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A NOTE ON w-SPLIT MODULES

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Abstract In this paper, we study some properties of w-split modules. Hence, we use these modules to characterize some classical rings. For example, we prove that a ring R is Von Neumann regular if and only if every finitely presented R-module is w-split, and R is semi-simple if and only if every R-module is w-split. We also introduce the w-split dimension for modules. The relation between projective dimension and w-split dimension will be discussed.

1 Introduction

Throughout, all rings are commutative with unity and all modules are unital. Let J be an ideal of R. Following [13], J is called a *Glaz-Vasconcelos ideal* (a GV-ideal for short) if J is finitely generated and the natural homomorphism $\varphi : R \to J^* = \text{Hom}_R(J, R)$ is an isomorphism. Note that the set $GV(R)$ of GV-ideals of R is a multiplicative system of ideals of R. Let M be an R-module. Set

$$\text{tor}_{GV}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in GV(R)\}.$$

It is clear that $\text{tor}_{GV}(M)$ is submodule of M. M is said to be GV-torsion (resp., GV-torsion-free) if $\text{tor}_{GV}(M) = M$ (resp., $\text{tor}_{GV}(M) = 0$). A GV-torsion-free module M is called a w-module if $\text{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$. Then, projective modules and reflexive modules are w-modules. In the recent paper [14], it was shown that flat modules are w-modules. Let $w - Max(R)$ denote the set of w-ideals of R maximal among proper integral w-ideals of R (maximal w-ideals). Following [13, Proposition 3.8], every maximal w-ideal is prime. For any GV-torsion free module M,

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the w-envelope of M, where $E(M)$ denotes the injective hull of M. It is clear that a GV-torsion-free module M is a w-module if and only if $M_w = M$.

Let M and N be R-modules and let $f : M \to N$ be a homomorphism. Following [10], f is called a w-monomorphism (resp., w-epimorphism, w-isomorphism) if $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is a monomorphism (resp., an epimorphism, an isomorphism) for all $\mathfrak{m} \in w - Max(R)$. A sequence $A \to B \to C$ of modules and homomorphisms is called w-exact if the sequence $A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m}$ is exact for all $\mathfrak{m} \in w - Max(R)$. An R-module M is said to be of finite type if there exists a finitely generated free R-module F and a w-epimorphism $g : F \to M$. Simi-larly, an R-module M is said to be of finitely presented type if there exists a w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_1 and F_0 are finitely generated free.

The introduction of the w-operation in the class of flat and projective modules has been successful. The notion of w-flat modules appeared first in [9] over a domain and was extended to arbitrary commutative rings in [4]; an R-module M is called a w-flat module if the induced map $1 \otimes f : M \otimes A \to M \otimes B$ is a w-monomorphism for any w-monomorphism $f : A \to B$. Certainly,

both flat modules and GV-torsion modules are w-flat.
Kim and Wang in [12] defined the w-projective modules as follows: An R-module M is called w-projective if $\mathrm{Ext}^1_R(L(M), N)$ is GV-torsion for every torsion-free w-module N where $L(M) = (M/\mathrm{tor}_{GV}(M))_w$.
Both projective modules and GV-torsion modules are w-projective. In what follows, we summarize some results about w-projective modules.

Lemma 1.1. *Let R be a ring. The following statements are satisfied:*

(i) ([12, Proposition 2.3 (2)]) Let M be a w-module. Then, M is w-projective if and only if $\mathrm{Ext}^1_R(M, N)$ is GV-torsion for any torsion-free w-module N.

(ii) ([12, Proposition 2.3 (3)]) Let M be an R -module. If $\mathrm{Ext}^1_R(M, N)$ is GV-torsion for any torsion-free w-module N, then M is w-projective.

(iii) ([12, Theorem 2.5, Theorem 2.8 & Proposition 2.9]) Let M be an R-module. If M is w-projective then $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all $\mathfrak{m} \in w - Max(R)$. The equivalence holds if M is of finitely presented type or R is a domain and M is of finite type.

(iv) ([12, Theorem 2.19]) Every w-projective module of finite type is of finitely presented type.

(v) ([11, Proposition 2.4]) Every w-projective module is w-flat.

In [9], a torsion-free module M over a domain R is said to be w-projective if M is of finite type and $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all $\mathfrak{m} \in w - Max(R)$. From the above Lemma, we see that the definition of w-projective modules in [12] extends that in [9] and the new definition coincides with the old one on a torsion-free module of finite type over a domain.
A short exact sequence of R-modules $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be w-split [21] if there exist $J =< d_1, \ldots, d_n >\in GV(R)$ and $h_1, \ldots, h_n \in \mathrm{Hom}_R(C, B)$ such that $d_k 1_C = gh_k$ for all $k = 1, \ldots, n$. An R-module M is said to be w-split [21] if there is a w-split short exact sequence of R-modules $0 \to K \to P \to M \to 0$ with P projective.
In section 2, we study some properties of w-split modules and we compare the classes of projective modules, w-split modules, and w-projective modules. In section 3, we introduce and study the w-split dimension of modules.

2 w-split modules

In [21, Proposition 2.3], it is proved that an R-module M is a w-split if and only if $\mathrm{Ext}^1_R(M, N)$ is GV-torsion for all R-modules N. Hence, by [21, Corollary 2.4], every w-split module is w-projective but the converse is not true (see [21, Example 2.6]). Recall that a ring R is called *DW-ring* if every ideal of R is a w-ideal, or equivalently every maximal ideal of R is w-ideal [6]. Examples of DW-rings are Prüfer domains, domains with Krull dimension one, and rings with Krull dimension zero. Clearly, over a DW-ring, the classes of projective, w-split and w-projective modules coincide (see [12, page 7]). However, in general, we have

{ Projective modules} $\subseteq$ { w-split modules} $\subseteq$ {w-Projective modules} $\subseteq$ { w-flat modules}

By [19, Proposition 2.5], if R is a perfect ring, then the four classes of modules above coincide. In [23], L. Mao and N. Ding proved that a ring R is a Von Neumann regular if and only if every FP-projective R-module is projective. Our first result characterize rings over which every FP-projective (resp., finitely presented) module is w-split.

Theorem 2.1. *Let R be a ring. The following are equivalent:*

(i) Every FP-projective R-module is w-split.

(ii) Every finitely presented R-module is w-split.

(iii) R is Von Neumann regular.

Proof. $(i) \Rightarrow (ii)$ Trivial, since every finitely presented R-module is FP-projective.
$(ii) \Rightarrow (iii)$ Let I is a finitely generated ideal of R, then R/I is finitely presented. So R/I is w-split, and so w-flat. Then, by [16, Proposition 3.3] and [11, Theorem 4.4], R is Von Neumann regular.
$(iii) \Rightarrow (i)$ Let M be a FP-projective R-module, so M is projective by [20, Remarks 2.2] . Thus, M is w-split. □

Next, we give an example of a FP-projective module which is not w-split.

Example 2.2. Consider the local quasi-Frobenius ring $R := k[X]/(X^2)$ where k is a field, and denote by $\overline{X}$ the residue class in R of X. Then, $(\overline{X})$ is a FP-projective R-module which is not w-split.

Proof. Since R is a quasi-Frobenius ring, every absolutely pure R-module is injective. Hence, for any absolutely pure R-module N, we have $\text{Ext}^1_R((\overline{X}), N) = 0$. So, $(\overline{X})$ is FP-projective. But, $(\overline{X})$ is not projective by [22, Example 2.2], and so not w-split since R is DW ring. □

Proposition 2.3. *Let M be a w-split module and let $\mathfrak{m}$ be a maximal w-ideal of R. For any $R_{\mathfrak{m}}$-module N, we have* $\text{Ext}^n_R(M, N) = 0$ *for all* $n \geq 1$.

Proof. By [21, Proposition 2.3], when N is seen as an R-module, $\text{Ext}^n_R(M, N)$ is GV-torsion. Thus, $\text{Ext}^n_R(M, N) = (\text{Ext}^n_R(M, N))_{\mathfrak{m}} = 0$. □

Proposition 2.4. *Let M be a finitely presented R-module. The following are equivalent:*

(i) M is w-split.

(ii) M is w-projective.

(iii) M is w-flat.

(iv) M_m is a free $R_{\mathfrak{m}}$-module for any maximal w-ideal $\mathfrak{m}$ of R.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear.
$(iii) \Rightarrow (iv)$ Since M is a finitely presented w-flat module, so M_m is a finitely presented flat $R_{\mathfrak{m}}$-module for any maximal w-ideal $\mathfrak{m}$ of R, and so free.
$(iv) \Rightarrow (i)$ Let $\mathfrak{m}$ be a maximal w-ideal of R and N an arbitrary R-module. We have the naturel homomorphism

$$\theta : \text{Hom}_R(M, N)_{\mathfrak{m}} \to \text{Hom}_{R_m}(M_{\mathfrak{m}}, N_{\mathfrak{m}}).$$

We also have the induced homomorphism from θ

$$\theta_1 : \text{Ext}^1_R(M, N)_{\mathfrak{m}} \to \text{Ext}^1_{R_m}(M_{\mathfrak{m}}, N_{\mathfrak{m}}).$$

From [12, Proposition 1.10], θ_1 is a monomorphism. Thus, since $\text{Ext}^1_{R_m}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$, we have $\text{Ext}^1_R(M, N)_{\mathfrak{m}} = 0$. Consequently, $\text{Ext}^1_R(M, N)$ is GV-torsion, and so M is w-split. □

Corollary 2.5. *Let M, N be a finitely presented R-modules and $f : M \to N$ be a w-isomorphism. Then, M is w-split if and only if N is w-split.*

Corollary 2.6. *For any $J \in GV(R)$, then R/J is w-split.*

Proof. Let $J \in GV(R)$, so R/J is finitely presented and w-flat by [19, Lemma 2.3]. Thus, by Proposition 2.4, R/J is w-split. □

The next result characterizes the rings over which every w-projective (resp., w-split) module is projective.

Theorem 2.7. *Let R be a ring. The following are equivalent:*

(i) Every w-projective R-module is projective.

(ii) Every w-split R-module is projective.

(iii) Every w-flat R-module is flat.

(iv) R is a DW-ring.

Proof. $(i) \Rightarrow (ii)$ Trivial.
$(ii) \Rightarrow (iv)$ (resp., $(iii) \Rightarrow (iv)$) Let $J \in GV(R)$. Then, R/J is a (finitely presented) GV-torsion module, and so w-split by Corollary 2.6 (hence w-flat). Thus, R/J is projective. Hence, R/J is a w-module, and so a GV-torsion free module. Thus, $R/J = 0$. Consequently, $GV(R) = \{R\}$. Thus, R is DW-ring (by [10, Theorem 3.8]).
$(iv) \Rightarrow (i)$ and $(iv) \Rightarrow (iii)$ follows immediately from [10, Theorem 3.8] and the definitions of w-projective modules and w-flat modules. □

Next, we give an example of a w-split (GV-torsion free) module with projective dimension $= n \geq 1$. Before that, recall that for a domain R and a nonzero fractional ideal I of R, the v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I. Hence, I is a t-ideal if $I_t = I$ and a t-finite (or v-finite) ideal if there exists a finitely generated fractional ideal J of R such that $I = J_t = J_v$. A domain R is called a Prüfer v-multiplication domain ($PvMD$) if the set of its t-finite t-ideals forms a group under ideal t-multiplication ($(I, J) \mapsto (IJ)_t$). A useful characterization is that R is a $PvMD$ if and only if each localization at a maximal t-ideal is a valuation domain [2, Theorem 5]. The class of $PvMD$s strictly contains the classes of Prüfer domains, Krull domains, and integrally closed coherent domains. The $PvMD$s have been considered by many authors (see for example [3, 18, 17]).

Example 2.8. Let $(R, \mathfrak{m})$ be a regular local ring with $\mathrm{gldim}(R) = n$ ($n \geq 2$). Then, $\mathrm{pd}_R(\mathfrak{m}) = n - 1$. On the other hand, it is known that R is a Krull domain and so $PVMD$ ring. Thus, by [16, Theorem 3.5], $\mathfrak{m}$ is w-flat, and so w-split since R is Noetherian (by Corollary 2.4).

The next example gives a GV-torsion w-split module.

Example 2.9. Let $(R, \mathfrak{m})$ be a regular local ring with $\mathrm{gldim}(R) = n$ ($n \geq 2$). It is clear form Example 2.8 and Theorem 2.7 that R is not a DW-ring. Hence, let $J \in GV(R)$ with $J \neq R$. Clearly, R/J is a finitely presented GV-torsion module, and so a finitely presented w-flat module. Consequently, R/J is an w-split module.

Proposition 2.10. *Let M be a finitely presented R-module. Then, M is w-split (and equivalently w-projective) if and only if for any w-exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$ is w-exact.*

Proof. $(\Rightarrow)$ Let m be a maximal w-ideal of R. We have the exact sequence of $R_\mathfrak{m}$-modules $0 \to A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m} \to 0$. Thus, since $M_\mathfrak{m}$ is free, we have the exact sequence

$$0 \to \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, A_\mathfrak{m}) \to \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}) \to \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m}) \to 0$$

Since M is finitely presented, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, A_\mathfrak{m}) & \to & \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}) & \to & \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m}) \\ \| \wr & & \| \wr & & \| \wr \\ \mathrm{Hom}_R(M, A)_\mathfrak{m} & \to & \mathrm{Hom}_R(M, B)_\mathfrak{m} & \to & \mathrm{Hom}_R(M, C)_\mathfrak{m} \end{array}$$

Thus, $0 \to \mathrm{Hom}_R(M, A)_\mathfrak{m} \to \mathrm{Hom}_R(M, B)_\mathfrak{m} \to \mathrm{Hom}_R(M, C)_\mathfrak{m} \to 0$ is exact, and so, $0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$ is w-exact.
$(\Leftarrow)$ Follows from [21, Proposition 2.3]. □

Proposition 2.11. *The class of all w-split modules is closed under direct sums and direct summands.*

Proof. Let M and N be two R-modules, and K an arbitrary R-module. We have $\mathrm{Ext}^1_R(M \oplus N, K) = \mathrm{Ext}^1_R(M, K) \oplus \mathrm{Ext}^1_R(N, K)$. Thus, $\mathrm{Ext}^1_R(M \oplus N, K)$ is GV-torsion if and only if $\mathrm{Ext}^1_R(M, K)$ and $\mathrm{Ext}^1_R(N, K)$ are GV-torsion. Hence, $M \oplus N$ is w-split if and only if M and N are w-split. □

In the following result we give a new characterization of semi-simple rings by using w-split modules and also w-injective modules. Recall that an R-module M is called w-injective if $0 \to \mathrm{Hom}_R(C, L(M)) \to \mathrm{Hom}_R(B, L(M)) \to \mathrm{Hom}_R(A, L(M)) \to 0$ is w-exact for any w-exact sequence $0 \to A \to B \to C \to 0$ (for more details, see [11]).

Theorem 2.12. *Let R be a ring. The following are equivalent:*

(i) Every R-module is w-split.

(ii) Every R-module is w-projective.

(iii) Every R-module is w-injective.

(iv) Every cyclic R-module is w-split.

(v) Every cyclic R-module is w-projective.

(vi) R is a semi-simple ring.

Proof. $(ii) \Rightarrow (v)$, $(i) \Rightarrow (iv)$, $(iv) \Rightarrow (v)$ and $(vi) \Rightarrow (i)$ are trivial, while $(v) \Rightarrow (vi)$ follows from [12, Theorem 3.15].
$(i) \Rightarrow (iii)$ Let M an R-module. For any R-module N, we have that $\mathrm{Ext}^1_R(N, L(M))$ is GV-torsion since N is w-split. Then, by [11, Corollary 3.4] M is w-injective.
$(iii) \Rightarrow (ii)$ Let M be an R-module and N any torsion free w-module. Since, $N = L(N)$ is w-injective, we have that $\mathrm{Ext}^1_R(L(M), N)$ is GV-torsion (by [11, Corollary 3.4]). Hence, M is w-projective. □

3 w-split dimension

In this section, we introduce and investigate the w-split dimension. Note that, any R-module can have a w-split resolution. One can just take any projective resolution.

Proposition 3.1. *Let M be any R-module and consider two exact sequences,*

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

$$0 \to K'_n \to G'_{n-1} \to \cdots \to G'_0 \to M \to 0,$$

where $G_0, \cdots, G_{n-1}$ and $G'_0, \cdots, G'_{n-1}$ are w-split modules. Then, K_n is w-split if and only if K'_n is w-split.

Proof. Using [1, Lemma 3.12], the stated result follows [21, Corollary 2.4] and Proposition 2.11. □

Now, we can introduce the w-split dimension as follows:

Definition 3.2. The w-split dimension of an R-module M, $w - \mathrm{sd}_R(M)$, is defined by declaring that $w - \mathrm{sd}_R(M) \leq n$ ($n \in \mathbb{N}$) if M has a w-split resolution of length n. Otherwise, we set $w - \mathrm{sd}_R(M) = \infty$.

Let M be an R-module. Recall that $w - \mathrm{fd}_R(M) \leq n$ if there exist an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

where $F_n, F_{n-1}, \cdots, F_0$ are w-flat ([16, Proposition 2.3]). It is clear that, for any R-module M, we have $w - fd_R(M) \leq w - \mathrm{sd}_R(M) \leq \mathrm{pd}_R(M)$. In the next result, we give some characterizations of the w-split dimension.

Proposition 3.3. *Let M be an R-module and let n be an integer. Then the following conditions are equivalent:*

(i) $w-\mathrm{sd}(M)\leq n$.

(ii) $\mathrm{Ext}_R^i(M,N)$ *is GV-torsion for any $i>n$ and any R-module N.*

(iii) $\mathrm{Ext}_R^{n+1}(M,N)$ *is GV-torsion for any R-module N.*

(iv) For every exact sequence $0\to K_n\to G_{n-1}\to\cdots\to G_0\to M\to 0$ *where* $G_0,\cdots,G_{n-1}$ *are w-split, K_n is also w-split.*

(v) For every exact sequence $0\to K_n\to P_{n-1}\to\cdots\to P_0\to M\to 0$ *where* $P_0,\cdots,P_{n-1}$ *are projective, K_n is w-split.*

Proof. Obviously $(ii)\Rightarrow(iii)$, $(iv)\Rightarrow(v)$ and $(v)\Rightarrow(i)$, while $(v\Rightarrow(iv)$ follows from Proposition 3.1.
$(i)\Rightarrow(ii)$ By definition there is an exact sequence

$$0\to G_n\to G_{n-1}\to\cdots\to G_0\to M\to 0$$

where $G_0,\cdots,G_n$ are w-split. Consider an exact sequence

$$0\to K_n\to P_{n-1}\to\cdots\to P_0\to M\to 0$$

where $P_0,\cdots,P_{n-1}$ are projective. Then, by using Proposition 3.1, K_n is w-split. On the other hand, for all $i>n$ and any R-module N, $\mathrm{Ext}_R^{i-n}(K_n,N)\cong\mathrm{Ext}_R^i(M,N)$. Thus, $\mathrm{Ext}_R^i(M,N)$ is GV-torsion.
$(iii)\Rightarrow(v)$. Let $0\to K_n\to P_{n-1}\to\cdots\to P_0\to M\to 0$ be an exact sequence where $P_0,\cdots,P_{n-1}$ are projective. Then, for any R-module N, we have $\mathrm{Ext}_R^1(K_n,N)\cong\mathrm{Ext}_R^{n+1}(M,N)$. Thus, $\mathrm{Ext}_R^1(K_n,N)$ is GV-torsion. Hence, K_n is w-split. □

Proposition 3.4. *Let R be a ring. The following are equivalent:*

(i) For every R-module M, $\mathrm{pd}_R(M)=w-\mathrm{sd}_R(M)$.

(ii) For every R-module M, $\mathrm{fd}_R(M)=w-\mathrm{fd}_R(M)$.

(iii) R is a DW-ring.

Proof. Follows directly from Theorem 2.7. □

Lemma 3.5. *Let R_1 and R_2 be two rings.*

(i) Every R_1-module which is GV-torsion as $R_1\times R_2$-module is a GV-torsion R_1-module.

(ii) If M_1 and M_2 are, respectively GV-torsion R_1-module and R_2-module, then $M_1\times M_2$ is a GV-torsion $R_1\times R_2$-module.

Proof. (1) Let M be an R_1-module and assume that M is GV-torsion as $R_1\times R_2$-module. Let $x\in M$. Then, there exist a GV-ideal J of $R_1\times R_2$ such that $Jx=0$. One can write $J=J_1\times J_2$ with J_1 and J_2 are, respectively, ideals of R_1 and R_2, and we have $0=Jx=J_1x$. By [13, Proposition 1.2 (5)], J_1 is a GV-ideal of R_1. thus, M is a GV-torsion R_1-module.
(2) Let $(a,b)\in M_1\times M_2$. There exist $J_1\in GV(R_1)$ and $J_2\in GV(R_2)$ such that $J_1a=0$ and $J_2b=0$. Thus, $J_1\times J_2(a,b)=0$ and $J_1\times J_2\in GV(R_1\times R_2)$ (by [13, Proposition 1.2 (5)]). Hence, $M_1\times M_2$ is a GV-torsion $R_1\times R_2$-module. □

Proposition 3.6. *Let R_1 and R_2 be two rings and M_1 and M_2 be R_1-module and R_2-module, respectively. Then,*

$$w-\mathrm{sd}_{R_1\times R_2}(M_1\times M_2)=\sup\{w-\mathrm{sd}_{R_1}(M_1),w-\mathrm{sd}_{R_2}(M_2)\}$$

Proof. Let n be a positive integer.
Suppose that $w-\mathrm{sd}_{R_1\times R_2}(M_1\times M_2)\leq n$ and consider an arbitrary R_1-module N. Then, by [8, Theorem 10.74],

$$\mathrm{Ext}^{n+1}_{R_1}(M_1,N)\cong \mathrm{Ext}^{n+1}_{R_1}((M_1\times M_2)\otimes R_1,N)\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,N)$$

Then, $\mathrm{Ext}^{n+1}_{R_1}(M_1,N)$ is a GV-torsion $R_1\times R_2$-module, and so it is a GV-torsion R_1-module (by Lemma 3.5). Hence, $w-\mathrm{sd}(M_1)\leq n$. Similarly, $w-\mathrm{sd}(M_2)\leq n$. Consequently,

$$\sup\{w-\mathrm{sd}_{R_1}(M_1),w-\mathrm{sd}_{R_2}(M_2)\}\leq w-\mathrm{sd}_{R_1\times R_2}(M_1\times M_2).$$

Now, suppose that $\sup\{w-\mathrm{sd}_{R_1}(M_1),w-\mathrm{sd}_{R_2}(M_2)\}\leq n$. Let N be an arbitrary $R_1\times R_2$-module, and set $N_i=N\otimes R_i$ for $i=1,2$. It is clear that $N\cong N_1\times N_2$. On the other hand, by [8, Theorem 10.74],

$$\begin{aligned}\mathrm{Ext}^{n+1}_{R_1}(M_1,N_1)\times \mathrm{Ext}^{n+1}_{R_2}(M_2,N_2) &\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,N_1)\times \mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,N_2)\\ &\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,N_1\times 0)\times \mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,0\times N_2)\\ &\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,N_1\times N_2)\\ &\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,N)\end{aligned}$$

On the other hand, $\mathrm{Ext}^{n+1}_{R_1}(M_1,N_1)$ and $\mathrm{Ext}^{n+1}_{R_2}(M_2,N_2)$ are, respectively, GV-torsion R_1-module and R_2-module. Then, by Lemma 3.5, $\mathrm{Ext}^{n+1}_{R_1\times R_2}(M_1\times M_2,N)$ is a GV-torsion $R_1\times R_2$-module. Thus, $w-\mathrm{sd}(M_1\times M_2)\leq n$. Consequently,

$$w-\mathrm{sd}_{R_1\times R_2}(M_1\times M_2)\leq \sup\{w-\mathrm{sd}_{R_1}(M_1),w-\mathrm{sd}_{R_2}(M_2)\}.$$

Finally, we have the desired equality. □
In the next example, for a given integers $n\leq k$, we construct an R-module M such that $w-\mathrm{sd}_R(M)=n$ and $\mathrm{pd}_R(M)=k\geq n$.

Example 3.7. Let n and k be two integer with $1\leq n\leq k$. Let $(R,\mathfrak{m})$ be a regular local ring with $gl.dim(R)=k+1\geq 2$ and consider a Von Neumann regular ring R' with global dimension $n\geq 1$ (for example take the free Boolean ring on $\aleph_{n-1}$ generators; see please [7, Corollary 5.2]). Let M be an R'-module such that $\mathrm{pd}_{R'}(M)=n$. Then, $\mathrm{pd}_{R\times R'}(\mathfrak{m}\times M)=k$ and $w-\mathrm{sd}_{R\times R'}(\mathfrak{m}\times M)=n$.

Proof. From Example 2.8, $\mathfrak{m}$ is w-split with $\mathrm{pd}_R(\mathfrak{m})=k$. Moreover, R' is Von Neumann regular. Thus, every R'-module is flat (and so w-module), and so $w-\mathrm{sd}(M)=\mathrm{pd}(M)=n$. Hence, by [5, Lemma 2.5 (2)] and Proposition 3.6, we have

$$\mathrm{pd}_{R\times R'}(\mathfrak{m}\times M)=\sup\{\mathrm{pd}_R(\mathfrak{m}),\mathrm{pd}_{R'}(M)\}=\sup\{k,n\}=k$$

and

$$w-\mathrm{sd}_{R\times R'}(\mathfrak{m}\times M)=\sup\{w-\mathrm{sd}_R(\mathfrak{m}),w-\mathrm{sd}_{R'}(M)\}=\sup\{0,n\}=n.\square$$

In the following example, we give a module with infinite w-split dimension.

Example 3.8. Let R be a quasi-Frobenius ring which is not semi-simple (for example $R:=k[x]/(x^2)$ with k is a field). Let M be an R-module such that $\mathrm{pd}_R(M)=\infty$. It is clear that R has a Krull dimension 0. Hence, R is a DW-ring (by [10, Corollary 3.4]). Thus, $w-\mathrm{sd}_R(M)=\mathrm{pd}_R(M)=\infty$.

Proposition 3.9. *Let M be an R-module. If M admits a finite free resolution (in particular if R is Noetherian and M is finitely generated) then*

$$w-\mathrm{sd}(M)=w-fd(M)=\sup\{\mathrm{pd}_{R_\mathfrak{m}}(M_\mathfrak{m})\mid \mathfrak{m}\in w-Max(R)\}$$

Proof. It is clear that $w-fd(M) \leq w-\mathrm{sd}(M)$. Now, suppose that $w-fd(M) \leq n$ and consider an exact sequence
$$0 \to N \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$
where F_i are finitely generated free module. Then, N is a finitely presented w-flat module. Hence, M is w-split. Thus, $w-\mathrm{sd}(M) \leq n$. Consequentially, $w-\mathrm{sd}(M) = w-fd(M)$. On the other hand,
$$w-fd(M) = \sup\{\mathrm{fd}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in w-Max(R)\}$$
But, $M_\mathfrak{m}$ admits finite free resolution over $R_\mathfrak{m}$. Hence, $\mathrm{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) = \mathrm{fd}_{R_\mathfrak{m}}(M_\mathfrak{m})$. □

Proposition 3.10. *Assume that* $0 \to A \to B \to C \to 0$ *is a short exact sequence of* R*-modules with finite* w*-split dimensions. Then,*
$$w-\mathrm{sd}_R A \leq \max\{\ w-\mathrm{sd}_R B,\ w-\mathrm{sd}_R C-1\ \};$$
$$w-\mathrm{sd}_R B \leq \max\{\ w-\mathrm{sd}_R A,\ w-\mathrm{sd}_R C\ \};$$
$$w-\mathrm{sd}_R C \leq \max\{\ w-\mathrm{sd}_R B,\ w-\mathrm{sd}_R A+1\ \}.$$

Proof. We have the following long exact sequence:
$$\cdots \longrightarrow \mathrm{Ext}_R^n(C,-) \longrightarrow \mathrm{Ext}_R^n(B,-) \longrightarrow \mathrm{Ext}_R^n(A,-) \longrightarrow \mathrm{Ext}_R^{n+1}(C,-) \longrightarrow \cdots$$
Assume that $w-\mathrm{sd}_R B \leq m$ and $w-\mathrm{sd}_R C \leq m+1$ (i.e. $m \geq \max\{\ w-\mathrm{sd}_R B,\ w-\mathrm{sd}_R C-1\ \}$). Then the above sequence, together with [15, Theorem 2.7], gives us that $\mathrm{Ext}_R^k(A,-)$ is a GV-torsion module for $k > m$. In particular $w-\mathrm{sd}_R A \leq m$ and thus
$$w-\mathrm{sd}_R A \leq \max\{\ w-\mathrm{sd}_R B,\ w-\mathrm{sd}_R C-1\ \}.$$
The proof of the other inequalities is analogous. □

Proposition 3.11. *Let* A *and* B *be two* R*-modules. Then,*
$$w-\mathrm{sd}_R(A \oplus B) = \sup\{w-\mathrm{sd}_R A, w-\mathrm{sd}_R B\}$$

Proof. The inequality $w-\mathrm{sd}_R(A \oplus B) \leq \sup\{w-\mathrm{sd}_R A, w-\mathrm{sd}_R B\}$ follows from the fact that the class of w-split modules is closed under direct sums. For the converse inequality, we may assume that $w-\mathrm{sd}_R(A \oplus B) = n$ is finite. Thus, for any R-module M,
$$\mathrm{Ext}_R^{n+1}(A \oplus B, M) \cong \mathrm{Ext}_R^{n+1}(A, M) \oplus \mathrm{Ext}_R^{n+1}(B, M)$$
is a GV-torsion module. Hence, $\mathrm{Ext}_R^{n+1}(A, M)$ and $\mathrm{Ext}_R^{n+1}(B, M)$ are GV-torsion modules. Thus, $\sup\{w-\mathrm{sd}_R A, w-\mathrm{sd}_R B\} \leq n$. □

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