Palestine Journal of Mathematics

Vol. 10(1)(2021) , 151–159

Multipliers for Gelfand Pairs

Abudulaï Issa and Yaogan Mensah

Communicated by Fuad Kittaneh

MSC 2010 Classifications: Primary 43A22 ; Secondary 43A20.

Keywords and phrases: convolution, Gelfand pair, multiplier, spherical Fourier transform, Banach algebra.

Abstract This paper addresses the characterization of the multipliers for Gelfand pairs. More precisely we obtain a theorem of Wendel type for the space of integrable bi-invariant functions on a locally compact group. We also obtain some results concerning the double multipliers for Gelfand pairs.

1 Introduction

In [16], Larsen pointed out the origin of the concept of a multiplier. It is extremely linked up with the summability of Fourier series. From then it became ubiquitous in harmonic analysis and other areas of pure and applied mathematics among which one can mention the general theory of Banach algebras, the theory of singular integrals and fractional integration, stochastic processes, time-frequency analysis, the theory of semigroups of operators and partial differential equations. The theory of multipliers is a very active field and it would be pretentious to name all those who contributed to its success. However let us mention some pioneers works in this area: multipliers for semisimple commutative Banach algebras (Helgason [10]), multipliers for commutative faithful Banach algebras (Wang [20] and Birtel [1]), multipliers for group algebras (Wendel [21, 22] and Edwards [7]).
More recently, A. Riazi and M. Adib [18] generalized the concept of multipliers on faithful Banach algebras to φ-multipliers where φ is an algebra homomorphism.
In this paper we study the multipliers and double multipliers related to Gelfand pairs. To do this we may consider a locally compact second countable group with a certain compact subgroup. Then the concept of Gelfand pair is linked to the commutativity of a certain Banach algebra. The main purpose of this paper is to study the multipliers of this commutative Banach algebra.
The rest of the paper is organized as follows. In Section 2 we collect some results on the multipliers for the group algebra $L^1(G)$ where G is a locally compact abelian group. Section 3 is devoted to the Gelfand transform, the Gelfand pairs and the spherical functions. In Section 4 we set our results on the multipliers for Gelfand pairs. Section 5 is concerned by double multipliers for Gelfand pairs. Finally, in Section 6, our main result is related to multipliers in the framework of hypergroups.

2 Multipliers for the group algebra $L^1(G)$

Let G be a locally compact group endowed with a fixed left Haar mesure dx. In this section we present the concept of multipliers for the group algebra $L^1(G)$, the latter being the linear space of equivalence classes of Lebesgue integrable complex valued functions on G. The set $L^1(G)$ has the natural norm

$$\|f\|_1 = \int_G |f(x)|dx, \tag{2.1}$$

and is endowed with the convolution product

$$f * g(x) = \int_G f(xy^{-1})g(y)dy,\ f, g \in L^1(G). \tag{2.2}$$

Under the above norm and product, the space $L^1(G)$ is a Banach algebra. This algebra is commutative if and only if G is a commutative group [9, page 49]. It is possible to extend the structure of Banach algebra to the set $M(G)$ of complex Radon measures on G. For this purpose one defines the norm

$$\|\mu\| = |\mu|(G) \tag{2.3}$$

where $|\mu|$ stands for the total variation of the measure μ. The total variation of μ is defined by

$$|\mu|(E) = \sup \sum_{A\in\Pi} |\mu(A)| \tag{2.4}$$

where the supremum is taken over all the partitions Π of E into pairwise disjoint Borel sets. One defines on the space $M(G)$ the convolution product

$$\mu * \nu(f) = \int_G \int_G f(xy)d\mu(x)d\nu(y),\ \mu,\nu \in M(G),\ f \in \mathcal{C}_c(G), \tag{2.5}$$

where $\mathcal{C}_c(G)$ is the space of continuous complex functions with compact support. Then $(M(G), \|\cdot\|, *)$ is a Banach algebra with identity which is the Dirac measure δ [9, page 50].

From now to the end of this section, G is assumed to be a locally compact abelian group. A group homomorphism from G into the unit circle $\mathbb{T}$ is called a *character* of G. All the characters of G form a group called the *dual group* of G and it is denoted by $\widehat{G}$. In $\widehat{G}$ the group law is the multiplication of functions. When $\widehat{G}$ is endowed with the compact-open topology then it becomes a locally compact abelian group and the Pontrjagin duality states that $\widehat{\widehat{G}} \simeq G$ [4, Chapter 3].
Now one can define the Fourier transformation. The Fourier transform of a function $f \in L^1(G)$ is defined by

$$\widehat{f}(\chi) = \int_G \overline{\chi(x)} f(x)dx,\ \chi \in \widehat{G}. \tag{2.6}$$

and the Fourier-Stieltjes transform of a measure $\mu \in M(G)$ is defined by

$$\widehat{\mu}(\chi) = \int_G \overline{\chi(x)} d\mu(x),\ \chi \in \widehat{G}. \tag{2.7}$$

The following relations are well-known in the literature. See for instance [9, page 94].

$$\widehat{f * g} = \widehat{f}\widehat{g},\ f,g \in L^1(G). \tag{2.8}$$

$$\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu},\ \mu,\nu \in M(G). \tag{2.9}$$

For $x \in G$ the translation operator associated with x and denoted by τ_x is defined by

$$(\tau_x f)(y) = f(yx^{-1}). \tag{2.10}$$

The following proposition characterizes linear transformations on $L^1(G)$ which commute with the translation operators. This proposition is the bedrock of the theory of multipliers for commutative Banach algebras [16].

Proposition 2.1. *Let $T : L^1(G) \longrightarrow L^1(G)$ be a continuous linear transformation. Then the following assertions are equivalent.*

(i) $\tau_x T = T\tau_x,\ \forall x \in G.$

(ii) $T(f * g) = Tf * g,\ \forall f,g \in L^1(G).$

(iii) There exists a unique function $\mathfrak{P}$ defined on $\widehat{G}$ such that $\widehat{Tf} = \mathfrak{P}\widehat{f},\ \forall f \in L^1(G)$.

(iv) There exists a unique measure $\mu \in M(G)$ such that $\widehat{Tf} = \widehat{\mu}\widehat{f},\ \forall f \in L^1(G)$.

(v) There exists a unique measure $\mu \in M(G)$ such that $Tf = f * \mu,\ \forall f \in L^1(G)$.

A *multiplier* for $L^1(G)$ is a continuous linear operator on $L^1(G)$ that satisfies one of the above equivalent assertions.

3 The Gelfand transform, Gelfand pairs and spherical functions

In this section, we are interested in the Gelfand transform on a commutative Banach algebra, the Gelfand pairs and the spherical functions. All this concepts will pave the way for the study of the multipliers for Gelfand pairs.
Let us briefly recall the definition of the Gelfand transform on a commutative Banach algebra $\mathfrak{A}$. For more details one may refer to [13]. An interesting treatment of the subject can also be found in the lecture notes [15]. Let $\Delta(\mathfrak{A})$ be the set of all the non-zero continuous algebra homomorphisms of $\mathfrak{A}$ into $\mathbb{C}$. One denotes by $\Delta(\mathfrak{A})$ *the structure space* of $\mathfrak{A}$. One has $\Delta(\mathfrak{A}) \subset \mathfrak{A}^*$. So we endow $\Delta(\mathfrak{A})$ with the relative w^*-topology of $\mathfrak{A}^*$ which is called the Gelfand topology. This topology turns $\Delta(\mathfrak{A})$ into a locally compact Hausdorff space. One denotes by $\mathcal{C}_0(\Delta(\mathfrak{A}))$ the set of all the continuous functions on $\Delta(\mathfrak{A})$ which vanish at infinity. Equipped with the sup-norm and the ordinary product of functions, $\mathcal{C}_0(\Delta(\mathfrak{A}))$ is a Banach algebra. The Gelfand transform of $\mathfrak{A}$ is the map $\mathfrak{G} : \mathfrak{A} \to \mathcal{C}_0(\Delta(\mathfrak{A}))$ defined by

$$(\mathfrak{G}a)(\omega) = \omega(a). \tag{3.1}$$

We will often denote the Gelfand transform of a by $\widehat{a}$ instead of $\mathfrak{G}a$. A Banach algebra is said to be semisimple if the Gelfand transformation $a \mapsto \widehat{a}$ is injective. For instance the convolution Banach algebra $L^1(G)$ seen in the above section is semisimple.

Let us notice that the Fourier transform is the Gelfand transform for a particular Banach algebra. For instance the Fourier transform on the locally compact abelian group G is the Gelfand transform for the commutative Banach algebra $L^1(G)$.

Now let G be a locally compact group and let K be a compact subgroup of G.

Definition 3.1. A function $f : G \to \mathbb{C}$ is said to be $K-$bi-invariant if

$$f(k_1xk_2) = f(x),\ \forall x \in G,\ \forall k_1, k_2 \in K. \tag{3.2}$$

We denote by $C_c(G)$ the set of compact supported complex functions on G and $C_c(G//K)$ stands for functions in $C_c(G)$ which are $K-$bi-invariant. The convolution product (2.2) is defined on $C_c(G//K)$. Let us denote by $L^1(G//K)$ the set of integrable bi-invariant functions. Since the convolution product is separately continuous and since $C_c(G//K)$ is dense in the space $L^1(G//K)$, the convolution product is uniquely extended to $L^1(G//K)$. The space $L^1(G//K)$ is a Banach algebra under the convolution product and the L^1- norm. The space $C_c(G//K)$ is a dense subalgebra of $L^1(G//K)$.

Herafter is the definition of a Gelfand pair. Interested readers may refer to [5], [8] or [12].

Definition 3.2. A Gelfand pair is a couple (G, K) such that the Banach algebra $C_c(G//K)$ is commutative.

By density of $C_c(G//K)$ in $L^1(G//K)$, the definition implies that $L^1(G//K)$ is also commutative. $L^1(G//K)$ is semisimple by the Gelfand-Raikov theorem.

The following proposition gives a necessary condition for a pair (G, K) to be a Gelfand pair [12].

Proposition 3.3. *If (G, K) is a Gelfand pair, then G is unimodular.*

The following proposition gives sufficient condition for a pair (G, K) to be a Gelfand pair. The proof can be found in [12, page 220].

Proposition 3.4. *Let G be a locally compact group and let K be a compact subgroup of G. Then the pair (G, K) is a Gelfand pair if there exists a continuous involutive automorphism θ of G such that $\theta(x) \in Kx^{-1}K$, $\forall x \in G$.*

Harmonic analysis on Gelfand pairs is based on the concept of spherical function, with particular emphasis on the bounded ones, and on the spherical Fourier transform.

Definition 3.5. Let (G,K) be a Gelfand pair. A spherical function is a K-bi-invariant continuous function φ on G, such that the mapping

$$f \longmapsto \chi(f) = \int_G f(x)\varphi(x^{-1})dx \tag{3.3}$$

is a non-zero continous homomorphism (character) of the convolution algebra $\mathcal{C}_c(G//K)$, that is χ is linear, continuous and

$$\chi(f*g) = \chi(f)\chi(g),\ \forall f,g \in \mathcal{C}_c(G//K). \tag{3.4}$$

For instance if $G = \mathbb{R}$ and $K = \{0\}$ then spherical functions are the exponential functions $\varphi_\lambda(x) = e^{i\lambda x}, \lambda \in \mathbb{R}$.
The following proposition is proved in [6].

Proposition 3.6. *A non-zero function $\varphi \in \mathcal{C}_c(G//K)$ is a spherical function for a Gelfand pair (G,K) if and only if*

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y),\ \forall x,y \in G. \tag{3.5}$$

The following proposition can be found in [6].

Proposition 3.7. *Let φ be a continuous K-bi-invariant function on G. Then φ is a spherical function if and only if $\varphi(e) = 1$ and $\forall f \in C_c(G//K),\ f*\varphi = \chi(f)\varphi$.*

There is a link between the bounded spherical functions and the characters of the Banach algebra $L^1(G//K)$.

Proposition 3.8. *Each non-zero character χ of the convolution Banach algebra $L^1(G//K)$ is of the form*

$$\chi(f) = \int_G f(x)\varphi(x^{-1})dx \tag{3.6}$$

where φ is a bounded spherical function on G.

We refer again to [6, page 208] for the proof.

In this paper, we denote by $S(G,K)$ the set of all the bounded spherical functions for the Gelfand pair (G,K). The latter is endowed with the topology of the uniform convergence on compact subsets defined by the seminorms

$$N_A(\varphi) = \sup_{x\in A} |\varphi(x)| \tag{3.7}$$

where A runs in the set of all the compact subsets of G. From the Proposition 3.8 it is clear that $S(G,K)$ can be identified with the structure space of $L^1(G//K)$. Therefore one obtains the spherical Fourier transform. The following definition precises the notion.

Definition 3.9. The spherical Fourier transform of a function $f \in L^1(G//K)$, denoted by $\widehat{f}$, is defined by

$$\widehat{f}(\varphi) = \int_G f(x)\varphi(x^{-1})dx,\ \varphi \in S(G,K). \tag{3.8}$$

The spherical Fourier transform satisfies the following relation:

$$\widehat{f*g} = \widehat{f} \times \widehat{g},\ \forall f,g \in L^1(G//K). \tag{3.9}$$

4 Multipliers for Gelfand pairs

We state our main results in this section. Our main goal is to characterize the multipliers of the commutative Banach algebra $L^1(G//K)$ by the means of the spherical Fourier transform.
We set the following definition of a multiplier.

Definition 4.1. A multiplier for a Gelfand pair (G, K) is a transformation

$$T : L^1(G//K) \to L^1(G//K)$$

such that

$$T(f * g) = Tf * g, \forall f, g \in L^1(G//K). \quad (4.1)$$

A note in [16, page 13] shows that the condition in the above definition implies the linearity and the contuinity of the multiplier T. We denote by $\mathcal{M}(G, K)$ the set of all the multipliers for the Gelfand pair (G, K). The following proposition gives a characterization of a multiplier for (G, K) by the means of the spherical Fourier transform.

Proposition 4.2. *Let (G, K) be a Gelfand pair. Let $T : L^1(G//K) \to L^1(G//K)$ be a transformation. Then the following assertions are equivalent.*

(i) T is a multiplier for (G, K).

(ii) There exists a unique function $\mathfrak{P}$ defined on $S(G, K)$ such that

$$\widehat{Tf} = \mathfrak{P}\widehat{f}, \forall f \in L^1(G//K).$$

Proof. (1) $\Rightarrow$ (2)

Assume $T(f * g) = Tf * g$ for all $f, g \in L^1(G//K)$. Since (G, K) is a Gelfand pair then $L^1(G//K)$ is commutative under the convolution product. Therefore we have

$$Tf * g = T(f * g) = T(g * f) = Tg * f.$$

Using a property of the spherical Fourier transform we obtain

$$\widehat{Tf} \times \widehat{g} = \widehat{Tg} \times \widehat{f}.$$

For each spherical function φ, let us choose g in $L^1(G//K)$ such that $\widehat{g}(\varphi) \neq 0$.
Now, define $\mathfrak{P}$ by $\mathfrak{P}(\varphi) = \dfrac{\widehat{Tg}(\varphi)}{\widehat{g}(\varphi)}$ (this definition does not depend on the choice of g because of the relation $\widehat{Tf} \times \widehat{g} = \widehat{Tg} \times \widehat{f}$). Therefore we have $\widehat{Tf}(\varphi) = \mathfrak{P}(\varphi)\widehat{f}(\varphi)$ for all $\varphi \in S(G, K)$. Hence $\widehat{Tf} = \mathfrak{P}\widehat{f}$.
Let's show the unicity of $\mathfrak{P}$. If $\mathfrak{R}$ is a second function on $S(G, K)$ such that $\widehat{Tf} = \mathfrak{R}\widehat{f} = \mathfrak{P}\widehat{f}$ for all $f \in L^1(G//K)$ then the equation $(\mathfrak{P} - \mathfrak{R})\widehat{f} = 0$ for all $f \in L^1(G//K)$ reveals that $\mathfrak{P} = \mathfrak{R}$.
(2) $\Rightarrow$ (1)
Let us assume that there exists a function $\mathfrak{P}$ defined on $S(G, K)$ such that $\widehat{Tf} = \mathfrak{P}\widehat{f}, \forall f \in L^1(G//K)$. For $f, g \in L^1(G//K)$, $f * g$ is in $L^1(G//K)$. Applying the hypothesis, one has $\widehat{T(f * g)} = \mathfrak{P}\widehat{(f * g)} = \mathfrak{P}\widehat{f}\widehat{g} = \widehat{Tf}\widehat{g} = \widehat{Tf * g}$. Since $L^1(G//K)$ is semisimple, we have $T(f * g) = Tf * g$. □

We denote by $\mathcal{L}(G, K)$ the space of all the continuous linear operators from $L^1(G//K)$ to $L^1(G//K)$. Then clearly $\mathcal{M}(G, K)$ is a closed commutative subalgebra of $\mathcal{L}(G, K)$ which contains the identity operator of $\mathcal{L}(G, K)$.

For $g \in L^1(G//K)$, consider the convolution operator

$$m_g : L^1(G//K) \to L^1(G//K), f \mapsto m_g f = g * f. \quad (4.2)$$

Proposition 4.3. *We have $m_g \in \mathcal{M}(G, K)$.*

Proof. Let $f, g, h \in L^1(G//K)$.

$$m_g f * h = (g * f) * h = g * (f * h) = m_g(f * h)$$

Thus $m_g \in \mathcal{M}(G, K)$.

□

Proposition 4.4. *$\mathcal{M}(G,K)$ is a maximal commutative subalgebra of $\mathcal{L}(G,K)$.*

Proof. Suppose that $\mathcal{M}(G,K)$ is not a maximal commutative subalgebra of $\mathcal{L}(G,K)$. Since $\mathcal{L}(G,K)$ contains an identity, there exists a maximal commutative subalgebra $\mathcal{N}(G,K)$ of $\mathcal{L}(G,K)$ which contains properly $\mathcal{M}(G,K)$.
Let $T \in \mathcal{N}(G,K)$. One has

$$f * Tg = m_f(Tg) = (m_f T)g.$$

Since $\mathcal{N}(G,K)$ is commutative then we have

$$(m_f T)g = (Tm_f)g.$$

But

$$\begin{aligned}(Tm_f)g &= T(m_f g)\\ &= T(f*g).\end{aligned}$$

Since $L^1(G//K)$ is commutative, it follows that

$$Tg * f = T(g*f)$$

Thus $T \in \mathcal{M}(G,K)$. This contradicts the construction of $\mathcal{N}(G,K)$. □

5 Double multipliers for Gelfand pairs

Here we study the double multipliers for the Banach algebra $L^1(G//K)$. We start by giving the general definition of a double multiplier.

Definition 5.1. Let $\mathfrak{A}$ be a Banach algebra. A pair (S,T) of maps $S : \mathfrak{A} \longrightarrow \mathfrak{A}$,
$T : \mathfrak{A} \longrightarrow \mathfrak{A}$ is called a double multiplier for $\mathfrak{A}$ if

$$f(Sg) = (Tf)g, \forall f, g \in \mathfrak{A}. \tag{5.1}$$

We are interested in the double multipliers for the convolution Banach algebra $L^1(G//K)$. For this algebra the definition is reformulated as follows.

Definition 5.2. Let (G,K) be a Gelfand pair. A pair (S,T) of maps $S : L^1(G//K) \longrightarrow L^1(G//K)$, $T : L^1(G//K) \longrightarrow L^1(G//K)$ is called a double multiplier for (G,K) if

$$f*(Sg) = (Tf)*g, \forall f, g \in L^1(G//K). \tag{5.2}$$

We denote by $\mathcal{M}_d(G,K)$ the set of all the double multipliers for (G,K).

Proposition 5.3. *Let (G,K) be a Gelfand pair and let $(S,T) \in \mathcal{M}_d(G,K)$. Then one has*

(i) $S, T \in \mathcal{M}(G,K)$.

(ii) $S = T$.

Proof. (1) Let $f, g, h \in L^1(G//K)$. We have

$$\begin{aligned}h*T(f*g) &= Sh*(f*g)\\ &= (Sh*f)*g\\ &= (h*Tf)*g\\ &= h*(Tf*g)\end{aligned}$$

Since $L^1(G//K)$ is semisimple, it implies that $T(f*g) = Tf*g$. Thus $T \in \mathcal{M}(G,K)$. Similarly, $S \in \mathcal{M}(G,K)$.
(2) Let $f, g \in L^1(G//K)$. We have

$$g*Sf = Tg*f = f*Tg = T(f*g) = T(g*f) = g*Tf.$$

Since $L^1(G//K)$ is semisimple, we have $Sf = Tf$ for all $f \in L^1(G//K)$. Hence $S = T$. □

Proposition 5.4. *Let (G,K) be a Gelfand pair. If T is a multiplier for (G,K) then the following assertions are equivalent.*

(i) T is involutive.

(ii) T^{-1} exists and $(T,T^{-1}) \in \mathcal{M}_d(G,K)$.

Proof. If (1) holds then T is bijective and $T = T^{-1}$. So T^{-1} exists and $T^{-1} \in \mathcal{M}(G,K)$. Since T is a multiplier we have $Tf * g = f * Tg$, $\forall f, g \in L(G//K)$.

$$\begin{aligned} T = T^{-1} \quad &\Longrightarrow \quad Tf * g = f * T^{-1}g \\ &\Longrightarrow \quad (T,T^{-1}) \in \mathcal{M}_d(G,K) \end{aligned}$$

Thus $(1) \Longrightarrow (2)$.

The converse is a consequence of the Proposition 5.3. □

A general Banach algebra $\mathfrak{A}$ is said to be *without order* if for $x \in \mathfrak{A}$,

$$xy = 0, \forall y \in \mathfrak{A} \Longrightarrow x = 0.$$

Via the Gelfand transform in $\mathfrak{A}$, one sees that if $\mathfrak{A}$ is a commutative semisimple Banach algebra then $\mathfrak{A}$ is without order.
To go further, we may need the following result in [11].

Lemma 5.5. *Let $\mathcal{A}$ be a Banach algebra without order . Let $x, z \in \mathcal{A}$ such that $xyz = zyx$ for all $y \in \mathcal{A}$. If $x \neq 0$ then there exists $\lambda \in \mathbb{C}$, such that $x = \lambda z$.*

Proposition 5.6. *Let (G,K) be a Gelfand pair. Let $T, T' \in \mathcal{M}(G,K)$ be such that*

$$Tf * T'g = T'f * Tg, \forall f, g \in L^1(G//K). \tag{5.3}$$

If $T \neq 0$ then there exists $\lambda \in \mathbb{C}$ such that $T = \lambda T'$.

Proof. For the proof we borrow the method from [23, Theorem 2.2]. Let $T, T' \in \mathcal{M}(G,K)$ and let $f, g, h \in L^1(G//K)$. Replacing g by $g * h$ in (5.3), we have

$$\begin{aligned} Tf * T'(g*h) = T'f * T(g*h) \quad &\Longleftrightarrow \quad Tf * (g * T'h) = T'f * (g * Th) \\ &\Longleftrightarrow \quad Tf * g * T'h = T'f * g * Th. \quad (\bigstar) \end{aligned}$$

In the case $f = h$, using the Lemma 5.5 under the assumption $Tf \neq 0$, we deduce the existence of a constant $\lambda(f) \in \mathbb{C}$ depending a priori on f such that $T'f = \lambda(f)Tf$. Now let us show that the constant $\lambda(f)$ is independent of f. If f_1 and f_2 are such that $T'f_1 = \lambda(f_1)Tf_1$ and $T'f_2 = \lambda(f_2)Tf_2$ then we have

$$\lambda(f_2)Tf_1 * g * Tf_2 = \lambda(f_1)Tf_1 * g * Tf_2, \forall g \in L^1(G//K).$$

That is $[\lambda(f_2) - \lambda(f_1)]Tf_1 * g * Tf_2 = 0$, $\forall g \in L^1(G//K)$. So $\lambda(f_2) = \lambda(f_1) = \lambda$.
Now if $Tf = 0$ the relation $(\bigstar)$ gives $T'f * g * Th = 0$, $\forall g, h \in L^1(G//K)$. So $T'f = 0$. Therefore $T'f = \lambda Tf$.
Finally we have $T'f = \lambda Tf, \forall f \in L^1(G//K)$.

□

6 Multipliers in the framework of hypergroups

The results of the previous sections are obtained in the framework of topological group theory. However, multipliers have been intensively studied in the framework of hypergroups; see [2, 3, 14] and references therein. The definiton of hypergroup can be found in [2]. For the convenience of the reader we recall it here.

A hypergroup is a locally compact Hausdorff space H with an involution $^-$ and a convolution $*$ on the space $M(H)$ of bounded Radon measures on H such that $(M(H), *)$ is an algebra and $\forall x, y \in H$,

(i) $\delta_x * \delta_y$ is a probability measure on H with compact support, where δ_x is the point measure at x,

(ii) The maps $H^2 \to M(H)$, $(x,y) \mapsto \delta_x * \delta_y$ and $H^2 \to C(H), (x,y) \mapsto$ supp$(\delta_x * \delta_y)$ are continuous, $C(H)$ being the set of nonvoid compact subsets of H with the Michael topology [17],

(iii) There exists an $e \in H$, called identity, satisfying $\delta_x*\delta_e = \delta_e*\delta_x = \delta_x$ and $e \in$ supp$(\delta_x*\delta_y)$ if and only if $x = y^-$,

(iv) $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$.

A hypergroup H is called commutative if for all $x, y \in H$, $\delta_x * \delta_y = \delta_y * \delta_x$.

Set

$$f(x*y) = \int_H f(z)d(\delta_x * \delta_y)(z). \tag{6.1}$$

For functions f, g and a measure μ on a hypergroup H, translation and convolution are defined as follows.

$$T^x f(y) = f(x*y), \tag{6.2}$$

$$(\mu * f)(x) = \int_H f(y^- * x)d\mu(y), \tag{6.3}$$

and

$$(f*g)(x) = \int_H f(y^- * x)g(y)dy, \tag{6.4}$$

where dy stands for the Haar measure of H, the existence of which was proved by Spector for commutative hypergroups [19].
Let G be a locally compact group and K a compact subgroup of G. Consider the set $H = G//K = \{KxK : x \in G\}$ of double cosets of K. Then $G//K$ is a hypergroup [2, Theorem 1.1.9]. If (G,K) is a Gelfand pair then $G//K$ is a commutative hypergroup.

Our result (Proposition 4.2) completes Theorem 1.6.24 in [2, page 68] applied to the double coset $G//K$ viewed as a hypergroup to give the following proposition.

Proposition 6.1. *If (G,K) is a Gelfand pair and if $T : L^1(G//K) \to L^1(G//K)$ is a bounded linear operator then the following assertions are equivalent.*

(i) $TT^x = T^xT$, $\forall x \in G//K$.

(ii) $T(f*g) = Tf*g$, $\forall f, g \in L^1(G//K)$.

(iii) There exists a unique bounded measure μ on $G//K$ such that $Tf = \mu * f$, $\forall f \in L^1(G//K)$.

(iv) There exists a unique function $\mathfrak{Q}$ defined on the set of spherical functions on G such that $\widehat{Tf} = \mathfrak{Q}\widehat{f}$, $\forall f \in L^1(G//K)$.

7 Conclusion

We obtain some results concerning the multipliers and the double multipliers for the convolution Banach algebra $L^1(G//K)$. Further developments of this subject will consist of dropping the commutativity provided by the fact that (G,K) is a Gelfand pair. Surely the notion of spherical function according to a group representation developped in [12] may play an important rôle.

References

[1] F. T. Birtel, Isomorphisms and isometric multipliers, *Proc. Amer. Math. Soc.*, 13(2), (1962) 204-210.

[2] R. Bloom, H. Heyer, *Harmonic analysis of probability measures on hypergroups*, de Gruyter Stud. Math., 20, Walter de Gruyter, Berlin and Hawthorne, (1995).

[3] S. N. Degenfeld-Schonburg, *Multipliers for hypergroups: concrete examples, application to time series*, PhD thesis, Technische Universität München, München (2012). Available at https://d-nb.info/1031513728/34

[4] A. Deitmar and S. Echterhoff, *Principles of harmonic analysis*, Springer International Publishing, 2nd Ed. (2014).

[5] J. Dieudonné, Gelfand pairs and spherical functions, *Internat. J. Math. and Math. Sci.*, 2(2), (1979) 155-162.

[6] G. van Dijk, *Introduction to harmonic analysis and generalized Gelfand pairs,* De Gruyter Studies in Mathematics 36, Berlin, Germany (2009).

[7] R. E. Edwards, Operators commuting with translations, *Pacific J. Math.*, 15, (1965) 85-95.

[8] J. Faraut, *Analyse harmonique sur les paires de Gelfand et les espaces hyperboliques*, Les Cours du CIMPA, (1980).

[9] G. B. Folland, *A course in abstract harmonic analysis*, CRC Press, (1995).

[10] S. Helgason, *Multipliers of Banach algebras*, Annals of Mathematics, Second Series, 64, (1956) 240-256.

[11] I. N. Herstein, *Rings with involution*, The University of Chicago Press, Chicago (1976).

[12] K. Kangni, *Analyse de Fourier non-commutative*, Edilivre (2018).

[13] E. Kaniuth, *A course in commutative Banach algebras*, Springer-Verlag, (2009).

[14] V. Kumar, R. Sarma, N. S. Kumar, Characterisation of multipliers on hypergroups, to appear in *Acta Math. Vietnamica*.

[15] N. P. Landsman, *C^*-algebras, Hilbert C^*-modules and Quantum mechanics*, arXiv:math-ph/9807030v1 24 Jul (1998).

[16] R. Larsen, *An introduction to the theory of multipliers*, Springer-Verlag, (1971).

[17] E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.*, 71, (1951), 152-182.

[18] A. Riazi, and M. Adib, φ-Multipliers on Banach algebras without order, *Int. Journal of Math. Analysis*, 3(3), (2009) 121-132.

[19] R. Spector, Mesures invariantes sur les hypergroupes, *Trans. Amer. Math. Soc.*, 239 (1978) 147-165.

[20] J. Wang , Multipliers of commutative Banach algebras, *Pacific J. Math.*, 11, (1961) 1131-1149.

[21] J. G. Wendel, On isometric isomorphism of group algebras, *Pacific J. Math.*, 1, (1951) 305-311.

[22] J. G. Wendel, Left centralizers and isomorphisms of group algebras, *Pacific J. Math.*, 2, (1952) 251-261.

[23] H. H. Zadeh, A. Jabbari and K. Moeini, A note on multipliers of Banach algebras, *Int. J. of Pure and Appl. Math.*, (75)2 (2012) 229-235.

Author information

Abudulaï Issa, Department of Mathematics, University of Lomé, 1 B.P. 1515 Lomé 1, Togo.
E-mail: `issaabudulai13@gmail.com`

Yaogan Mensah, Department of Mathematics, University of Lomé, 1 B.P. 1515 Lomé 1, Togo,
and International Chair in Mathematical Physics and Applications (ICMPA)-Unesco Chair, University of Abomey-Calavi,, Benin.
E-mail: `mensahyaogan2@gmail.com, ymensah@univ-lome.tg`

Received: April 11, 2019.
Accepted: September 28, 2019.