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BOUNDS FOR POISSON AND NEUTROSOPHIC POISSON DISTRIBUTIONS ASSOCIATED WITH CHEBYSHEV POLYNOMIALS

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Abstract The paper investigates Poisson and neutrosophic Poisson distribution series. The first few coefficient bounds for Poisson distribution whose parameter takes a definite and determined values were studied, while coefficient bounds for neutrosophic Poisson distribution whose parameter takes undetermined values or inaccurate statistical data were investigated. Examples to demonstrate our argument for neutrosophic Poisson distribution were provided.

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C, |z| < 1\}$, and let $S \in A$ consisting of univalent functions in U normalized with $f(0) = f'(z) - 1 = 0.$.
A function f of the form (1.1) is said to be starlike of order α if

$$Re\frac{zf'(z)}{f(z)} > \alpha, z \in U$$

and convex of order α if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in U$$

and are respectively denoted by $S^*(\alpha)$ and $K(\alpha)$ for $0 \leq \alpha < 1$.
Let P denote the well known class of Caratheodory functions with the positive real part in U. Let $P(p_k), 0 \leq k < \infty$ denote the family of functions p, such that $p \in P$, and $p \prec p_k$ in U; where the function p_k maps the unit disk conformally onto the region Ω_k such that $1 \in \Omega_k$ and

$$\delta\Omega = \{u + iv : u^2 = k^2(u-1)^2 + k^2v^2\}.$$

The domain Ω_k is elliptic, hyperbolic, parabolic and possibly covers the right half plane whenever $k > 1, 0 \leq k < 1, k = 1$, and $k = 0$ respectively.
Recently, several authors have investigated various sublcasses of analytic functions associated with Chebyshev polynomials due to its importance in both theoretical and practical applications (see for detail [2,5,11]) and the references therein. In this work we shall concern ourselves with Chebyshev polynomial of the first and second kinds. In the case of real variable $x \in [-1, 1]$ we have

$$T_n(x) = cosn\theta, \quad U_n(x) = \frac{sin(n+1)\theta}{sin\theta},$$

where n is the degree of polynomial and $x = cos\theta$. It is observed that for $t = cos\alpha, \alpha \in (\frac{-\pi}{3}, \frac{\pi}{3})$, then

$$H(z,t) = \frac{1}{1-2tz+z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n.$$

That is

$$H(z,t) = 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + ..., z \in U.$$

The above can be further expressed as follows

$$H(z,t) = 1 + U_1(t) + U_2(t)z^2 + ...(z \in U, t \in [-1,1]). \tag{1.2}$$

It is observed that

$$U_1(t) = 2t,\ U_2(t) = 4t^2 - 1,\ U_3(t) = 8t^3 - 4t, ... \tag{1.3}$$

It is important to recall that Chebyshev polynomials of the first kind and second kind are connected as follows

$$\frac{T_n(t)}{dt} = nU_{n-1}(t),\ T_n = U_n(t) - tU_{n-1}(t),\ 2T_n(t) = U_n(t) - U_{n-2}(t).$$

Recently, Poisson distribution series gathered momentum in geometric function theory. Porwal and Srivastava [12], Murugusundaramoorthy et. al [8], Porwal [9] Srivastava and Porwal [3], extensively studied Poisson distribution series by establishing various geometric properties in terms of univalency, starlikeness, convexity and harmonic structures. Detail can be found in the above mentioned literatures and the references therein.
Meanwhile, Poisson distribution series is defined as

$$K(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n \tag{1.4}$$

and by ratio test the radius of convergence of the above series is infinity.
The convolution of two series $f(z)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined by power series given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z); [1].$$

Let the linear operator $I(m) : A \to A$ define by

$$\Phi(z) = K(m,z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n. \tag{1.5}$$

Let f and g be analytic in U, then f is said to be subordinate to g written as $f(z) \prec g(z)$, if there exists a function $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$ [7].
Definition 1.1 Let $\lambda \geq 0,\ t \in (\frac{1}{2}, 1], m > 0$, a function $f \in A$ of the form (1.1) is said to belong to the class $M_\lambda^m(H(z,t))$ if the following condition holds

$$(1-\lambda)\frac{z\Phi'(z)}{\Phi(z)} + \lambda\left(1 + \frac{z\Phi''(z)}{\Phi'(z)}\right) \prec H(z,t) = \frac{1}{1-2tz+z^2}, (z \in U).$$

The author is motivated by the earlier works of Srivastava and Porwal [3], we make use of Chebyshev polynomials expansions to estimate the first few coefficients of functions in $M_\lambda^m(H(z,t))$. The authors in [3] investigated coefficient inequalities of Poisson distribution series in conic domain related to uniformly convex, k-spiralike and starlike functions. In their results m is a precise parameter for example we can say $m = 1$, that is m is accurately defined. Sectin 2 of the present investigation further the work in [3] by estimating the bound on the first few coefficients and classical Fekete-Szege theorem via subordination principle in connection with Chebyshev polynomials. In section 3 of the present investigation the work in [3] is further extended to a situation when m is not precisely defined. Classical probability distributions only deals with specified data and its parameters are always given with a specified value,while neutrosophic probability distribution gives a more general and clearity of the study issues when m is an interval. Summarily the work in [3] and section 2 of the present investigation are particular cases or one solution of our investigation in section 3.

2 Bounds for Poisson Distribution Series

For the purpose of our investigation the following lemma shall be employed.
Lemma 2.1 [7] If $\omega \in \Omega$ then

$$\left|\omega_2 - t\omega_1^2\right| \leq max\{1, |t|\}$$

for any complex number t. The result is sharp for the function $\omega(z) = z$ or $\omega(z) = z^2$, (see also [4,6]).
Next we shall consider the first few coefficient bounds for the class defined in Definition 2.1.
Theorem 2.1 Let the function f given by (1.1) be in the class $M_\lambda^m(H(z,t)), m > 0, \lambda \geq 0$. Then we have

$$|a_2| \leq \frac{2t}{m(1+\lambda)e^{-m}}$$

$$|a_3| \leq \frac{2t}{m^2(1+2\lambda)e^{-m}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(1+3\lambda)}{(1+\lambda)^2}\right|\right\}$$

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2t}{m^2(1+2\lambda)e^{-m}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(e^{-m}-\mu) - 2t\lambda(3e^{-m}-2\mu)}{e^{-m}(1+\lambda)^2}\right|\right\}$$

Proof. If $f \in M_\lambda^m(H(z,t))$, and $\omega \in \Omega$ such that

$$(1-\lambda)\frac{z\Phi'(z)}{\Phi(z)} - \lambda\left(1 + \frac{z\Phi''(z)}{\Phi'(z)}\right) = H(z,t). \tag{2.1}$$

It is observed from (1.3) and (2.1) that

$$1 + (1+\lambda)me^{-m}a_2 z + [(1+2\lambda)m^2e^{-m}a_3 - (1+3\lambda)m^2e^{-2m}a_2^2]z^2 + ...$$

$$= 1 + U_1(t)c_1 z + (U_1(t)c_2 + U_2(t))z^2 + ... \tag{2.2}$$

Hence by (2.2) we have

$$a_2 = \frac{U_1(t)c_1}{(1+\lambda)me^{-m}} \tag{2.3}$$

and

$$a_3 = \frac{U_1(t)}{m^2(1+2\lambda)e^{-m}}\left(c_2 + \left(\frac{U_2}{U_1} + \frac{(1+3\lambda)U_1(t)}{(1+\lambda)^2}\right)c_1^2\right). \tag{2.4}$$

By (2.3) and (2.4) we have

$$a_3 - \mu a_2^2 = \frac{U_1}{m^2(1+2\lambda)e^{-m}}\left(c_2 + \rho c_1^2\right)$$

where

$$\rho = \frac{U_2}{U_1} + \frac{(e^{-m}(1+3\lambda) - \mu(1+2\lambda))U_1(t)}{(1+\lambda)^2}$$

The desired result is obtained by applying Lemma 2.1
Corolllary 2.1 $k \in [0,\infty),\ 0 \leq \alpha < 1,\ \lambda = 0$ in Theorem 2.1 and if $f \in M_0^m(H(z,t))$ then we have

$$|a_2| \leq \frac{2t}{me^{-m}}$$

$$|a_3| \leq \frac{2t}{m^2e^{-m}} max\left\{1, \frac{8t^2-1}{2t}\right\}$$

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2t}{m^2 e^{-m}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(e^{-m}-\mu)}{e^{-m}}\right|\right\}.$$

Corollary 2.2 For $k \in [0,\infty),\ 0 \leq \alpha < 1,\ \lambda = 1,\ t = \frac{1}{2}$ in Theorem 3.1, and if $f \in M_1^m(H(z,\frac{1}{2}))$ then we have

$$|a_2| \leq \frac{e^m}{2m}$$

$$|a_3| \leq \frac{e^m}{3m^2}$$

$$\left|a_3 - \mu a_2^2\right| \leq \frac{e^m}{3m^2} max\left\{1, \left|\frac{\mu - 2e^{-m}}{4e^{-m}}\right|\right\}.$$

3 Neutrosophic Poisson Distribution Series

Recently, precisely in 1995, Smarandache introduced the concept of neutrosophic theory. It is a new branch of philosophy as a generalization for the fuzzy logic, also as a generalization of the intrinstic fuzzy logic. This new branch of philosophy in the fuzzy logic provides a new foundation for dealing with issues that have indeterminate data, which may be numbers, (see [10] for neutrosophic numbers) and the references therein.
The use of neutrosophic crisp sets theory with the classical probabilty distributions particularly Poisson distribusion, exponential distibutions and uniform distributions opens a new stairway for dealing with issues that follow the classical distributions and at the same time contain data not specified accurately. The extension of classical distributions according to logic, means that parameters of classical distribution take undetermined values, which allows dealing with every situations that are encountered while dealing with statistical data, especially when working with vague and inaccurate statistical data.
Neutrosophic Poisson distribution of a discrete variable X is a classical Poisson distribution of x, but its parameter is imprecise. For example and from our earlier discussion in Section 2 m can be set with two or more elements. The most common of such distribution is when m is interval. Let

$$NP(x=k) = e^{-m_N}.\frac{(m_N)^k}{k!}, k = 0,1,2...$$

where m_N is the distribution parameter and is equal to the expected value and the variance. That is

$$NE(x) = NV(x) = m_N$$

and $N = d + I$ is a neutrosophic statistical number (see [10]) and the references therein.
Now we modifiy (1.4) as follows

$$K(m_N, z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!} e^{-m_N} z^n. \tag{3.1}$$

Our quest in this section is to investigate the possibility of having a range of values for the first few coefficients for neutrosophic Poisson distribution series whose parameter is indeterminate.
We shall use examples to demonstrate the concept of neutrosophic Poisson distribution using Theorem 2.1.
Theorem 3.1 Let the function f given by (1.1) be in the class $M_\lambda^m(H(z,t)), m \in [1,\infty]$. Then we have

$$|a_2| \leq \frac{2t}{m_N(1+\lambda)e^{-m_N}}$$

$$|a_3| \leq \frac{2t}{m_N^2(1+2\lambda)e^{-m_N}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(1+3\lambda)}{(1+\lambda)^2}\right|\right\}$$

$$\left|a_3 - \mu a_2^2\right| \le \frac{2t}{m_N^2(1+2\lambda)e^{-m_N}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(e^{-m_N}-\mu) - 2t\lambda(3e^{-m_N}-2\mu)}{e^{-m_N}(1+\lambda)^2}\right|\right\}$$

Example for case study
In a company, phone employee receives phone calls, the calls arrive with rate of [1,3] calls per minute, we will calculate the probability that the employee will not receive any call within a minute.

Corollary 3.1 Let the function f given by (1.1) be in the class $M_\lambda^m(H(z,t))$, $m \in [1,3]$. Then we have

$$|a_2| \le \frac{2t}{[1,3](1+\lambda)e^{-[1,3]}}$$

$$|a_3| \le \frac{2t}{[1,3]^2(1+2\lambda)e^{-[1,3]}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(1+3\lambda)}{(1+\lambda)^2}\right|\right\}$$

$$\left|a_3 - \mu a_2^2\right| \le \frac{2t}{[1,3]^2(1+2\lambda)e^{-[1,3]}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(e^{-[1,3]}-\mu) - 2t\lambda(3e^{-[1,3]}-2\mu)}{e^{-[1,3]}(1+\lambda)^2}\right|\right\}.$$

Corollary 3.2 Let the function f given by (1.1) be in the class $M_0^m(H(z,t))$, $m \in [1,3]$ and $\lambda = 0$. Then we have

$$|a_2| \le \frac{2t}{[1,3]e^{-[1,3]}}$$

$$|a_3| \le \frac{2t}{[1,3]^2 e^{-[1,3]}} max\left\{1, \left|\frac{4t^2-1}{2t} + 2t\right|\right\}$$

$$\left|a_3 - \mu a_2^2\right| \le \frac{2t}{[1,3]^2 e^{-[1,3]}} max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t(e^{-[1,3]}-\mu)}{e^{-[1,3]}}\right|\right\}$$

Conclusion: We conclude that the neutrosophic probability distribution gives a more general and clarity study of the problem under investigation. That is the classical probability distribution discussed in section 2 is one solution among the solutions resulting from the study in section 3, this is as a result of giving the distribution parameters several options possible which does not remain linked to a single value. We hope this work will serve as a stairway to study other types of probability distributions according to the neutrosophic logic.

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