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# A Class of Polynomial Semirings which are not $B$-Type Stable

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**Abstract** The notion and some properties of (strongly) $B$-rings, in a natural way, are extended to (strongly) $B$- and (strongly) $B_J$-semirings which is somewhat similar to the notion of rings having stable range 2. Results are given showing the connection between several types of semirings whose finite sequences satisfy some stability condition, some involving the Jacobson $k$-radical of the semiring $R$. Besides some examples and other results, our main objective is to study the conditions under which $R[x]$, the semiring of polynomials over a semiring $R$, is not a $B$-type semiring [i.e., (strongly) $B$-, (strongly) $B_J$-semirings, $S$-relative $B$-, and $S$-relative $B_J$-semirings].

## 1 Introduction

The main goal of this paper is to discuss some conditions under which $R[x]$ (*the semiring of polynomials over a semiring* $R$) is not a $B$-*type semiring* (see the $*$ statement below). We will recall the notion of *stable range in commutative semirings* from [8] (here Definition 1.3) for comparing the stable range of $B$-type and $n$-stable semirings. We merely focus on those properties of $B$-type semirings that are required only for our main results related to nonstability conditions of $R[x]$ (See the last section). In the next section, we will review some properties of commutative semirings that are required in this paper and mainly follow [5] throughout.

The concept of *stable range* was initiated by H. Bass in his investigation of *the stability properties of the general linear group in algebraic* $K$-*theory* [2]. In ring theory, stable range provides an arithmetic invariant for rings that is related to interesting issues such as cancellation, substitution, and exchange. The simplest case of stable range 1 has especially proved to be important in the study of many ring-theoretic topics.

$*$ In this paper *a semiring (ring)* $R$, unless otherwise indicated, is commutative with identity $1 \neq 0$ and $0a = 0$ for all $a \in R$; and $U(R)$ denotes the set of units of $R$. By *a* $B$-*type semiring*, we mean *a (strongly)* $B$-*, or a (strongly)* $B_J$-*, or an* $S$-*relative* $B$-*, or an* $S$-*relative* $B_J$-*semiring*, where $S$ is a nonempty subset of $R$. Also by a sequence of elements of $R$, we mean *a finite sequence* and will use it implicitly without any confusion in the context.

**Definition 1.1.** Let $R$ be a commutative semiring (ring) and $s \geq 1$ an integer. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of $R$ is said to be *stable* if $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + b_1a_{s+1}, a_2+b_2a_{s+1}, \ldots, a_s+b_sa_{s+1})$ for some $b_1, b_2, \ldots, b_s \in R$. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of $R$ is said to be a *unimodular sequence* if 1 is in the ideal $(a_1, a_2, \ldots, a_s, a_{s+1})$.

**Remark 1.2.** As in [4], we use $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 1$, to denote both a sequence and the ideal generated by the elements of the sequence; but the context will always make our meaning clear. Also, we follow [4] for the term “unimodular sequence" instead of “primitive vector" as used in [6]. For a detailed study of stable range in commutative rings and (strongly) $B$-rings; see [4], [6], [7], [9], and [10].

We continue this section by recalling some definitions and results from [8] (Definition 1.3 and Remark 1.5) and [6] and will end the section with a note about the organization of the paper.

In Section 2, we also recall some properties of $B$- and $B_J$-semirings from [8] (Definition 3.1 and Remark 3.5). Note that the main result of [8] is a classification of $B_J$-semirings which follows with two examples that can only occur in a nonring commutative semiring.

**Definition 1.3.** Let $R$ be a commutative semiring and $s \geq 1$ an integer. A fixed integer $n \geq 1$ is said to be in the *stable range* of $R$ (or simply, $R$ is $n$*-stable*) if every unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq n$, of elements of $R$ is stable. The semiring $R$ is said to be $n_J$*-stable* if every unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq n$, of elements of $R$ with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J_k(R)$ is stable.

Note that in the above definition for the $1_J$-stable case, it is obvious that we assume $a_1 \notin J_k(R)$ when a sequence $(a_1, a_2)$ of size two is under consideration.

**Remark 1.4.** It is clear that if R is $n$-stable, then it is $m$-stable for any integer $m \geq n$. Note that the term "$R$ is $n$-stable" is used in [7] (for convenience) and is exactly the same as the statement "$n$ is in the stable range of $R$", which is used by D. Estes and J. Ohm [4, p. 345]. A ring $R$ is said to be *2-stable* provided that any unimodular sequence of elements in $R$ of size strictly larger than 2 is stable.

In the following remark, we recall some results (related to $n$-stable semirings) from [8] for the sake of comparison and completeness.

**Remark 1.5.** The following facts are true in a commutative semiring.

(a) If all unimodular sequences of size $n+1$ ($n \geq 1$ a fixed integer) of a commutative semiring $R$ are stable, then any unimodular sequence of size larger than $n$ is stable (see [8, Theorem 2.6]).

(b) Let $n \geq 1$ be a fixed integer and $R$ a commutative semiring in which every maximal ideal is subtractive. Then $R$ is $n$-stable if and only if $R$ is $n_J$-stable (see [8, Theorem 2.8]).

(c) Let $R$ be a commutative semiring in which every maximal ideal is subtractive [in particular, $R$ is a subtractive semiring (i.e., a semiring in which every ideal is subtractive)] and $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, a unimodular sequence of $R$. Then $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n) = R$ provided that $a_i \in J_k(R)$ for some $1 \leq i \leq n$. Further, $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n) = R$ for each $1 \leq i \leq n$ provided that $a_{n+1} \in J_k(R)$ (see [8, Proposition 2.18]).

We now recall some definitions and results from [6] which are related to this paper.

• Let $J(R)$ be the *Jacobson radical* of a commutative ring $R$. A ring $R$ is said to be a $B$*-ring* if for any unimodular sequence $(a_1, \ldots, a_{n+1})$, $n \geq 2$ with $(a_1, \ldots, a_{n-1}) \nsubseteq J(R)$, there exists an element $b$ in $R$ such that $(a_1, \ldots, a_n + ba_{n+1}) = R$.

• Similarly, $R$ is defined to be a *strongly* $B$*-ring* (or $SB$*-ring* for short) if $d \in (a_1, \ldots, a_n, a_{n+1})$, $n \geq 2$, and $(a_1, a_2, \ldots, a_{n-1}) \nsubseteq J(R)$ implies that there exists $b \in R$ such that $d \in (a_1, a_2, \ldots, a_{n-1}, an + ba_{n+1})$.

In [6], Moore and Steger have studied some properties of (strongly) $B$-rings in detail. Besides many other results regarding $B$- and $SB$-rings, they showed that $R[X]$ is a $B$-ring ([6, Theorem 2.7]) [resp. an $SB$-ring ([6, Theorem 3.4])] if and only if $R$ is *completely primary* (a ring consisting of units and nilpotents) [resp. a field]; and we will discuss some of these results for $B$-type semirings in the sequel.

$**$ The organization of this paper is as follows: In Section 1 we recall some standard definitions from semiring theory and prove some results on semirings that will be used in the sequel. Section 2 is devoted to those properties of $B$-type semirings [i.e., (strongly) $B$-, (strongly) $B_J$-semirings (Definitions 3.1 and 3.2), $S$-relative $B$-, and $S$-relative $B_J$-semirings (Definition 3.9)] that are required for our main results in Section 3. In our study of these semirings, it suffices only to consider (arbitrary) unimodular triples instead of (arbitrary) unimodular $(n+1)$-tuples

(Theorems 3.6 and 3.12) [note that for $S$-relative case (Theorem 3.12), sequences of size 3 need not be unimodular (i.e., should satisfy a special condition)]. We also in this section, study the homomorphic image of (strongly $B$- and (strongly) $B_J$-semirings (Theorem 3.13). Finally, in Section 3, besides some examples, we study the conditions under which $R[x]$ (the semiring of polynomials over a semiring $R$) is not a $B$-type semiring (Theorems 4.1], 4.4, and Corollary 4.6). In Theorem 4.4 [resp. Theorem 4.1], we show that $R[x]$ can not be a $B$-semiring [resp. $B_J$-semiring] with respect to $I[x]$ when $I$ is a strong proper ideal of $R$.

## 2 Commutative Semirings

In this section we recall some definitions and prove some results concerning semirings which will be used in the sequel and mainly follow Golan [5]. By a semiring $(R, +, \cdot)$, we will mean a nonempty set $R$ with two binary operations of addition and multiplication defined on $R$ such that $(R, +)$ and $(R, \cdot)$ are commutative monoids with identity elements 0 and 1, respectively, where Multiplication distributes over addition (from either side) and $0a = 0$ for all $a \in R$ and $1 \neq 0$.

• A nonempty subset $I$ of a semiring $R$ will be called an *ideal* if $a, b \in I$ and $r$ in $R$ implies $a + b$ in $I$ and $ra$ in $I$. A *prime ideal* of $R$ is a *proper ideal* $P$ of $R$ in which $x \in P$ or $y \in P$ whenever $xy \in P$ (see also [5, Corollary, 6.5]). Note that in [5, Chapter 5], Golan defines an ideal $I$ of a semiring $R$ to be (proper) different from $R$, but we don't follow this assumption and make it clear when there is any confusion in the context.

**Definition 2.1.** A *subtractive ideal (= $k$-ideal) $I$* of a semiring $S$ is an ideal such that if $a, a+b \in I$, then $b \in I$. An ideal $I$ of $S$ is said to be a *strong ideal (= a strongly $k$-ideal)* if and only if $a + b \in I$ implies that $a \in I$ and $b \in I$.

**Remark 2.2.** From the above definition, it is clear that $(0)$ is a $k$-ideal of $S$. Also, every strongly $k$-ideal of a semiring $S$ is a $k$-ideal of $S$. But the converse need not be true in general. For example, the set $2N$ of all nonnegative even integers is a subtractive ideal of the semiring of all nonnegative integers. But it is not a strongly $k$-ideal since $3 + 5 \in 2N$ while neither 3 nor 5 belong to $2N$. Note that in [5], Golan uses the term "subtractive ideal", [resp. strong] for a $k$-ideal [resp. strongly $k$-ideal] but in the literature of semirings, authors use equivalently the term "$k$-ideal" [resp. strongly $k$-ideal] as well. Throughout this work, except for some cases in this section, we mainly follow Golan in [5]. Also, for some examples of nonsubtractive ideals in a semiring, see Chapter 5 of [5].

• A non-zero element $a$ of a semiring $R$ is said to be a *semiunit* in $R$ if there exist $r, s \in R$ such that $1 + ra = sa$.

• We define the *Jacobson $k$-radical* of a semiring $R$, denoted by $J_k(R)$ (= Jac$(R)$ as used in [8]), to be the intersection of all maximal $k$-ideals of $R$. Note that by [11, Corollary 2.2], the Jacobson $k$-radical of $R$ always exists and it can easily be seen that it is a $k$-ideal since the intersection of any number of $k$-ideals is a $k$-ideal.

We now follow Golan [5, Chapter 8, p. 92] to define a morphism of semirings as follows.

**Definition 2.3.** If $R$ and $S$ are semirings then a function $f : R \to S$ is a morphism of semirings if and only if:

(a) $f(0_R) = 0_S$;

(b) $f(1_R) = 1_S$; and

(c) $f(r + r') = f(r) + f(r')$ and $f(rr') = f(r)f(r')$ for all $r$ and $r'$ in $R$.

We now begin considering some properties of morphisms of semirings.

**Proposition 2.4.** (cf. [5, Proposition 8.37]) Let $f : R \to S$ be a morphism of semirings.

(a) If $H$ is an ideal of $S$, then $f^{-1}(H)$ is an ideal of $R$. Moreover, if H is subtractive then so is $f^{-1}(H)$.

(b) If $f$ is a surjective morphism and if $I$ is an ideal of $R$, then $f(I)$ is an ideal of $S$.

(c) If $f$ is a surjective morphism, then the kernel of $f$ is a subtractive ideal of $R$.

(d) If $f$ is a surjective morphism, then $u$ is a unit in $R$ if and only if $f(u)$ is a unit in $S$.

*Proof.* Parts (a) and (b) follows from [5, Proposition 8.37] and (c) follows from (a) since $\ker(f) = f^{-1}(\{0\})$. The necessary part of (d) is clear since $1_S = f(1_R) = f(uu^{-1}) = f(u)f(u^{-1})$. Conversely, let $u \in R$, $I = (u)$ an ideal of $R$, and $f(u)$ be a unit in $S$. Clearly $f(I) = S$ since $f(u)$ is a unit in $S$ and so $I = R$. Otherwise, $I \neq R$ implies $1_R \in R \setminus I$, which implies $1_S = f(1_R) \notin f(I) = S$, yielding a contradiction. Thus $u$ is a unit in $R$. □

**Remark 2.5.** As defined on page 68 of [5]), an ideal $I$ of a semiring $R$ defines an *equivalence relation* $=_I$ on $R$ called the *Bourne relation*, given by $r =_I r'$ if and only if there exist elements $a$ and $a'$ of $I$ satisfying $r+a = r'+a'$. Note that if $r =_I r'$ and $s =_I s'$ in $R$, then $r+s =_I r'+s'$ and $rs =_I r's'$. We denote the set of all *equivalence classes* of elements of $R$ under this relation by $R/I$ and will denote the equivalence class of an element $r$ of $R$ by $r/I$. Clearly this relation is a *congruence* (i.e., an equivalence relation which is compatible with two binary operations of $R$) and, consequently, $R/I$ is well-defined for any ideal $I$ of $R$. Also, $a \in I$ implies $a \in 0/I$ since $a =_I 0$ by the fact that $a + 0 = 0 + a$. Thus $I \subseteq 0/I$. Moreover, if $I$ is a subtractive ideal of $R$, then $0/I = I$ since $a+i = 0+j \in I$ implies $a \in I$. Thus, for any subtractive ideal $I$ of $R$, the factor semiring $R/I$ and the surjective morphism $f : R \to R/I$, given by $r \mapsto r/I$, is well defined and its kernel is $I$. See also Example 9.1 and Proposition 9.10 in [5].

We will use the following lemma for the proof of Lemma 2.7, which will be used in the proof of our main result (Theorem 4.1).

**Lemma 2.6.** (cf. [3, Lemma 3.4]) Let $R$ be a semiring and $r \in R$. Then:

(a) If $r$ is a nilpotent element of $R$, then it is not a semiunit.

(b) If $r \in J_k(R)$, then for every $a \in R$, the element $1 + ra$ is a semiunit of $R$.

*Proof.* See Lemma 3.4 in [3]. □

We will use the following lemma For the proof of our main result (Theorem 4.1).

**Lemma 2.7.** *Let $R[x]$ be the semiring of polynomials over a commutative semiring $R$. Suppose $1 + r \neq r$ for any $r \in R$ [in particular, $R$ is a cancellative semiring]. Then $x^2$ can not lie in $J_k(R[x])$.*

*Proof.* Suppose to the contrary that $x^2 \in J_k(R[x])$. Then by Lemma 2.6, $1 + x^2$ is a semiunit in $R[x]$. Thus, $1 + s(x)(1 + x^2) = t(x)(1 + x^2)$ for some $s(x)$ and $t(x)$ in $R[x]$. Let $s(x) = s_0 + s_1x + \cdots + s_nx^n$ and $t(x) = t_0 + t_1x + \cdots + t_mx^m$, where $s_i, t_j \in R$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. without loss of generality (by inserting zeros for coefficients if it is required), assume $m = n$. Let $r = s_0 + s_1 + \cdots + s_n$. Now it is not difficult to see that for $n = 0, 1, 2, 3$, we get $1 + r = r$, which is a contradiction by hypothesis. Next, in order to complete the proof, it suffices to consider the following four different cases. That is, when $n$ [resp. $n-1$, $n-2$, or $n-3$] is congruent mod 4 and we just discuss when 4 divides $n$ (or $n-3$ and leave the other cases to the reader.

Suppose $1 + s(x)(1 + x^2) = t(x)(1 + x^2)$. Thus $1 + s_0 + s_1x + (s_0 + s_2)x^2 + (s_1 + s_3)x^3 + (s_2 + s_4)x^4 + \cdots + (s_{n-2} + s_n)x^n + s_{n-1}x^{n+1} + s_nx^{n+2} = t_0 + t_1x + (t_0 + t_2)x^2 + (t_1 + t_3)x^3 + (t_2 + t_4)x^4 + \cdots + (t_{n-2} + t_n)x^n + t_{n-1}x^{n+1} + t_nx^{n+2}$. Hence, $1 + s_0 = t_0$, $s_1 = t_1$, $(s_0+s_2) = (t_0+t_2)$, $(s_1+s_3) = (t_1+t_3)$, $(s_2+s_4) = (t_2+t_4)$, $\ldots$, $(s_{n-2}+s_n) = (t_{n-2}+t_n)$, $s_{n-1} = t_{n-1}$, and $s_n = t_n$.

Next by a proper partitioning of the summations of $s_i$'s and $t_i$'s, we get a contradiction as follows:

**Case 1:** $n$mod4. $r = s_0+s_1+s_2+\cdots+s_n = ((s_0+s_2)+(s_1+s_3))+((s_4+s_6)+(s_5+s_7))+\cdots+((s_{n-4}+(s_{n-2})+(s_{n-3}+s_{n-1}))+s_n = ((t_0+t_2)+(t_1+t_3))+((t_4+t_6)+(t_5+t_7))+\cdots+((t_{n-4}+(t_{n-2})+(t_{n-3}+t_{n-1}))+t_n = t_0+(t_2+t_1+t_3+t_4)+(t_6+t_5+t_7+t_8)+\cdots+(t_{n-2}+t_{n-3}+t_{n-1}+t_n) = (1+s_0)+(s_1+s_2+s_3+s_4)+(s_5+s_6+s_7+s_8)+\cdots+(s_{n-3}+s_{n-2}+s_{n-1}+s_n) = 1+r$, yielding a contradiction.

**Case 4:** $n - 3\text{mod}4$. In this case since $t_1 = s_1$, we can write $r = s_0 + s_1 + s_2 + \cdots + s_n = \cdots = t_0 + t_1 + (t_2 + t_3 + t_4 + t_5) + (t_6 + t_7 + t_8 + t_9) + \cdots$
= $(1+s_0 + s_1) + s_2 + s_3 + s_4 + s_5 + \cdots + s_n = 1 + r$, yielding a contradiction.

Note that for any $n \geq 4$, $s_{n-1} = t_{n-1}$ and $s_n = t_n$ and for any case, partitioning into four-element groups, leaves out at most two elements, namely $t_{n-1}$ and $t_n$, which can be replaced by $s_{n-1}$ and $s_n$, respectively. □

We end this section by recalling some more definitions from [5] and write them here for the sake of completeness as follows.

• A semiring with no nonzero zero divisors is called an *entire (= semidomain). A semifield is a semiring in which every nonzero element has a multiplicative inverse. A semiring $R$ is zerosumfree if and only if $r + r' = 0$ implies that $r = r' = 0$. A semiring $R$ is said to be simple if $1 + r = 1$ for each $r \in R$. Let $R$ be a semiring and $G(R) = \{r \in R \mid 1 + r \in U(R)\}$. A semiring $R$ is called a Gelfand semiring when $G(R) = R$. Clearly, every simple semiring is Gelfand. Of course, bounded distributive lattices are among Gelfand semirings. But the class of the Gelfand semirings is quite wider as Example 1.4 in [8] shows (cf. [5, Example 3.38]).*

## 3 Some Results on $B$-type Semirings

*In this section, we merely focus on $B$-type semirings [i.e., (strongly) $B$-, (strongly) $B_J$-semirings (Definitions 3.1 and 3.2), $S$-relative $B$-, and $S$-relative $B_j$-semirings (Definition 3.9)] and study those properties of them (Theorems 3.6, 3.13, and 3.12) that are required for the proof of our main results in the next section on some nonstability conditions of $R[x]$. We also in Remark 3.5, recall some results related to $B$- and $B_J$-semirings from [8].*

**Definition 3.1.** A semiring $R$ is said to be a $B$- *[resp $B_J$-semiring]* whenever for any unimodular sequence $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 2$, of elements in $R$ [resp. with $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J_k(R)$], there exists an element $b$ in $R$ such that $(a_1, a_2, \ldots, a_n + ba_{n+1}) = R$.

**Definition 3.2.** A semiring $R$ is said to be a *strongly $B$-semiring* (or $SB$-semiring for short) [resp. *strongly $B_J$-semiring* (or $SB_J$-semiring for short)] if $d \in (a_1, \ldots, a_n, a_{n+1})$, $n \geq 2$, [resp. with $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J_k(R)$] implies that there exists $b$ in $R$ such that $d \in (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$.

**Remark 3.3.** From the above definitions, it is clear that the definition of a $B_J$-semiring [resp. an $SB_J$-semiring] is exactly a natural extension of the definition of a $B$-ring [resp. an $SB$-ring] to semirings whenever $R$ is assumed to be a ring as defined in [6] and $B$-semirings can be regarded as a generalization of a subclass (special case) of *2-stable rings* (see Remark 1.4). Obviously, any $B$-semiring [resp. $SB$-semiring] is a $B_J$-semiring [resp. $SB_J$-semiring]. Also, it is clear that any $SB$-semiring [resp. $SB_J$-semiring] is a $B$-semiring [resp. $B_J$-semiring]. See the following diagram.

$$SB\text{-semiring} \to B\text{-semiring} \to B_J\text{-semiring}$$

$$SB\text{-semiring} \to SB_J\text{-semiring} \to B_J\text{-semiring}$$

*The following example provides a trivial instance of a class of $B$-semirings.*

**Example 3.4.** A semifield is a $B$-semiring (consequently, a $B_J$-semiring). That is, $1 \in (a_1, a_2, \ldots, a_n, a_{n+1}) = (a_1, a_2, \ldots, a_n + ba_{n+1})$, where $b = 0$ when $a_n \neq 0$; or $b = 1$ when $a_n = 0$. Moreover, besides some trivial examples of semifields such as semifields of nonnegative reals and nonnegative rationals, see [5, Proposition 7.8] that states: If $I$ is a subtractive maximal ideal of a commutative semiring $R$, then $R/I$ is a semifield.

*In the following remark, we recall some results (related to $B$- and $B_J$-semirings) from [8] for the sake of comparison and completeness.*

**Remark 3.5.** The following facts are true in a commutative semiring.

(a) Let $R$ be a $B_J$-semiring in which every maximal ideal is subtractive. Then $R$ is 2-stable (see [8, Corollary 2.9]).

(b) Let $R$ be a Gelfand semiring and let $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, be a unimodular sequence of $R$. Then $1 \in (a_1, a_2, \ldots, a_n + ba_{n+1})$ for some $b \in R$ (i.e., one is in the stable range of $R$). In other words, we may simply say $R$ is a $B$-semiring when $n \geq 2$ (see [8, Theorem 2.10]).

(c) A simple semiring is a $B$-semiring (consequently, a $B_J$-semiring), see [8, Corollary 2.11].

(d) A semiring $R$ is a $B$-semiring (consequently, a $B_J$-semiring) provided that $R$ is a semiring in which every maximal ideal is strong (see [8, Corollary 2.11 and Proposition 2.15]).

*We now provide a criterion for the study of (strongly) $B$- and (strongly) $B_J$-semirings.*

**Theorem 3.6.** (cf. [8, Theorem 2.7] and [9, Theorem 2]) The following hold:

(a) A semiring $R$ is a $B$-semiring [resp. $B_J$-semiring] if and only if for any unimodular sequence $(a_1, a_2, a_3)$ of $R$ [resp. with $a_1 \notin J_k(R)$], there exists an element $b \in R$ such that $(a_1, a_2 + ba_3) = R$.

(b) A semiring $R$ is an $SB$-semiring [resp. $SB_J$-semiring] if and only if for every $s, c_1, c_2, c_3 \in R$ with $s \in (c_1, c_2, c_3)$ [resp. $c_1 \notin J_k(R)$], it follows that $s \in (c_1, c_2 + bc_3)$ for some $b \in R$.

*Proof.* (a) The proof of Part (a) is exactly the same as the proof of [8, Theorem 2.7] and we write it here since the proof of Theorem 3.12 (below) is refered to this theorem. We just write a proof for the $B_J$-case. The necessary part is quite clear. To prove the sufficient part, let $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 2$, be a unimodular sequence in $R$ with $(a_1, a_2, \ldots, a_n - 1) \nsubseteq J_k(R)$. Without loss of generality, assume that $a_1 \notin J_k(R)$. Now, $1 \in (a_1, a_2, \ldots, a_n, a_{n+1})$ implies $1 = \sum_{i=1}^{n+1} a_i x_i$ for some $x_1, x_2, \ldots, x_n, x_{n+1} \in R$. Thus, $1 \in (a_1, a_n, l)$, where $l = a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_{n+1}x_{n+1}$. Now by the hypothesis, there exists $b \in R$ such that $1 \in (a_1, a_n + bl) \subseteq (a_1, a_2, \ldots, a_{n-1}, a_n + bx_{n+1}a_{n+1})$.

(b) The proof is essentially similar to the proof of Lemma 3.1 of [6]. The necessity clearly follows from the definition of an $SB$-semiring [resp. $SB_J$-semiring]. We just give a proof for the $SB_J$-semiring case and leave the other part to the reader. To prove the sufficient part, assume that $a_1, a_2, \ldots, a_n, a_{n+1}$, $n \geq 2$, is a sequence in $R$ with $(a_1, a_2, \ldots, a_{n-1}) \nsubseteq J_k(R)$ and let $r \in (a_1, a_2, \ldots, a_n, a_{n+1})$. Without loss of generality, we may assume that $a_{n-1} \notin J_k(R)$. Suppose $r = \sum_{i=1}^{n+1} a_i x_i$ and let $s = a_{n-1}x_{n-1} + a_n x_n + a_{n+1}x_{n+1}$ for some $x_i \in R$. Then $r \in (a_1, a_2, \ldots, a_{n-2}, s)$ and $s \in (a_{n-1}, a_n, a_{n+1})$. Since $a_{n-1} \notin J_k(R)$, $s \in (a_{n-1}, a_n + ba_{n+1})$ for some $b \in R$. Therefore $r \in (a_1, a_2, \ldots, a_{n-2}, s) \subseteq (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$, and the proof is complete. □

**Remark 3.7.** We can also prove Part (b) of the above theorem by using the same argument as in Part (a), which is taken from [8, Theorem 2.7].

*$bullet$ In view of the above theorem, we need only consider the unimodular [resp. arbitrary] triples instead of arbitrary unimodular [resp. arbitrary] $(n + 1)$-tuples, $n > 2$, in our study of $B$- and $B_J$-semirings [resp. $SB$- and $SB_J$-semirings].*

*We now provide a sharper result than Theorem 3.6(b) for the study of $SB$- and $SB_J$-semirings when the underlying semiring is subtractive.*

**Theorem 3.8.** *Let $R$ be a subtractive semiring. Then $R$ is an $SB$-semiring [resp. $SB_J$-semiring] if and only if for every sequence $(c_1, c_2, c_3)$ of $R$ [resp. with $c_1 \notin J_k(R)$], there exists $b \in R$ such that $c_3 \in (c_1, c_2 + bc_3)$. Further, if $A = (a_1, a_2, \ldots, a_n, a_{n+1})$, then $A = (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$ for some $b \in R$.*

*Proof.* The necessity clearly follows from the definition of an $SB$-semiring [resp. $SB_J$-semiring]. We just give a proof for the $SB_J$-semiring case and leave the other part to the reader. To prove the sufficient part, assume that $a_1, a_2, \ldots, a_n, a_{n+1}$, $n \geq 2$, is a sequence in $R$ with $(a_1, a_2, \ldots, a_{n-1}) \nsubseteq J_k(R)$ and let $r \in (a_1, a_2, \ldots, a_n, a_{n+1})$. Without loss of generality, we may assume that $a_{n-1} \notin J_k(R)$. Since $a_{n+1} \in (a_{n-1}, a_n, a_{n+1})$, there exist $b \in R$ such that $a_{n+1} \in (a_{n-1}, a_n + ba_{n+1})$ by hypothesis. Hence $r \in (a_1, a_2, \ldots, a_{n-1}, a_n, a_{n+1}) = (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$, where the equality holds by the subtractive assumption and the proof is complete. □

• *In view of the above theorem, we see that a subtractive semiring $R$ is an $SB$-semiring [resp. $SB_J$-semiring] if for any sequence $(a_1, a_2, a_3)$ of $R$ [resp. with $a_1 \notin J_k(R)$], there exists $b \in R$ such that $a_3 \in (a_1, a_2 + ba_3)$. Clearly, the above theorem is a good criterion to check whether a ring is an $SB$-ring or not since every ideal in a ring is subtractive.*

*We now introduce a class of $B$-type semirings that are defined with respect to a nonempty subset $S$ of a semiring $R$.*

**Definition 3.9.** Let $S$ be a nonempty subset of a semiring $R$. $R$ is said to be a *$B$-semiring [resp. $B_J$-semiring]* with respect to $S$ or $R$ is an *$S$-relative $B$-semiring [resp. an $S$-relative $B_J$-semiring]* if for any ideal $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 2$, of $R$ and $a \in S$ [resp. with $(a_1, a_2, \ldots, a_{n-1}) \nsubseteq J_k(R)$] such that $1 + a \in (a_1, a_2, \ldots, a_n, a_{n+1})$, then there exists $b \in R$ such that $1 + a \in (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$.

**Remark 3.10.** From the above definition, a $B$-semiring [resp. $B_J$-semiring] is a $\{0\}$-relative (or simply, 0-relative) $B$-semiring [resp. 0-relative) $B_J$-semiring]. Clearly, every $B$-semiring [resp. $B_J$-semiring] with respect to a nonempty subset $S$ of $R$ is a $B$-semiring [resp. $B_J$-semiring] provided $0 \in S$. Moreover, every $SB$-semiring [resp. $SB_J$-semiring] (Definition 3.2) is an $S$-relative $B$-semiring [resp. $S$-relative $B_J$-semiring] for each nonempty subset $S$ of $R$. Also, let $S \subseteq T$ be two nonempty subsets of a semiring $R$. Then $R$ is an $S$-relative $B$-semiring [resp. $S$-relative $B_J$-semiring] if $R$ is a $T$-relative $B$-semiring [resp. $T$-relative $B_J$-semiring]. Clearly, $R$ is a $B$-semiring [resp. $B_J$-semiring] if and only if $R$ is a $G(R)$-relative $B$-semiring [resp. $G(R)$-relative $B_J$-semiring], where $G(R) = \{a \in R \mid 1 + a \in U(R)\}$.

• *From the above remark, it is clear that the class of $SB$- and $SB_J$-semirings are contained in the class of $S$-relative $B$- and $S$-relative $B_J$-semirings, respectively. Further, the class of $S$-relative $B$- and $S$-relative $B_J$-semirings are contained in the class of $B$- and $B_J$-semirings, respectively, provided that* $0 \in S$.

**Example 3.11.** In [8, Theorem 2.10 and Corollary 2.11] (see Remark 3.5(b and c)), it is shown that a Gelfand semiring $R$ [in particular, a simple semiring] is a $B$-semiring. Thus from the above remark, $R$ is an $R$-relative $B$-semiring or equivalently an $S$-relative $B$-semiring for any nonempty subset $S$ of $R$.

*We now provide a criterion (similar to Theorem 3.6(a)) for the study of $S$-relative $B$- and $S$-relative $B_J$-semirings.*

**Theorem 3.12.** *Let $S$ be a nonempty subset of a semiring $R$. A semiring $R$ is an $S$-relative $B$-semiring [resp. an $S$-relative $B_J$-semiring] if and only if for every $a \in S$ and $c_1, c_2, c_3 \in R$ with $1 + a \in (c_1, c_2, c_3)$ [resp. $c_1 \notin J_k(R)$], it follows that $1 + a \in (c_1, c_2 + bc_3)$ for some $b \in R$.*

*Proof.* The proof is similar to the proof of Theorem 3.6(a) by replacing 1 with $1 + a$. □

• *In view of the above theorem, we need only consider the sequences of size three that satisfy $B$-stability condition with respect to a nonempty subset $S$ of $R$ in our study of $S$-relative $B$- and $S$-relative $B_J$-semirings.*

*We now consider the homomorphic image of (strongly) $B$- and (strongly) $B_J$-semirings. Also, Abdolyousefi and Chen in [1, Lemma 2.9] show the similar result for $J$-stable rings and they refer to the work of the author [9, Theorem 3] that shows the homomorphic image of a $B$-ring is a $B$-ring. Further, they show how the classes of $J$-stable rings and $B$-rings coincide with each other (see the paragraph preceding Theorem 2.5 and Remark 2.6 in [1]).*

**Theorem 3.13.** (cf. [9, Theorem 3]) Let $f : R \to S$ be a surjective morphism of semirings.

(a) If $R$ is a $B$-semiring, then so is $S$.

(b) If $R$ is a $B_J$-semiring, then so is $S$ provided that $f(J_k(R)) \subseteq J_k(S)$.

(c) If $R$ is an $SB$-semiring, then so is $S$.

(d) If $R$ is an $SB_J$-semiring, then so is $S$ provided that $f(J_k(R)) \subseteq J_k(S)$.

*Proof.* We write a proof for Parts (b) and (d) and leave the other parts to the reader.
(b): By virtue of Theorem 3.6(a), it suffices to argue only for unimodular sequences of size three. Suppose $R$ is a $B_J$-semiring and let $1_S \in (x_1, x_2, x_3)$ with $x_1 \notin J_k(S)$, where $x_1, x_2, x_3 \in S$. Thus $f(1_R) = 1_S = \sum s_i x_i$ for some $s_i \in S$, where $i = 1, 2, 3$. Therefore, $f(1_R) = \sum f(r_i) f(a_i) = \sum f(r_i a_i) = f(\sum r_i a_i)$ for some $r_i$ and $a_i$ in $R$, where $f(r_i) = s_i$ and $f(a_i) = x_i$ and $i = 1, 2, 3$. Clearly $1_R \in (a_1, a_2, a_3)$ and $a_1 \notin J_k(R)$ by hypothesis (see also Proposition 2.4). Thus $1_R \in (a_1, a_2 + ba_3)$ for some $b \in R$ since $R$ is a $B_J$-semiring. Consequently $f(1_R) = 1_S \in (f(a_1), f(a_2) + f(b)f(a_3)) = (x_1, x_2 + sx_3)$, where $s = f(b)$.

(d): Let $\overline{R}$ be the image of $R$ under the homomorphism $f$, and let $\overline{d} \in (\overline{a_1}, \overline{a_2}, \overline{a_3}$ with $\overline{a_1} \notin J_k(\overline{R})$, where $\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{d} \in \overline{R}$. suppose that $\overline{d} = \sum_{i=1}^{3} \overline{a_i x_i}$ for some $\overline{x_i} \in \overline{R}$ and let $f(a_i) = \overline{a_i}$, $fx_i = \overline{x_i}$ for $i = 1, 2, 3$. Let $d = \sum_{i=1}^{3} a_i x_i$. Since by hypothesis $f(J_k(R)) \subseteq J_k(\overline{R})$, we have $a_1 \notin J_k(R)$ and so $d \in (a_1, a_2 + ba_3)$ for some $b \in R$. Since $f(d) = \overline{d}$, we have $\overline{d} \in (\overline{a_1}, \overline{a_2} + \overline{b}\overline{a_3})$, where $f(b) = \overline{b}$. Hence by Theorem 3.6(b), $S = \overline{R}$ is an $SB_J$-semiring. □

**Corollary 3.14.** *Let $I$ be a proper ideal of a semiring $R$. Then $R/I$ is a $B$-semiring [resp. an $SB$-semiring] when $R$ is a $B$-semiring [resp. an $SB$-semiring].*

*Proof.* Clearly $r \mapsto r/I$ defines a surjective morphism from $R$ to $R/I$, where $r \in R$. Now the proof follows directly from Part (a) [resp. (c)] of the above theorem. See also Example 9.1 and Proposition 9.10 in [5]. □

*We end this section by extending the above corollary to an $S$-relative $B$-semiring.*

**Theorem 3.15.** *Let $I$ be a proper ideal of a semiring $R$ and $S$ a nonempty subset of $R$. Then $R/I$ is an $S/I$-relative $B$-semiring when $R$ is an $S+I$-relative $B$-semiring. Further, if $S \cap I \neq \emptyset$ [in particular, if $0 \in S$], then $R/I$ is also a $B$-semiring.*

*Proof.* By virtue of Theorem 3.12, it suffices to argue only for sequences of size three. Suppose $1/I + s/I \in (a_1/I, a_2/I, a_3/I)$, where $s \in S$. Thus $(1+s)/I = \sum (r_i/I)(a_i/I) = \sum (r_i a_i)/I$ for some $r_i \in R$, where $i = 1, 2, 3$. Therefore, $1 + s + a = r_1 a_1 + r_2 a_2 + r_3 a_3 + a'$ for some $a, a' \in I$ by definition. Thus $1 + s + a \in (a_1, a_2, r_3 a_3 + a')$. Now by hypothesis, there exists $b \in R$ such that $1 + s + a \in (a_1, a_2 + b(r_3 a_3 + a'))$. Consequently, $(1/I) + (s/I) \in (a_1/I, (a_2/I) + (br_3)/I(a_3/I))$ and the proof of the first part is complete. The “further" part is clear since $a/I = 0/I$ when $a \in I$ (see also Remarks 2.5 and 3.10). □

## 4 Non-$B$-type Stability of $R[x]$

*In this section, besides some examples, we apply the results from previous sections to provide a sufficient condition for which $R[x]$ is not a $B$-type semiring (Theorems 4.1, 4.4; and Corollary 4.6). See also Examples 4.5, 4.7, 4.8, and 4.9.*

*In the following theorem, we partially characterize the $B$-stability condition of $R[x]$ (the semiring of polynomials over a semiring $R$), which is somewhat a counterpart to [6, Theorem 2.7] and [resp. [6, Theorem 3.4]] that states: $R[x]$ is a $B$-ring [resp. an $SB$-ring] if and only if $R$ is completely primary (i.e., a ring consisting of units and nilpotents) [resp. a field].*

**Theorem 4.1.** (cf. [6, Theorems 2.7 and 3.4]) Let $R[x]$ be the semiring of polynomials over a semiring $R$ and $1 + r \neq r$ for all $r \in R$.

(a) Let $I$ be a proper ideal of $R$. If $R[x]$ is a $B_J$-semiring with respect to the ideal $I[x]$ of $R[x]$, then $I$ is not a strong ideal of $R$. In other words, if $I$ is a strong proper ideal of $R$, then $R[x]$ can not be a $B_J$-semiring with respect to the ideal $I[x]$ in $R[x]$.

(b) If $R[x]$ is a $B_J$-semiring, then $R$ is not a zerosumfree semiring.

(c) If $R[x]$ is an $SB_J$-semiring, then $R$ is not a zerosumfree semiring.

**Remark 4.2.** Clearly, $\mathbb{Z}$ (ring of integers) is not zerosumfree as a semiring since rings can not be zerosumfree by the fact that $-1+1=0$. Thus, the converse of Parts (b) and (c) of the above theorem are not true in general since by Theorem 2.7 of [6], $\mathbb{Z}[x]$ is not a $B$-ring [consequently, not an $SB$-ring]. We can also directly conclude from [6, Theorem 3.4] that $\mathbb{Z}[x]$ is not an $SB$-ring since $\mathbb{Z}$ is not a field.

*Proof.* (a): Suppose to the contrary that $I$ is a strong proper ideal of $R$. Let $a \in I$ and $1+ax \in (x^2, x, 1+ax)$. Clearly by Lemma 2.7, $X^2 \notin J_k(R[x])$. If $R[x]$ is a $B_J$-semiring with respect to $I[x]$, then $1+ax \in (x^2, x+b(x)(1+ax))$ for some $b(x) \in R[x]$ by definition. Let $1+ax = x^2 f(x) + (x+b(x)(1+ax))g(x)$, where $f(x), g(x) \in R[x]$. Let $f_i$, $g_i$, and $b_i$ represent the coefficient of $x^i$ in the polynomials $f(x)$, $g(x)$, and $b(x)$, respectively. Now by equating the corresponding coefficients in the above equation, we get $(x+b(x)+axb(x))g(x) = (x+b_0+b_1x+\cdots+ab_0x+\cdots)g(x)$, which implies $g_0x+b_0g_0+b_1g_0x+ab_0g_0x+b_0g_1x$ and so $1=g_0b_0$ and $a=g_0+b_1g_0+ab_0g_0+b_0g_1$. Thus if $I$ is strong in $R$, then $g_0 \in I$ since $a \in I$ by the assumption, which implies $1 \in I$ and leads to a contradiction.

(b): The proof follows directly from Part (a) by setting $I=\{0\}$ and using the fact that $\{0\}$ is a strong ideal of $R$ if and only if $R$ is a zerosumfree semiring.

(c): The proof is very much similar to Part (a) by replacing $1+ax$ with $r$ and we write it here for the sake of comparison and completeness. Notice that (c) is an immediate consequence of (b) since an $SB_J$-semiring is a $B_J$-semiring. Suppose to the contrary that $R$ is a zerosumfree semiring. Let $r \in R$ with $r \neq 0$. Then $r \in (x^2, x, r)$ and $x^2 \notin J_k(R[x])$ by Lemma 2.7. If $R[x]$ is an $SB_J$-semiring, then $r \in (x^2, x+rb(x))$ for some $b(x) \in R[x]$. Let $r = x^2 f(x) + (x+rb(x))g(x)$, where $f(x), g(x) \in R[x]$. Let $f_i$, $g_i$, and $b_i$ represent the coefficient of $x^i$ in the polynomials $f(x)$, $g(x)$, and $b(x)$, respectively. Equating coefficients in the above equation gives $r = rb_0g_0$ and $0 = g_0 + r(b_0g_1 + g_0b_1)$. Now if $R$ is zerosumfree, then $g_0 = 0$, which implies $r=0$ and leads to a contradiction. □

**Corollary 4.3.** *Let $R$ be an additively cancellative zerosumfree semiring. Then $R[x]$ is not a $B_J$-semiring (consequently, not an $SB_J$-semiring).*

**Theorem 4.4.** *Let $R[x]$ be the semiring of polynomials over a semiring $R$.*

(a) *Let $I$ be a proper ideal of $R$. If $R[x]$ is a $B$-semiring with respect to the ideal $I[x]$ of $R[x]$, then $I$ is not a strong ideal of $R$. In other words, if $I$ is a strong proper ideal of $R$, then $R[x]$ can not be a $B$-semiring with respect to the ideal $I[x]$ in $R[x]$.*

(b) *If $R[x]$ is a $B$-semiring, then $R$ is not a zerosumfree semiring.*

(c) *If $R[x]$ is an $SB$-semiring, then $R$ is not a zerosumfree semiring.*

*Proof.* The proof is similar to the proof of Theorem 4.1. Note that for the proof of this theorem, we don't need Lemma 2.7 since we don't need to show that $x^2 \notin J_k(R[x])$. □

**Example 4.5.** Let $R$ be the semiring of nonnegative reals, or nonnegative rationals, or nonnegative integers, respectively, with usual addition and multiplication. Clearly, $R$ is a commutative, zerosumfree semiring which is not additively idempotent and by Theorem 4.4(b), $R[x]$ is not a $B$-semiring [consequently, not an $SB$-semiring]. For more examples of zerosumfree semirings, see [5].

**Corollary 4.6.** *If $R$ is an additively idempotent semiring [in particular, a simple semiring], then $Rx$ is not a $B$-semiring [consequently, not an $SB$-semiring].*

*Proof.* The proof is immediate from Part (b) of the above theorem since every additively idempotent semiring [in particular, a (simple semiring], which is not a ring, is zerosumfree. Note that $a+b=0$ implies $a=a+a+b=a+b+b=b$ in any additively idempotent semiring and also a simple semiring is additively idempotent since $1+1=1$. □

**Example 4.7.** Let $R$ be a semiring (ring) and $S = \text{ideal}(R)$ be the semiring of ideals of $R$ under the addition and multiplication of the ideals of $R$. Then by the above Corollary, $S[x]$ is not a $B$-semiring [consequently, not an $SB$-semiring] since $S$ is a simple semiring (i.e., $R + A = R$ for any ideal $A$ of $R$).

**Example 4.8.** (cf. [5, Example 5.1]) If $A$ is an infinite set, then the family $\text{fsub}(A)$ of all finite subsets of $A$ is a strong ideal of the semiring $(\text{sub(A)}, \cup, \cap)$. Thus, by Theorem 4.4(a), $R[x]$ is not a $B$-semiring with respect to $I[x]$ when $I = \text{fsub}(A)$ and $R = (\text{sub(A)}, \cup, \cap)$.

*Finally, we close this paper with an example and a discussion regarding a nonstability condition of a polynomial semiring.*

**Example 4.9.** Clearly, from Theorem 3.13(a) [resp. 3.13(c)], if $R[x]$ (the semiring of polynomials over a semiring $R$) is a $B$-semiring [resp. an $SB$-semiring], then so is $R$ under the morphism $\phi : R[x] \to R$ given by $a_0 + a_1x + \cdots + a_nx^n \mapsto a_0$, where $a_i \in R$ for each $0 \leq i \leq n$. See also Theorem 4.4 that shows $R[x]$ is not a $B$-semiring [resp. an $SB$-semiring] when $R$ is zerosumfree. Further, since a simple semiring $R$ is a $B$-semiring ([8, Corollary 2.11]; see Remark 3.5(c)), the converse of this example need not be true in general since by Corollary 4.6, $R[x]$ is not a $B$-semiring when $R$ is a simple semiring. We also, by a trivial example of a $B$-semiring, show that the converse of this example need not be true in general. Let $R$ be the semiring of nonnegative reals, which is a $B$-semiring by Example 3.4. Now, by Theorem 4.4, $R[x]$ is not a $B$-semiring since $R$ is zerosumfree.

## References


[1] M. S. Abdolyousefi, H. Chen, Elementary matrix reduction over $J$-stable rings, Comm. Algebra(2017), 45:5, 1983-1995.

[2] H. Bass, $K$-theory and stable algebra, Publ. IHES 22, (1964), 5-60.

[3] S. Ebrahimi Atani, Zero-divisor graph with respect to ideals of a commutative semiring, Glas. Mat. Ser. III 43(63), (2008), 309-320.

[4] D. Estes and J. Ohm, Stable range in commutative rings, Jour. of Alg. 7, (1967), 343-362.

[5] J. S. Golan, The theory of semirings (with applications in mathematics and theoretical computer science), Pitman Monographs & Surveys in Pure and Appl. Math., vol. 54, Longman Scientific & Tech., Essex, (1992).

[6] M. Moore, A. Steger, Some results on completability in commutative rings, Pacific Journal of Mathematics Vol. 37, (1971), 453-460.

[7] A. M. Rahimi, Some Results on Stable Range in Commutative Rings, PhD dissertation, University of Texas at Arlington, (1993).

[8] A. M. Rahimi, A class of commutative semirings with stable range 2, Quaestiones Mathematicae, to appear.

[9] A. M. Rahimi, Some improved results on $B$-rings,, Missouri J. Math. Sci. 9 no. 3, (1997) 167-169.

[10] A. M. Rahimi, Some results on $n$-stable rings, Missouri J. Math. Sci. 15 no. 2, (2003) 129-139.

[11] M. K. Sen and M. R. Adhikari, On maximal $k$-ideals of semirings, Proc. Amer. Math. Soc, 118, (1993), 699-703.


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