{k-1}\left[f\left(\frac{x_i+x_{i+1}}{2}\right) + f\left(\frac{3x_i+x_{i+1}}{4}\right)\right]\frac{(x_{i+1}-x_i)}{2}$$

is the trapezium version and $E(f,Q)$ is denote their associated approximation error. Also

$$\int_{a_1}^{a_2} f(x)dx = M(f,Q) + E^{\star}(f,Q),$$

where

$$M(f,Q) = \sum_{i=0}^{k-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$

is the midpoint version and $E^{\star}(f,Q)$ is denote their associated approximation error.

Proposition 3.10. *Let $f : [a_1, a_2] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2), where $a_1 < a_2$. If $|f'|^q$ is convex on $[a_1, a_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality holds:*

$$\left|E(f,Q)\right| \leq \frac{1}{8\sqrt[q]{8}\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.9}$$

$$\times\left\{\sqrt[q]{3|f'(x_i)|^q + \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q} + \sqrt[q]{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}\right\}.$$

Proof. Applying Theorem 2.4 for $\lambda = \frac{1}{2}$, $m = 1$, $n = 0$, $x = \frac{ma_1+a_2}{2}$, $\eta(x, ma_1) = x - ma_1$, $\eta(a_2, mx) = a_2 - mx$, on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \ldots, k-1)$ of the partition Q, we have

$$\Bigg|f\left(\frac{x_i+x_{i+1}}{2}\right) + f\left(\frac{3x_i+x_{i+1}}{4}\right)$$

$$-\frac{2}{(x_{i+1}-x_i)}\left[\int_{x_i}^{\frac{3x_i+x_{i+1}}{4}} f(x)dx + \int_{\frac{x_i+x_{i+1}}{2}}^{\frac{x_i+3x_{i+1}}{4}} f(x)dx\right]\Bigg|$$

$$\leq \frac{(x_{i+1}-x_i)}{2^{\frac{3q+2}{q}}\sqrt[p]{p+1}} \tag{3.10}$$

$$\times\left\{\sqrt[q]{3|f'(x_i)|^q + \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q} + \sqrt[q]{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}\right\}.$$

Hence from (3.10), we get

$$\begin{aligned}
\left|E(f,Q)\right| &= \left|\int_{a_1}^{\frac{3a_1+a_2}{4}} f(t)dt + \int_{\frac{a_1+a_2}{2}}^{\frac{a_1+3a_2}{4}} f(x)dx - T(f,Q)\right| \\
&\leq \left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{\frac{3x_i+x_{i+1}}{4}} f(x)dx + \int_{\frac{x_i+x_{i+1}}{2}}^{\frac{x_i+3x_{i+1}}{4}} f(x)dx\right.\right. \\
&\left.\left. -\left[f\left(\frac{x_i+x_{i+1}}{2}\right)+f\left(\frac{3x_i+x_{i+1}}{4}\right)\right]\frac{(x_{i+1}-x_i)}{2}\right\}\right| \\
&\leq \sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{\frac{3x_i+x_{i+1}}{4}} f(x)dx + \int_{\frac{x_i+x_{i+1}}{2}}^{\frac{x_i+3x_{i+1}}{4}} f(x)dx\right.\right. \\
&\left.\left. -\left[f\left(\frac{x_i+x_{i+1}}{2}\right)+f\left(\frac{3x_i+x_{i+1}}{4}\right)\right]\frac{(x_{i+1}-x_i)}{2}\right\}\right| \\
&\leq \frac{1}{8\sqrt[q]{8}\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \\
&\times\left\{\sqrt[q]{3|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.
\end{aligned}$$

The proof of Proposition 3.10 is completed. □

Proposition 3.11. *Let $f:[a_1,a_2]\longrightarrow\mathbb{R}$ be a differentiable function on (a_1,a_2), where $a_1<a_2$. If $|f'|^q$ is convex on $[a_1,a_2]$ for $q\geq 1$, then the following inequality holds:*

$$\left|E(f,Q)\right| \leq \frac{1}{32\sqrt[q]{6}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\sqrt[q]{x_{i+1}-x_i} \tag{3.11}$$

$$\times\left\{\sqrt[q]{4|f'(x_i)|^q+2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{5\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

Proof. The proof is analogous as to that of Proposition 3.10 but use Theorem 2.14. □

Proposition 3.12. *Let $f:[a_1,a_2]\longrightarrow\mathbb{R}$ be a differentiable function on (a_1,a_2), where $a_1<a_2$. If $|f'|^q$ is convex on $[a_1,a_2]$ for $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality holds:*

$$\left|E^{\star}(f,Q)\right| \leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.12}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

Proof. Applying Theorem 2.4 for $\lambda=m=1$, $n=0$, $x=\frac{ma_1+a_2}{2}$, $\eta(x,ma_1)=x-ma_1$, $\eta(a_2,mx)=a_2-mx$, on the subintervals $[x_i,x_{i+1}]$ $(i=0,\ldots,k-1)$ of the partition Q, we have

$$\left|f\left(\frac{x_i+x_{i+1}}{2}\right)-\frac{1}{(x_{i+1}-x_i)}\int_{x_i}^{x_{i+1}}f(x)dx\right| \leq \frac{(x_{i+1}-x_i)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{3.13}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}+\sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}\right\}.$$

Hence from (3.10), we get

$$\left|E^{\star}(f,Q)\right| = \left|\int_{a_1}^{a_2} f(t)dt - M(f,Q)\right|$$

$$\leq \left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|$$

$$\leq \sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right|$$

$$\leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q + \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q} + \sqrt[q]{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}\right\}.$$

The proof of Proposition 3.12 is completed. □

Proposition 3.13. *Let $f : [a_1, a_2] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2), where $a_1 < a_2$. If $|f'|^q$ is convex on $[a_1, a_2]$ for $q \geq 1$, then the following inequality holds:*

$$\left|E^{\star}(f,Q)\right| \leq \frac{1}{8\sqrt[q]{6}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2\sqrt[q]{x_{i+1}-x_i} \tag{3.14}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q} + \sqrt[q]{2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}\right\}.$$

Proof. The proof is analogous as to that of Proposition 3.12 but use Theorem 2.14. □

Remark 3.14. Applying our Theorems 2.4 and 2.14 for value $m = 1$, for special parameter values λ and various suitable choices of function $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}, \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi(t) = t(a_2-t)^{\alpha-1}$ and $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$, such that $|f'|^q$ to be convex, we can deduce some new bounds for the trapezium and midpoint formula using above ideas and techniques. We omit the proof here and the details are left to the interested reader.

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