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On Mosco convergence of conditional expectation for sequences of Pettis-integrable random sets

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Abstract. In this paper, we give some new convergence results of reversed martingale consisting of strongly measurable and Pettis-integrable random variable (resp. random set). We also present a new version of Mosco convergence of conditional expectation for Pettis-integrable multifunctions of the form $E^{\mathfrak{B}_n}X_n$, where $(\mathfrak{B}_n)_{n\geq 1}$ is a decreasing sequence of sub-σ-algebras of $\mathfrak{F}$ and $(X_n)_{n\geq 1}$ is a sequence of Pettis-integrable convex and weakly compact random sets in a separable Banach space.

1 Introduction

The convergence of conditional expectation of random variables (resp. random sets) plays a crucial role in the probability theory. This concept was used in many concrete problems, more precisely in the theory of martingales, optimization and control, stochastic geometry and in the mathematical economics.

The conditional expectation of random variables (resp. random sets) Bochner-integrable always exists and has been treated by several authors (see. for example Neveu [13], Hiai [10], Hess [9], Hiai and Umegaki [11], Ezzaki [7], Wei-an [17], Ezzaki et al [3] and others). However, the conditional expectations of Pettis-integrable random variables (resp. random sets) doesn't generally exist (see. Rybakov [14] and Talagrand [15]). Recently, several authors have studied the existence of this operator (see. El harami and Ezzaki [6], Akhiat et al [2], Oulghazi and Ezzaki [19][20] and Castaing et al [1]).

By using the recent tools established in this theory, we prove an extension of results stated by Hiai [10] and by Ezzaki el al [3] in Pettis integration. More precisely the aim of this work is the study of almost sure Mosco convergence of a sequences of random sets $E^{\mathfrak{B}_n}X_n$, where $(\mathfrak{B}_n)_{n\geq 1}$ is a decreasing sequence of sub-σ-algebra of $\mathfrak{F}$ and $(X_n)_{n\geq 1}$ is a sequence of Pettis-integrable convex and weakly compact random sets in a separable Banach space. It is interesting to know that the same result for vector valued Pettis-integrable function is not established yet.

Our paper is organized as follows. In Section 2, we present some definitions, preliminaries and needed results. Section 3 is devoted to the convergence of the conditional expectation for sequences of Pettis-integrable vector valued random variables; we state a new version of Levy's theorem and dominated convergence theorem in Pettis integration. At the last section of this paper, we present the almost sure Mosco convergence of Pettis-integrable random sets $(E^{\mathfrak{B}_n}X)_{n\geq 1}$ and $(E^{\mathfrak{B}_n}X_n)_{n\geq 1}$, where $(\mathfrak{B}_n)_{n\geq 1}$ is a decreasing sequence of sub-σ-algebras of $\mathfrak{F}$ and $(X_n)_{n\geq 1}$ as well as X are the convex and weakly compact Pettis-integrable random sets.

2 Notations and preliminaries

Throughout this paper $(\Omega, \mathfrak{F}, P)$ is a complete probability space, $(\mathfrak{B}_n)_{n\geq 1}$ is a decreasing sequence of sub-σ-algebras of $\mathfrak{F}$ and $\mathfrak{B}_\infty = \cap_{n\geq 1}\mathfrak{B}_n$. Let E be a separable Banach space with the dual space E^*. Let $cwk(E)$ (resp. $cc(E)$) be the family of all nonempty convex and weakly compact subsets of E (resp. the family of all nonempty closed convex subsets of E). Let $L^1_E(\mathcal{F})$ (resp. $L^1_{cwk(E)}(\mathcal{F})$) be the family of all $\mathcal{F}$-measurable and Bochner-integrable functions $X : \Omega \longrightarrow E$ (resp. the family of integrably bounded random sets with values in $cwk(E)$).

Given B in $cwk(E)$, the distance function and the support function of B are defined by

$$d(x,B)=\inf\{\|x-y\|, y\in B\},\quad (x\in E).$$

$$\delta^*(x^*,B)=\sup\{\prec x^*,y\succ,\;\; y\in B\},\quad (x^*\in E^*).$$

For any B in $cwk(E)$, we get

$$|B|=\sup\{\|x\|,\quad x\in B\}.$$

A $cwk(E)$-valued sequence $(A_n)_{n\geq 1}$ is called Mosco convergent to a closed convex and weakly compact set A_∞ (see. [12]), if $A_\infty=s-li\;A_n=w-ls\;A_n$, where

$$s-li\;A_n=\{y\in E,\;\; y_n\longrightarrow y,\;\; y_n\in A_n,\quad \forall n\geq 1\}$$

$$w-ls\;A_n=\{y\in E,\;\; y_k\overset{w}{\longrightarrow} y,\;\; y_k\in A_{n_k},\forall k\geq 1\},$$

where $(A_{n_k})_{k\geq 1}$ is a subsequence of $(A_n)_{n\geq 1}$. If $(A_n)_{n\geq 1}$ Mosco converges to A_∞ in $cwk(E)$, we write $M-\lim A_n=A_\infty$.
The topology determined by the convergence of support functions is denoted by τ_s. A sequence A_n is τ_s-convergent to subset A_∞ if $\lim_{n\to+\infty}\delta^*(x^*,A_n)=\delta^*(x^*,A_\infty)$ for all $x^*\in E^*$.
A measurable function $g:\Omega\longrightarrow E$ is Pettis-integrable, if g is scalarly integrable (i.e. $<x^*,g>$ is integrable), and for each $A\in\mathfrak{F}$, there exists x_A in E, such that

$$\int_A <x^*,g>dP=<x^*,x_A>,\;\; \forall x^*\in E^*.$$

x_A is called the Pettis-integral of g over A. We will denote by $P^1_E(\mathfrak{F})$ the space of all $\mathfrak{F}$-measurable and Pettis-integrable E-valued function defined on $(\Omega,\mathfrak{F},P)$. We consider the space $P^1_E(\mathfrak{F})$ provided with the following topologies:

- The topology of the usual Pettis norm

$$\|g\|_{Pe}=\sup_{x^*\in B_{E^*}}\int_\Omega |<x^*,g>|dP.$$

- The topology induced by the duality $(P^1_E(\mathfrak{F}),L^\infty\otimes E^*)$. Recall that a sequence $(g_n)_{n\geq 1}$ in $P^1_E(\mathfrak{F})$ converges to g in this topology, if for each $h\in L^\infty(\mathfrak{F})$ and for all $x^*\in E^*$, one has

$$\lim_{n\to+\infty}\int_\Omega h(\omega)<x^*,g_n(\omega)>dP(\omega)=\int_\Omega h(\omega)<x^*,g(\omega)>dP(\omega).$$

  This topology is known as the weak topology and is denoted by w-Pe.

A multifunction $X:\Omega\longrightarrow cc(E)$ is said to be measurable, if for every open set U of E the subset

$$X^-(U)=\{\omega\in\Omega/X(\omega)\cap U\neq\emptyset\}.$$

is an element of $\mathfrak{F}$. The Effros σ-field ξ on $cc(E)$ is generated by the subsets $U^-=\{F\in cc(E),/F\cap U\neq\emptyset\}$, so the multifunction $X:\Omega\longrightarrow cc(E)$ is measurable if, for any $B\in\xi$, one has $X^-(B)\in\mathfrak{F}$. A measurable multifunction is called a random set.
A measurable function $f:\Omega\longrightarrow E$ is said to be a selection of X, if, for any $\omega\in\Omega$, $f(\omega)\in X(\omega)$. We denote by $S^1_X(\mathfrak{F})$ the set of all $\mathfrak{F}$-measurable and integrable selections of X. It is known that if E is a complete space (see. theorem 1.0 in [11]) a closed and convex valued multifunction X is measurable if and only if $dom(X)\in\mathfrak{F}$ and X has a Castaing representation (i.e. there exists a sequence $(f_n)_{n\geq 1}$ of measurable selections of X such that for all $\omega\in\Omega$, $X(\omega)=cl\{f_n(\omega),\; n\geq 1\}$), or if and only if the real function $d(x,X(.))$ is measurable for any x in E .
The random set $X:\Omega\longrightarrow cwk(E)$ is scalarly integrable, if for any $x^*\in E^*$, the real function $\delta^*(x^*,X(.))$ is integrable. We say that the random set X is Pettis-integrable, if X is scalarly integrable and for each $A\in\mathfrak{F}$, there exists $K_A\in cwk(E)$ such that

$$\int_A \delta^*(x^*,X)dP=\delta^*(x^*,K_A),\;\forall x^*\in E^*.$$

$K_A := \int_A XdP$ is called the Pettis-integral of X over A.
A random set X is said to be Aumann-Pettis integrable if $S_X^{Pe} \neq \phi$ and the multivalued Aumann Pettis-integral of a random set X over Ω is defined by

$$\int_\Omega XdP = \{\int_\Omega fdP, f \in S_X^{Pe}(\mathfrak{F})\}.$$

We will denote by $P^1_{cwk(E)}(\mathfrak{F})$ the set of all $cwk(E)$-valued Pettis-integrable random set.
Let X be in $L^1_E(\mathfrak{F})$ and $\mathcal{B}$ a sub-σ-algebra of $\mathfrak{F}$. It is known in the literature (see. Neuveu [13]) that the conditional expectation of X relative to $\mathcal{B}$ exists and is the unique almost surely $\mathcal{B}$-measurable and Bochner integrable random variable such that

$$\int_A E^{\mathcal{B}} XdP = \int_A XdP,\ \forall A \in \mathcal{B}.$$

The extension of this result is stated by Hiai and Umegaki [11] of integrable random sets;
If X is an integrable random set and $\mathcal{B}$ is a sub-σ-algebra of $\mathfrak{F}$, then there exists a unique almost surely $\mathcal{B}$-measurable and integrable random set denoted $E^{\mathcal{B}}X$ such that

$$S^1_{E^{\mathcal{B}}X}(\mathcal{B}) = cl\{E^{\mathcal{B}}f,\ f \in S^1_X(\mathfrak{F})\},$$

where the closure is taken with respect to the norm in $L^1_E(\mathfrak{F})$.
The conditional expectation for Pettis-integrable random variable not always exists (see. Rybakov [14] and Talagrand [15]). Recently Akhiat et al ([1], [2]), Ezzaki and El harami [6] and Uhl [16] gave a sufficient conditions of the existence of this operator for Pettis-integrable vector random variables and Pettis-integrable random sets. For the convenience of the reader, we recall the following propositions which will be used after.

Proposition 2.1. *([1]) Assume that E is a separable Banach space. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathfrak{F}$ and let X be a Pettis-integrable E-valued function such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then there exists a unique $\mathcal{B}$-measurable and Pettis-integrable E-valued function denote by $E^{\mathcal{B}}X$ which enjoys the following property, for every $h \in L^\infty(\mathcal{B})$, one has*

$$\int_\Omega hE^{\mathcal{B}}XdP = \int_\Omega hXdP.$$

$E^{\mathcal{B}}X$ is called the Pettis conditional expectation of X relative to $\mathcal{B}$.

Proposition 2.2. *([1], [6]) Assume that E^* is separable. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathfrak{F}$ and let X be a $cwk(E)$-valued Pettis-integrable random set such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then there exists a unique $\mathcal{B}$-measurable $cwk(E)$-valued Pettis-integrable random set which enjoys the following property, for every $h \in L^\infty(\mathcal{B})$, one has*

$$\int_\Omega hE^{\mathcal{B}}XdP = \int_\Omega hXdP.$$

$E^{\mathcal{B}}X$ is called the Pettis conditional expectation of X relative to $\mathcal{B}$ such that

$$S^{Pe}_{E^{\mathcal{B}}X} = \{E^{\mathcal{B}}f,\ f \in S^{Pe}_X(\mathfrak{F})\}.$$

We close this section with the following lemmas. the first one is known as Levy's theorem for decreasing sequence of sub-σ-algebras $(\mathfrak{B}_n)_{n\geq 1}$ and the second is the dominated convergence theorem for the conditional expectation of Bochner integrable random variables.

Lemma 2.3. *([18]) Let $(\mathfrak{B}_n)_{n\geq 1}$ be a decreasing sequence of a sub σ algebras of $\mathfrak{F}$ and let $f \in L^1_E(\mathfrak{F})$, set $\mathfrak{B}_\infty = \cap_{n\geq 1}\mathfrak{B}_n$. Then*

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} f(.) = E^{\mathfrak{B}_\infty} f(.)\ a.s.$$

Lemma 2.4. *([3]) Let $(\mathfrak{B}_n)_{n\geq 1}$ be a decreasing sequence of a sub-σ-algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \cap_{n\geq 1}\mathfrak{B}_n$. Let Y be a positive random variable satisfying $E^{\mathfrak{B}_\infty}Y \in [0, +\infty[$ and $(X_n)_{n\geq 1}$ in $L^1_E(\mathfrak{F})$ such that:*

(i) $\forall n \geq 1, |X_n| \leq Y$.

(ii) $\lim_{n\to+\infty} X_n(.) = X_\infty(.)\ a.s.$

Then the following equality hold true

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} X_n(.) = E^{\mathfrak{B}_\infty} X_\infty(.)\ a.s.$$

3 Levy's theorem and dominated convergence theorem of conditional expectation for a sequences of E-valued Pettis-integrable random variables

In this section, we give an extension of Lemma 2.3 and Lemma 2.4 in Pettis integration case.

Theorem 3.1. *Let $(\mathfrak{B}_n)_{n\geq 1}$ be a decreasing sequence of a sub-σ-algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \cap_{n\geq 1}\mathfrak{B}_n$. Let $f \in P^1_E(\mathfrak{F})$ such that $E^{\mathfrak{B}_\infty}|f| \in [0, +\infty[$. Then*

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} f(.) = E^{\mathfrak{B}_\infty} f(.)\ a.s.$$

Proof. Since $E^{\mathfrak{B}_\infty}|f| \in [0, +\infty[$, so is $E^{\mathfrak{B}_n}|f|$ for each $n \geq 1$. Then by Proposition 2.1, $E^{\mathfrak{B}_n} f$ exists and is in $P^1_E(\mathfrak{B}_n)$. The condition $E^{\mathfrak{B}_\infty}|f| < +\infty$ provides a $\mathfrak{B}_\infty$-measurable partition $(A_k)_{k\geq 1}$ of Ω such that $f_k = f1_{A_k} \in L^1_{\mathbf{E}}(\mathfrak{F})$. Using Lemma 2.3, we obtain

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_\infty} f_k\ a.s.$$

As $A_k \in \mathfrak{B}_\infty \subset \mathfrak{B}_n$, for every $n \in \mathbf{N}^*$ and for every $k \in \mathbf{N}^*$, then by using Proposition 2.1, it is easy to see that

$$E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_n} 1_{A_k} f = 1_{A_k} E^{\mathfrak{B}_n} f\ a.s.$$

On the other hand

$$\begin{aligned} E^{\mathfrak{B}_\infty} f &= \textstyle\sum_{k=1}^{+\infty} E^{\mathfrak{B}_\infty} f_k = \sum_{k=1}^{+\infty} \lim_{n\to+\infty} E^{\mathfrak{B}_n} f_k\ a.s. \\ &= \textstyle\sum_{k=1}^{+\infty} 1_{A_k} \lim_{n\to+\infty} E^{\mathfrak{B}_n} f = \lim_{n\to+\infty} E^{\mathfrak{B}_n} f\ a.s. \end{aligned}$$

We conclude that

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} f(.) = E^{\mathfrak{B}_\infty} f(.)\ \ a.s.$$

□

Now, we give a new version of dominated convergence theorem for a sequences of E-valued Pettis-integrable random variables.

Theorem 3.2. *Let $(\mathfrak{B}_n)_{n\geq 1}$ be a decreasing sequence of a sub-σ-algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \cap_{n\geq 1}\mathfrak{B}_n$. Let Y be a positive random variable satisfying $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$. Let $(X_n)_{n\geq 1}$ be a sequence in $P^1_E(\mathfrak{F})$ such that*

(i) $\forall n \geq 1, |X_n(.)| \leq Y(.)$

(ii) $\lim_{n\to+\infty} X_n(.) = X_\infty(.)\ a.s.$

Then

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} X_n(.) = E^{\mathfrak{B}_\infty} X_\infty(.)\ a.s.$$

Proof. By assymption (i), the condition $E^{\mathfrak{B}_\infty} Y < +\infty$ and Proposition 2.1, we have $E^{\mathfrak{B}_n} X_n$ exists and it is in $P^1_E(\mathfrak{B}_n)$. Since $E^{\mathfrak{B}_\infty}|Y| < +\infty$ then, there exists a $\mathfrak{B}_\infty$-measurable partition $(A_m)_{m\geq 1}$ of Ω such that $\int_{A_m} Y dP < +\infty$ for all $m \geq 1$.
On the other hand

- $|X_n| 1_{A_m} \leq Y 1_{A_m}$.
- $\lim_{n\to+\infty} X_n 1_{A_m} = X_\infty 1_{A_m}\ a.s.$

Then $X_n 1_{A_m}$ and $Y 1_{A_m}$ satisfied all conditions of Lemma 2.4, consequently

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n}(X_n 1_{A_m}) = E^{\mathfrak{B}_\infty}(X_\infty 1_{A_m})\ a.s. \tag{3.1}$$

Since $A_m \in \mathfrak{B}_\infty \subset \mathfrak{B}_n$, for every $n \in \mathbf{N}^*$ and for every $m \in \mathbf{N}^*$, then

$$E^{\mathfrak{B}_n}(1_{A_m} X_n) = 1_{A_m} E^{\mathfrak{B}_n} X_n \ \ a.s. \tag{3.2}$$

By (3.1) and (3.2), we have $E^{\mathfrak{B}_\infty} X_\infty = \sum_{m\geq 1} E^{\mathfrak{B}_\infty}(X_\infty) 1_{A_m} = \sum_{m\geq 1} E^{\mathfrak{B}_\infty}(X_\infty 1_{A_m})\ a.s.$
$= \sum_{m\geq 1} \lim_{n\to+\infty} E^{\mathfrak{B}_n}(X_n 1_{A_m}) = \sum_{m\geq 1} \lim_{n\to+\infty} 1_{A_m} E^{\mathfrak{B}_n}(X_n) =$
$= \lim_{n\to+\infty} E^{\mathfrak{B}_n} X_n\ a.s.$

We conclude that

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} X_n(.) = E^{\mathfrak{B}_\infty} X_\infty(.)\ a.s.$$

□

4 Levy's theorem and dominated convergence theorem of conditional expectation for a $cwk(E)$-valued Pettis-integrable random sets

Now, we provide an extension of the Levy's theorem for a $cwk(E)$-valued Pettis-integrable random sets which extends Theorem 3.1.

Theorem 4.1. *Assume that E^* is separable. Let X be a $cwk(E)$-valued Pettis-integrable random set and let $(\mathfrak{B})_{n\geq 1}$ be a decreasing sequence of a sub-σ-algebras of $\mathfrak{F}$. Set $\mathfrak{B}_\infty = \cap_{n\geq 1}\mathfrak{B}_n$. If $E^{\mathfrak{B}_\infty}|X| \in [0, +\infty[$, we have*

$$M - \lim_{n\to\infty} E^{\mathfrak{B}_n} X(.) = E^{\mathfrak{B}_\infty} X(.)\ a.s.$$

Proof. • Step1: we prove that $E^{\mathfrak{B}_\infty} X(.) \subset\ s - li\ E^{\mathfrak{B}_n} X(.)\ a.s.$

Since X is $cwk(E)$-valued Pettis-integrable random set, then by [5], $S_X^{Pe}(\mathfrak{F}) \neq \emptyset$. On the other hand $E^{\mathfrak{B}_\infty}|X| \in [0, +\infty[$, then by Proposition 2.2, $E^{\mathfrak{B}_n} X$ exists and it is in $P^1_{cwk(E)}(\mathfrak{B}_n)$. Let $h \in S_X^{Pe}(\mathfrak{F})$, so $E^{\mathfrak{B}_n} h$ exists and by [6], $E^{\mathfrak{B}_n} h(.) \in E^{\mathfrak{B}_n} X(.)\ a.s.$ Using Theorem 3.1, we have

$$\lim_{n\to\infty} E^{\mathfrak{B}_n} h(.) = E^{\mathfrak{B}_\infty} h(.) \quad a.s.$$

Since $E^{\mathfrak{B}_\infty} h$ is Pettis-integrable, then

$$E^{\mathfrak{B}_\infty} h \in S^{Pe}_{s-liE^{\mathfrak{B}_n}X}(\mathfrak{B}_\infty),\ \forall h \in S_X^{Pe}(\mathfrak{F}).$$

By Proposition 2.2, we have $S^{Pe}_{E^{\mathfrak{B}_\infty}X}(\mathfrak{B}_\infty) = \{E^{\mathfrak{B}_\infty} f, f \in S_X^{Pe}(\mathfrak{F})\}$.
Therefore $S^{Pe}_{E^{\mathfrak{B}_\infty}X}(\mathfrak{B}_\infty) \subset S^{Pe}_{s-liE^{\mathfrak{B}_n}X}(\mathfrak{B}_\infty)$.
Consequently

$$E^{\mathfrak{B}_\infty} X(\omega) \subset s - li\ E^{\mathfrak{B}_n} X(\omega)\ a.s.$$

• Step2: we chow that $w - ls\ E^{\mathfrak{B}_n} X(.) \subset E^{\mathfrak{B}_\infty} X(.)\ a.s.$

Let $(e_k^*)_{k\geq 1}$ be a dense sequence in E^* for the Mackey topology, so by [2] for all $k \geq 1$ and for all $n \geq 1$, we have
$\delta^*(e_k^*, E^{\mathfrak{B}_n} X(.)) = E^{\mathfrak{B}_n} \delta^*(e_k^*, X(.))\ a.s.$ and $\delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(.)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(.))\ a.s.$
This yields the existence of a negligible set $N_1^{'}$ such that, for every $\omega \in \Omega \setminus N_1^{'}$, for every $n \geq 1$ and for every $k \geq 1$ one has
$\delta^*(e_k^*, E^{\mathfrak{B}_n} X(\omega)) = E^{\mathfrak{B}_n} \delta^*(e_k^*, X(\omega))$ and $\delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(\omega))$.
Using, the Lemma 2.3 applied to integrable real function $\delta^*(e_k^*, X)$, so there exists a negligible set $N_2^{'}$ such that for every $k \geq 1$ and $\omega \in \Omega \setminus N_2^{'}$

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} \delta^*(e_k^*, X(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(\omega)). \tag{4.1}$$

So, let $\omega \in \Omega \setminus N_1^{'} \cup N_2^{'}$ and $x \in w - ls\ E^{\mathfrak{B}_n} X(\omega)$, then there exists $(x_m)_{m\geq 1}$ in $(E^{\mathfrak{B}_{n_m}} X)_{m\geq 1}$ such that $x_m \xrightarrow{w} x$, where $(E^{\mathfrak{B}_{n_m}} X)_{m\geq 1}$ is a subsequence of $(E^{\mathfrak{B}_n} X)_{n\geq 1}$. Then by (4.1)

$$< e_k^*, x >= \lim_{m\to+\infty} < e_k^*, x_m >\leq \limsup_{m\to\infty} \delta^*(e_k^*, E^{\mathfrak{B}_{n_m}} X(\omega)) = \limsup_{m\to\infty} E^{\mathfrak{B}_{n_m}} \delta^*(e_k^*, X(\omega)) =$$

$$= E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(\omega)), \ \ \forall k \geq 1.$$

Therefore, for all $k \geq 1$ and for almost surely $\omega \in \Omega$, $< e_k^*, x >\leq \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(\omega))$.
Then $x \in E^{\mathfrak{B}_\infty} X(\omega)$, which implies that

$$w - lsE^{\mathfrak{B}_n} X(.) \subset E^{\mathfrak{B}_\infty} X(.)\ a.s.$$

This yields

$$M - \lim_{n\to\infty} E^{\mathfrak{B}_n} X(.) = E^{\mathfrak{B}_\infty} X(.)\ a.s.$$

□

Now, we give a new version of Fatou's lemma for the conditional expectation of the strong lower limit of a $cwk(E)$-valued Pettis-integrable random sets. The following theorem is an extension of Theorem 3.2 to multivalued case.

Theorem 4.2. *Assume that E^* is separable. Let $(\mathcal{B}_n)_{n\geq 1}$ be a decreasing sequence of a sub-σ-algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \cap_{n\geq 1} \mathfrak{B}_n$. Let Y be a positive random variable satisfying $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$. Let $(X_n)_{\mathbb{N}^* \cup \{+\infty\}}$ be a sequence of a $cwk(E)$-valued Pettis-integrable random sets such that,*

(i) $\forall n \in \mathbb{N}^* \cup \{+\infty\},\ |X_n(.)| \leq Y(.)$.

(ii) $X_\infty(.) \subset s - li\ \ X_n(.)$ *a.s.*

Then

$$E^{\mathfrak{B}_\infty} X_\infty(.) \subset s - liE^{\mathfrak{B}_n} X_n(.)\ \ a.s.$$

Proof. By (i) and the condition $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$, $E^{\mathfrak{B}_n} X_n$ exists and it is in $P^1_{cwk(E)}(\mathfrak{F})$ and by [6], we have

$$S^{Pe}_{E^{\mathfrak{B}_n} X_n} = \{E^{\mathfrak{B}_n} g, \quad g \in S^{Pe}_{X_n}(\mathfrak{F})\} \tag{4.2}$$

Since X_n is $cwk(E)$-valued Pettis-integrable random set, then by [8] there exists a sequence $(f_k)_{k\geq 1}$ of Pettis-integrable selections of X_n such that, $\forall \omega \in \Omega$, $X_n(\omega) = cl\{f_k(\omega),\ k \geq 1\}$.
Let $g \in S^{Pe}_{X_\infty}(\mathfrak{F})$ and $\omega \in \Omega$, set

$$H_n(\omega) = \{y \in X_n(\omega),\ \|g(\omega) - y\| \leq d(g(\omega), X_n(\omega)) + \frac{1}{n}\}.$$

On the other hand, we have

$$d(g(\omega), X_n(\omega)) = \inf\{\|g(\omega) - z\|,\ \ z \in X_n(\omega)\} = \inf_{k\geq 1} d(g(\omega), f_k(\omega)).$$

Then, according to the lower bound property, there exists $p \in \mathbb{N}^*$ such that

$$\|g(\omega) - f_p(\omega)\| \leq d(g(\omega), X_n(\omega)) + \frac{1}{n},$$

which implies that $H_n(\omega) \neq \phi$.
It is not hard to see that $H_n(\omega)$ is closed and convex. Since $X_n(\omega)$ is convex and weakly compact so is $H_n(\omega)$, then $H_n(\omega) \in cwk(E)$. It follows from theorem $III.41$ in [4] that H_n is measurable and by theorem III.6 in [4], there is a measurable selection g_n of H_n. So g_n is also a measurable selection of X_n. Then by theorem 5.4 in [5], g_n is Pettis-integrable and for all $\omega \in \Omega$, we have

$$\|g(\omega) - g_n(\omega)\| \leq d(g(\omega), X_n(\omega)) + \frac{1}{n}.$$

By (ii), we have $X_\infty(.) \subset s-li\ \ X_n(.)$ a.s. and since $g(.) \in X_\infty(.)$ a.s. then $\lim_{n\to+\infty} d(g(.), X_n(.)) = 0\ a.s.$ Therefore

$$\lim_{n\to+\infty} \|g(.) - g_n(.)\| = 0\ a.s.$$

From the conditions (i), we deduce that the sequence $(g_n)_{n\geq 1}$ satisfied all conditions of the Theorem 4.1, which implies that

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} g_n(.) = E^{\mathfrak{B}_\infty} g(.)\ \ a.s.$$

By the definition of the strong upper limit (s-li) and (4.2), we conclude that

$$E^{\mathfrak{B}_\infty} g(.) \in s-li\ \ E^{\mathfrak{B}_n} X_n(.)\quad a.s.$$

Then

$$E^{\mathfrak{B}_\infty} g \in S^{Pe}_{s-liE^{\mathfrak{B}_n} X_n}(\mathfrak{B}_\infty),\ \forall g \in S^{Pe}_{X_\infty}(\mathfrak{F}).$$

On the other hand $S^{Pe}_{E^{\mathfrak{B}_\infty} X_\infty}(\mathfrak{B}_\infty) = \{E^{\mathfrak{B}_\infty} f,\ \ f \in S^{Pe}_{X_\infty}(\mathfrak{F})\}$.
Therefore

$$S^{Pe}_{E^{\mathfrak{B}_\infty} X_\infty}(\mathfrak{B}_\infty) \subset S^{Pe}_{s-liE^{\mathfrak{B}_n} X_n}(\mathfrak{B}_\infty).$$

Consequently

$$E^{\mathfrak{B}_\infty} X_\infty(.) \subset s-li\ \ E^{\mathfrak{B}_n} X_n(.)\quad a.s.$$

□

Now, let us state the Mosco convergence of the conditional expectation for a $cwk(E)$-valued Pettis-integrable random sets.

Theorem 4.3. *Assume that E^* is separable. Let $(\mathcal{B}_n)_{n\geq 1}$ be a decreasing sequence of a sub-σ-algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \cap_{n\geq 1}\mathfrak{B}_n$. Let Y be a positive random variable satisfying $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$. Let $(X_n)_{\mathbb{N}^*\cup\{+\infty\}}$ be a sequence of a $cwk(E)$-valued Pettis-integrable random sets such that*

(i) $\forall n \in \mathbb{N}^* \cup \{+\infty\},\ |X_n(.)| \leq Y(.)$.

(ii) $X_\infty(.) \subset s-li\ \ X_n(.)$ *a.s.*

(iii) $\lim_{n\to+\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty)$ *a.s.* $\quad \forall x^* \in E^*$.

Then

$$M - \lim_{n\to+\infty} E^{\mathfrak{B}_n} X_n = E^{\mathfrak{B}_\infty} X_\infty\ a.s.$$

Proof. By condition (i), (ii) and Theorem 4.2, we conclude that

$$E^{\mathfrak{B}_\infty} X_\infty(.) \subset s-li\ \ E^{\mathfrak{B}_n} X_n(.)\quad a.s.$$

So, it is enough to check that $w-ls\ E^{\mathfrak{B}_n} X_n(.) \subset E^{\mathfrak{B}_\infty} X_\infty(.)\ a.s.$
By the condition (i) and (iii), it is easy to check that, the sequence $(\delta^*(x^*, X_n))_{n\geq 1}$ satisfied all conditions of Lemma 2.4, then

$$\forall x^* \in E^*,\qquad \lim_{n\to+\infty} E^{\mathfrak{B}_n} \delta^*(x^*, X_n(.)) = E^{\mathfrak{B}_\infty} \delta^*(x^*, X_\infty(.))\ \ a.s.$$

Let $(e_k^*)_{k\geq 1}$ be a dense sequence in E^* for the Mackey topology, so by [2] for all $k \geq 1$ and for every $n \geq 1$, we have
$E^{\mathfrak{B}_n} \delta^*(e_k^*, X_n(.)) = \delta^*(e_k^*, E^{\mathfrak{B}_n} X_n(.))\ a.s.$ and $E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty(.)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty(.))\ a.s.$
This yields the existence of a negligible set $N_3^{'}$ such that, for every $\omega \in \Omega \setminus N_3^{'}$ and every $k \geq 1$, one has
$E^{\mathfrak{B}_n} \delta^*(e_k^*, X_n(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_n} X_n(\omega)),\ \ E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty(\omega))$ and

$$\lim_{n\to+\infty} E^{\mathfrak{B}_n} \delta^*(e_k^*, X_n(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty(\omega)).$$

Let $\omega \in \Omega \setminus N_3'$ and $x \in w - ls\ E^{\mathfrak{B}_n} X_n(\omega)$, then there exists $(x_m)_{m\geq 1}$ in $(E^{\mathfrak{B}_{n_m}} X_{n_m})_{m\geq 1}$ such that $x_m \xrightarrow{w} x$, where $(E^{\mathfrak{B}_{n_m}} X_{n_m})_{m\geq 1}$ is a subsequence of $(E^{\mathfrak{B}_n} X_n)_{n\geq 1}$. Then

$$< e_k^*, x >= \lim_{m\to+\infty} < e_k^*, x_m > \leq \limsup_{m\to\infty} \delta^*(e_k^*, E^{\mathfrak{B}_{n_m}} X_{n_m}(\omega)) =$$

$$= \limsup_{m\to\infty} E^{\mathfrak{B}_{n_m}} \delta^*(e_k^*, X_{n_m}(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty(\omega)), \forall k \geq 1.$$

Therefore $x \in E^{\mathfrak{B}_\infty} X_\infty(\omega)$. This yields $w - lsE^{\mathfrak{B}_n} X_n(.) \subset E^{\mathfrak{B}_\infty} X_\infty(.)\ a.s.$ Hence

$$M - \lim_{n\to+\infty} E^{\mathfrak{B}_n} X_n(.) = E^{\mathfrak{B}_\infty} X_\infty(.)\ a.s.$$

□

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