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# Computation of three theorems of Srinivasa Ramanujan associated with definite integrals connected with Gauss sums

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**Abstract:** In this paper, we have analysed numerically and graphically the three doubtful theorems of Srinivasa Ramanujan without giving any analytical proofs [*Mess.Math.*, XLIV, pp. 75-86, 1915; p.75, eqns(3 and $3^{'}$); p.76, eqns(6 and $6^{'}$); pp.76-77, eqns(10 and $10^{'}$); see also Ramanujan's Papers, Published by Prism Book Pvt. Ltd. Mumbai, 2000] by using the Wolfram Mathematica and Matlab softwares. Here we have finalised that the same three theorems of Srinivasa Ramanujan edited by Hardy- Aiyar- Wilson without giving any analytical proofs [see Collected Papers of Srinivasa Ramanujan, Published by Cambridge University Press 1927; Reprinted by AMS Chelsea Publ. Co. 1962; p.59, eqns(3and $\acute{3}$); p.60, eqns(6 and $\acute{6}$); p.60, eqns(10 and $10^{'}$); p.66, after eq.(44)] are correct , by the comparative study shown in the tables.

## 1 Introduction and Preliminaries

Throughout the present paper, we shall adopt the following notation [2, p.33]

$$\mathbb{R}_{>} := \{x \in \mathbb{R} : x > 0\}, \tag{1.1}$$

where, as usual, the symbol $\mathbb{R}$ denotes the set of real numbers and the symbol $\circeq$ denotes that the corresponding equations are doubtful.

The motivation for this paper arises from the following three doubtful (erroneous/ misprint) theorem of Ramanujan. Although, all three theorems are inaccurate in the paper of Ramanujan [4] and we have to found the accuracy of all these three Ramanujan's theorems in the Collected papers edited by G.H. Hardy et.al. [3]. No analytical proofs of the following three theorems, were given by Ramanujan.

**Theorem I**: The following theorem is given by Ramanujan ([4, p.75, eqns(3 and $3^{'}$) ] see also [5, p.75, eqns(3 and $3^{'}$)]):

If

$$\Phi_1(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{\cosh(\pi x)} dx, \tag{1.2}$$

and

$$\Psi_1(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{\cosh(\pi x)} dx, \tag{1.3}$$

then

$$\Phi_1(n) \circeq \sqrt{\left\{\left(\frac{2}{n}\right)\Psi_1\left(\frac{1}{n}\right) + \Psi_1(n)\right\}}, \tag{1.4}$$

and

$$\Psi_1(n) \circeq \sqrt{\left\{\left(\frac{2}{n}\right)\Phi_1\left(\frac{1}{n}\right) - \Phi_1(n)\right\}}, \tag{1.5}$$

where $n \in \mathbb{R}_{>}$ .

**Theorem II**: The following theorem is given by Ramanujan ( [4, p.76, eqns(6 and $6^{'}$)] see also [5, p.76, eqns(6 and $6^{'}$)] ):
If

$$\Phi_2(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{\{1 + 2\cosh\left(2\pi x/\sqrt{3}\right)\}} dx, \tag{1.6}$$

and

$$\Psi_2(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{\{1 + 2\cosh\left(2\pi x/\sqrt{3}\right)\}} dx, \tag{1.7}$$

then

$$\Phi_2(n) \overset{\circ}{=} \sqrt{\left\{\left(\frac{2}{n}\right)\Psi_2\left(\frac{1}{n}\right) + \Psi_2(n)\right\}}, \tag{1.8}$$

and

$$\Psi_2(n) \overset{\circ}{=} \sqrt{\left\{\left(\frac{2}{n}\right)\Phi_2\left(\frac{1}{n}\right) - \Phi_2(n)\right\}}, \tag{1.9}$$

where $n \in \mathbb{R}_>$ .

**Theorem III**: The following theorem is given by Ramanujan ([4, p.76-77, eqns(10 and $10^{'}$)] see also [5, p.76-77, eqns(10 and $10^{'}$); see also p.84, last line] ):
If

$$\Phi_3(n) = \int_0^\infty \frac{\cos(\pi n x)}{\{-1 + \exp\left(2\pi\sqrt{x}\right)\}} dx, \tag{1.10}$$

and

$$\Psi_3(n) = \frac{1}{2\pi n} + \int_0^\infty \frac{\sin(\pi n x)}{\{-1 + \exp\left(2\pi\sqrt{x}\right)\}} dx, \tag{1.11}$$

then

$$\Phi_3(n) \overset{\circ}{=} \frac{1}{n}\sqrt{\left\{\left(\frac{2}{n}\right)\Psi_3\left(\frac{1}{n}\right) - \Psi_3(n)\right\}}, \tag{1.12}$$

and

$$\Psi_3(n) \overset{\circ}{=} \frac{1}{n}\sqrt{\left\{\left(\frac{2}{n}\right)\Phi_3\left(\frac{1}{n}\right) + \Phi_3(n)\right\}}, \tag{1.13}$$

where $n \in \mathbb{R}_>$ .

Inspired by some of these aforesaid enquiries of the three Ramanujan's theorems [4]. We have found computational work of the three Ramanujan's theorems [4, p.75, eqns(3 and $3^{'}$)], [4, p.76, eqns(6 and $6^{'}$)] and [4, p.76-77, eqns(10 and $10^{'}$)] in terms of numerical values by using Wolfram Mathematica software and each of the three theorems are plotted by using Matlab software. Taking $n = 1/6,\ 1/4,\ 2/5,\ 1/2,\ ,2/3,\ \sqrt{3}/\sqrt{5},\ 1,\ \sqrt{5}/\sqrt{3},\ 3/2,\ 2,\ 5/2,\ 4,\ 6, ...$ as a scale on $x - axis$ with corresponding values of $\Phi_1(n),\ \Psi_1(n);\ \Phi_2(n),\ \Psi_2(n)$ given in the second and third columns of the theorems (I,II) and values $\Phi_3(n),\ \Psi_3(n)$ given in the fourth and fifth column of the theorem III as a scale mentioned on Y-axis.
The plan of this paper as follows: Comparative study of Ramanujan's theorem I, theorem II, theorem III are contained in the *Sections 2-4* and its Corrected forms theorem IV, theorem V, theorem VI are presented in the *Section 5*. Moreover, we have verified theorem IV; theorem V; theorem VI via Wolfram Mathematica software shown in the *sections 6-8* and some concluding remarks are discussed in *Section 9*.

## 2 Comparative study of Ramanujan's theorem I (see [4, 5]) via Wolfram Mathematica software

| Comparative study of $\Phi_1(n) = \sqrt{\{(\frac{2}{n})\Psi_1(\frac{1}{n}) + \Psi_1(n)\}}$...(1.4) | | | |
|---|---|---|---|
| *Ramanujan's integral* $\Phi_1(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{\cosh(\pi x)} dx.$ *Column-I* | Numerical values of the integral $\Phi_1(n)$ by using Wolfram Mathematica software. *Column-II* | Numerical values of $\Phi_1(n)$ from (1.4) by using the values of $\Psi_1(n)$,$\Psi_1(\frac{1}{n})$ given in second column of the table (2.2). *Column-III* | Remark for Ramanujan's theorem (1.4). *Column-IV* |
| $\Phi_1(1) = \int_0^\infty \frac{\cos(\pi x^2)}{\cosh(\pi x)} dx$ | 0.353553 | 0.6628280... | inaccuracy |
| $\Phi_1(2) = \int_0^\infty \frac{\cos(2\pi x^2)}{\cosh(\pi x)} dx$ | 0.270598 | 0.5201903... | inaccuracy |
| $\Phi_1(4) = \int_0^\infty \frac{\cos(4\pi x^2)}{\cosh(\pi x)} dx$ | 0.19352 | 0.421003... | inaccuracy |
| $\Phi_1(6) = \int_0^\infty \frac{\cos(6\pi x^2)}{\cosh(\pi x)} dx$ | 0.15623 | 0.376978... | inaccuracy |
| $\Phi_1(\frac{1}{2}) = \int_0^\infty \frac{\cos(\frac{\pi x^2}{2})}{\cosh(\pi x)} dx$ | 0.42388 | 0.854658... | inaccuracy |
| $\Phi_1(\frac{1}{4}) = \int_0^\infty \frac{\cos(\frac{\pi x^2}{4})}{\cosh(\pi x)} dx$ | 0.468769 | 1.0872847... | inaccuracy |
| $\Phi_1(\frac{1}{6}) = \int_0^\infty \frac{\cos(\frac{\pi x^2}{6})}{\cosh(\pi x)} dx$ | 0.483342 | 1.2376550... | inaccuracy |
| $\Phi_1(\frac{2}{3}) = \int_0^\infty \frac{\cos(\frac{2\pi x^2}{3})}{\cosh(\pi x)} dx$ | 0.397336 | 0.7693698... | inaccuracy |
| $\Phi_1(\frac{3}{2}) = \int_0^\infty \frac{\cos(\frac{3\pi x^2}{2})}{\cosh(\pi x)} dx$ | 0.305333 | 0.5734343... | inaccuracy |
| $\Phi_1(\frac{2}{5}) = \int_0^\infty \frac{\cos(\frac{2\pi x^2}{5})}{\cosh(\pi x)} dx$ | 0.441331 | 0.9257245... | inaccuracy |
| $\Phi_1(\frac{5}{2}) = \int_0^\infty \frac{\cos(\frac{5\pi x^2}{2})}{\cosh(\pi x)} dx$ | 0.244361 | 0.4841888... | inaccuracy |
| $\Phi_1(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{\cosh(\pi x)} dx$ | 0.323448 | 0.6045059... | inaccuracy |
| $\Phi_1(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{\cosh(\pi x)} dx$ | 0.381883 | 0.7279927... | inaccuracy |
| Table 2.1 of Theorem I. | | | |

*Table 2.1* contains numerical values of Ramanujan's integral $\Phi_1(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{\cosh(\pi x)} dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (1.4)

| Comparative study of $\Psi_1(n) = \sqrt{\{(\frac{2}{n})\Phi_1(\frac{1}{n}) - \Phi_1(n)\}}$...(1.5) | | | |
|---|---|---|---|
| *Ramanujan's integral* $\Psi_1(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{\cosh(\pi x)} dx.$ *Column-I* | Numerical values of the integral $\Psi_1(n)$ by using the Wolfram Mathematica software. *Column-II* | Numerical values of $\Psi_1(n)$ from (1.5) by using the values of $\Phi_1(n)$,$\Phi_1(\frac{1}{n})$ given in second column of the table (2.1). *Column-III* | Remark for Ramanujan's theorem (1.5). *Column-IV* |
| $\Psi_1(1) = \int_0^\infty \frac{\sin(\pi x^2)}{\cosh(\pi x)} dx$ | 0.146447 | 0.5946032... | inaccuracy |
| $\Psi_1(2) = \int_0^\infty \frac{\sin(2\pi x^2)}{\cosh(\pi x)} dx$ | 0.153281 | 0.3915124... | inaccuracy |
| $\Psi_1(4) = \int_0^\infty \frac{\sin(4\pi x^2)}{\cosh(\pi x)} dx$ | 0.13795 | 0.202149... | inaccuracy |
| $\Psi_1(6) = \int_0^\infty \frac{\sin(6\pi x^2)}{\cosh(\pi x)} dx$ | 0.122828 | 0.0698856... | inaccuracy |
| $\Psi_1(\frac{1}{2}) = \int_0^\infty \frac{\sin(\frac{\pi x^2}{2})}{\cosh(\pi x)} dx$ | 0.117317 | 0.8114875... | inaccuracy |
| $\Psi_1(\frac{1}{4}) = \int_0^\infty \frac{\sin(\frac{\pi x^2}{4})}{\cosh(\pi x)} dx$ | 0.0785882 | 1.03893743... | inaccuracy |
| $\Psi_1(\frac{1}{6}) = \int_0^\infty \frac{\sin(\frac{\pi x^2}{6})}{\cosh(\pi x)} dx$ | 0.0578541 | 1.17958382... | inaccuracy |

| | | | |
|---|---|---|---|
| $\Psi_1(\frac{2}{3}) = \int_0^\infty \frac{\sin(\frac{2\pi x^2}{3})}{\cosh(\pi x)}dx$ | 0.131517 | 0.7201826... | inaccuracy |
| $\Psi_1(\frac{3}{2}) = \int_0^\infty \frac{\sin(\frac{3\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.153471 | 0.4737597... | inaccuracy |
| $\Psi_1(\frac{2}{5}) = \int_0^\infty \frac{\sin(\frac{2\pi x^2}{5})}{\cosh(\pi x)}dx$ | 0.105076 | 0.8834443... | inaccuracy |
| $\Psi_1(\frac{5}{2}) = \int_0^\infty \frac{\sin(\frac{5\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.150378 | 0.3297025... | inaccuracy |
| $\Psi_1(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\sin(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{\cosh(\pi x)}dx$ | 0.151868 | 0.5178441... | inaccuracy |
| $\Psi_1(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{sin(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{cosh(\pi x)}dx$ | 0.137852 | 0.6732430... | inaccuracy |

Table 2.2 of Theorem I.

*Table 2.2* contains numerical values of Ramanujan's integral $\Psi_1(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{\cosh(\pi x)}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (1.5).

$\Phi_1(n)$ of Ramanujan's theorem I , represented by the graphical Method:

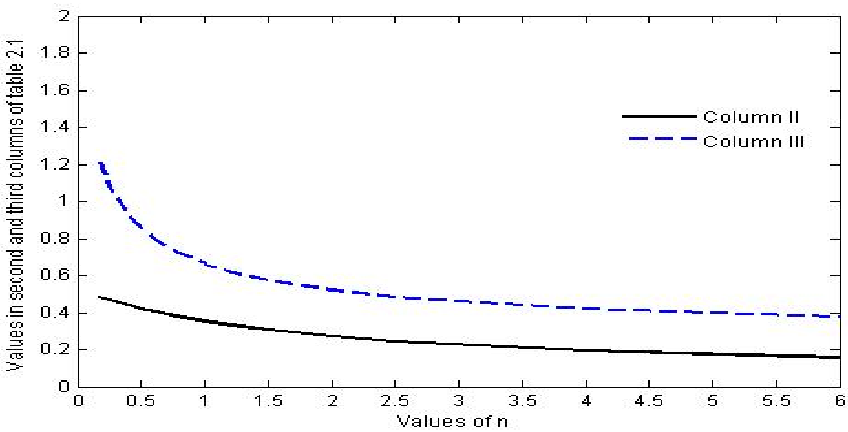


**Figure 1.** Graph of the table 1

$\Psi_1(n)$ of Ramanujan's theorem I, represented by the graphical Method:

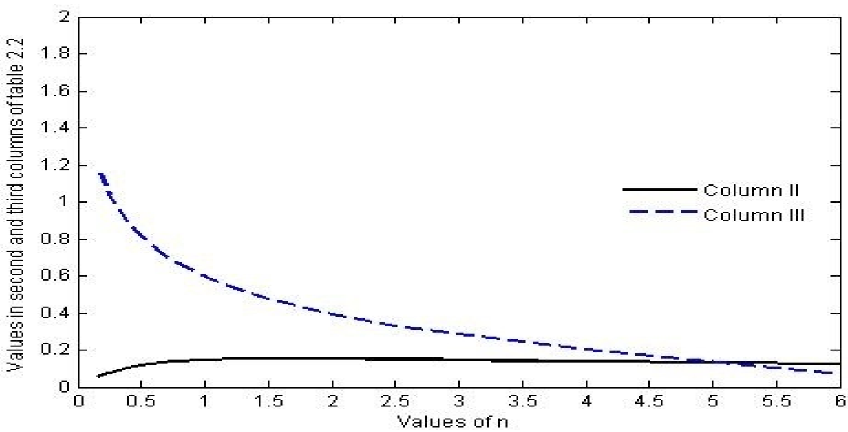


**Figure 2.** Graph of the table 2

## 3 Comparative study of Ramanujan's theorem II (see [4, 5]) via Wolfram Mathematica software

| Comparative study of $\Phi_2(n) = \sqrt{\{\left(\frac{2}{n}\right)\Psi_2\left(\frac{1}{n}\right) + \Psi_2(n)\}}$...(1.8) | | | |
|---|---|---|---|
| *Ramanujan's integral* $\Phi_2(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx.$ *Column-I* | Numerical values of the integral $\Phi_2(n)$ by using the Wolfram Mathematica software. *Column-II* | Numerical values of $\Phi_2(n)$ from (1.8) by using the values of $\Psi_2(n)$,$\Psi_2(\frac{1}{n})$ given in second column of the table (3.2). *Column-III* | Remark for Ramanujan's theorem (1.8). *Column-IV* |
| $\Phi_2(1) = \int_0^\infty \frac{\cos(\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.12059 | 0.387105... | inaccuracy |
| $\Phi_2(2) = \int_0^\infty \frac{\cos(2\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.0915064 | 0.30250008... | inaccuracy |
| $\Phi_2(4) = \int_0^\infty \frac{\cos(4\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.0647048 | 0.24408174... | inaccuracy |
| $\Phi_2(6) = \int_0^\infty \frac{\cos(6\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right))} dx$ | 0.0520162 | 0.21833124... | inaccuracy |
| $\Phi_2(\frac{1}{2}) = \int_0^\infty \frac{\cos(\frac{\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.144338 | 1.4669434... | inaccuracy |
| $\Phi_2(\frac{1}{4}) = \int_0^\infty \frac{\cos(\frac{\pi x^2}{4})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.158248 | 0.6342822... | inaccuracy |
| $\Phi_2(\frac{1}{6}) = \int_0^\infty \frac{\cos(\frac{\pi x^2}{6})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.162372 | 0.7201181... | inaccuracy |
| $\Phi_2(\frac{2}{3}) = \int_0^\infty \frac{\cos(\frac{2\pi x^2}{3})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.135566 | 0.4500416... | inaccuracy |
| $\Phi_2(\frac{3}{2}) = \int_0^\infty \frac{\cos(\frac{3\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.10372 | 0.334057... | inaccuracy |
| $\Phi_2(\frac{2}{5}) = \int_0^\infty \frac{\cos(\frac{2\pi x^2}{5})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.149914 | 0.5413050... | inaccuracy |
| $\Phi_2(\frac{5}{2}) = \int_0^\infty \frac{\cos(\frac{5\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.0823142 | 0.28122937... | inaccuracy |
| $\Phi_2(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.110082 | 0.3525009... | inaccuracy |
| $\Phi_2(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.130338 | 0.4256609... | inaccuracy |
| Table 3.1 of Theorem II. | | | |

*Table 3.1* contains numerical values of Ramanujan's integral $\Phi_2(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (1.8).

| Comparative study of $\Psi_2(n) = \sqrt{\{\left(\frac{2}{n}\right)\Phi_2\left(\frac{1}{n}\right) - \Phi_2(n)\}}$...(1.9) | | | |
|---|---|---|---|
| *Ramanujan's integral* $\Psi_2(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx.$ *Column-I* | Numerical values of the integral $\Psi_2(n)$ by using the Wolfram Mathematica software. *Column-II* | Numerical values of $\Psi_2(n)$ from (1.9) by using the values of $\Phi_2(n)$,$\Phi_2(\frac{1}{n})$ given in second the table (3.1). *Column-III* | Remark for Ramanujan's theorem (1.9). *Column-IV* |
| $\Psi_2(1) = \int_0^\infty \frac{\sin(\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)} dx$ | 0.0499502 | 0.34726070... | inaccuracy |

| | | | |
|---|---|---|---|
| $\Psi_2(2) = \int_0^\infty \frac{\sin(2\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0528312 | 0.22985125... | inaccuracy |
| $\Psi_2(4) = \int_0^\infty \frac{\sin(4\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0471937 | 0.12007997... | inaccuracy |
| $\Psi_2(6) = \int_0^\infty \frac{\sin(6\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0417294 | 0.04591078... | inaccuracy |
| $\Psi_2(\frac{1}{2}) = \int_0^\infty \frac{\sin(\frac{\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0386751 | 0.47083712... | inaccuracy |
| $\Psi_2(\frac{1}{4}) = \int_0^\infty \frac{\sin(\frac{\pi x^2}{4})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0247644 | 0.59949178... | inaccuracy |
| $\Psi_2(\frac{1}{6}) = \int_0^\infty \frac{\sin(\frac{\pi x^2}{6})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0178174 | 0.67957516... | inaccuracy |
| $\Psi_2(\frac{2}{3}) = \int_0^\infty \frac{\sin(\frac{2\pi x^2}{3})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0440817 | 0.41903937... | inaccuracy |
| $\Psi_2(\frac{3}{2}) = \int_0^\infty \frac{\sin(\frac{3\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0528186 | 0.27755119... | inaccuracy |
| $\Psi_2(\frac{2}{5}) = \int_0^\infty \frac{\sin(\frac{2\pi x^2}{5})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0341462 | 0.51152419... | inaccuracy |
| $\Psi_2(\frac{5}{2}) = \int_0^\infty \frac{\sin(\frac{5\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.051773 | 0.1939510... | inaccuracy |
| $\Psi_2(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\sin(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{1+2cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0521457 | 0.30304580... | inaccuracy |
| $\Psi_2(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{\sin(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0465476 | 0.39229134... | inaccuracy |

Table 3.2 of Theorem II.

*Table 3.2* contains numerical values of Ramanujan's integral $\Psi_2(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (1.9).

$\Phi_2(n)$ of Ramanujan's theorem II , represented by the graphical Method:

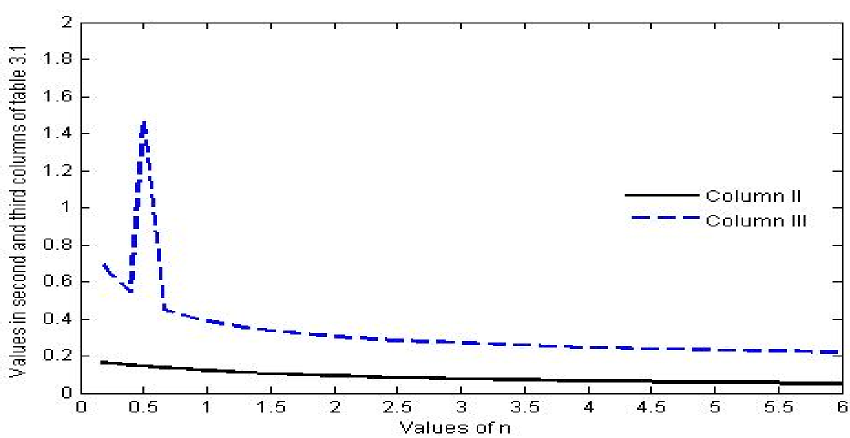


**Figure 3.** Graph of the table 3

$\Psi_2(n)$ of Ramanujan's theorem II, represented by the graphical Method:

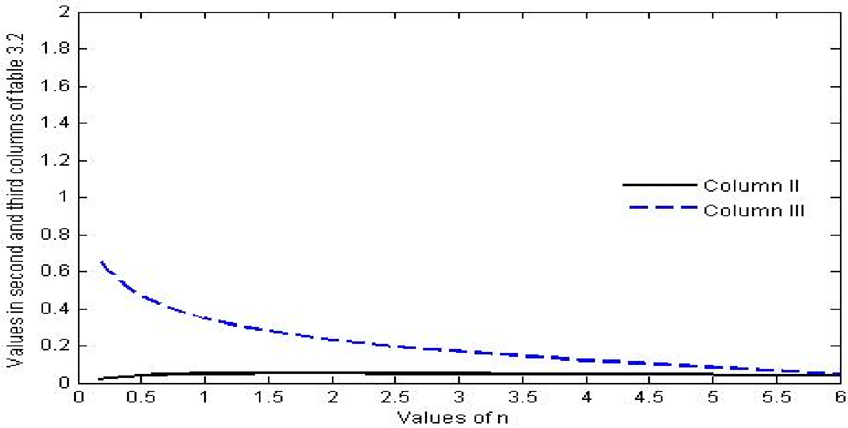


**Figure 4.** Graph of the table 4

## 4 Comparative study of Ramanujan's theorem III (see [4, 5]) via Wolfram Mathematica software

| Comparative study of $\Phi_3(n) = \sqrt{\{(\frac{2}{n})\Psi_3(\frac{1}{n}) - \Psi_3(n)\}}$...(1.12) | | | |
|---|---|---|---|
| $Ramanujan's\ integral$ $\Phi_3(n) = \int_0^\infty \frac{\cos(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx$. *Column-I* | Numerical values of the integral $\Phi_3(n)$ by using the Wolfram Mathematica software. *Column-II* | Numerical values of $\Phi_3(n)$ from (1.12) by using the values of $\Psi_3(n)$,$\Psi_3(\frac{1}{n})$ given in fourth column of the table (4.2). *Column-III* | Remark for Ramanujan's theorem (1.12). *Column-IV* |
| $\Phi_3(1) = \int_0^\infty \frac{\cos(\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0732233 | 0.42044826... | inaccuracy |
| $\Phi_3(2) = \int_0^\infty \frac{\cos(2\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0625 | 0.23837... | inaccuracy |
| $\Phi_3(4) = \int_0^\infty \frac{\cos(4\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0495558 | 0.12682327... | inaccuracy |
| $\Phi_3(6) = \int_0^\infty \frac{\cos(6\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0421653 | 0.08653827... | inaccuracy |
| $\Phi_3(\frac{1}{2}) = \int_0^\infty \frac{\cos(\frac{\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0795775 | 0.5641900... | inaccuracy |
| $\Phi_3(\frac{1}{4}) = \int_0^\infty \frac{\cos(\frac{\pi x}{4})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0822095 | 1.4431151$i$... | inaccuracy |
| $\Phi_3(\frac{1}{6}) = \int_0^\infty \frac{\cos(\frac{\pi x}{6})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0828111 | 3.5879535$i$... | inaccuracy |
| $\Phi_3(\frac{2}{3}) = \int_0^\infty \frac{\cos(\frac{2\pi x}{3})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0774899 | 0.53781552... | inaccuracy |
| $\Phi_3(\frac{3}{2}) = \int_0^\infty \frac{\cos(\frac{3\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0673963 | 0.30531551... | inaccuracy |
| $\Phi_3(\frac{2}{5}) = \int_0^\infty \frac{\cos(\frac{2\pi x}{5})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0807373 | 0.42728743... | inaccuracy |
| $\Phi_3(\frac{5}{2}) = \int_0^\infty \frac{\cos(\frac{5\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0584207 | 0.19536590... | inaccuracy |
| $\Phi_3(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{5}\pi x}{\sqrt{3}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0697198 | 0.42310970... | inaccuracy |
| $\Phi_3(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{3}\pi x}{\sqrt{5}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.076099 | 0.49786670... | inaccuracy |

Table 4.1 of Theorem III.

*Table 4.1* contains numerical values of Ramanujan's integral $\Phi_3(n) = \int_0^\infty \frac{\cos(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (1.12).

| Comparative study of $\Psi_3(n) = \frac{1}{n}\sqrt{\left\{\left(\frac{2}{n}\right)\Phi_3\left(\frac{1}{n}\right) + \Phi_3(n)\right\}}$ ...(1.13) | | | | |
|---|---|---|---|---|
| Auxiliary integral $\Psi_3^*(n) = \int_0^\infty \frac{sin(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx.$ *Column-I* | Numerical values of the auxiliary integral $\Psi_3^*(n)$ by using the Wolfram Mathematica software. *Column-II* | $Ramanujan's\ integral$ $\Psi_3(n) = \Psi_3^*(n) + \frac{1}{2\pi n}.$ *Column-III* | Numerical values of the Ramanujan integral $\Psi_3(n)$. *Column-IV* | Numerical values of $\Psi_3(n)$ from (1.13) by using the values of $\Phi_3(n)$,$\Phi_3(\frac{1}{n})$ given in second column of the table (4.1). *Column-V* |
| $\Psi_3^*(1) = \int_0^\infty \frac{\sin(\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0176218 | $\Psi_3(1) = \Psi_3^*(1) + \frac{1}{2\pi}$ | 0.17677674.. | 0.46868955... |
| $\Psi_3^*(2) = \int_0^\infty \frac{\sin(2\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0227113 | $\Psi_3(2) = \Psi_3^*(2) + \frac{1}{4\pi}$ | 0.10228877.. | 0.18846584... |
| $\Psi_3^*(4) = \int_0^\infty \frac{\sin(4\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0242998 | $\Psi_3(4) = \Psi_3^*(4) + \frac{1}{8\pi}$ | 0.06408853.. | 0.07527472... |
| $\Psi_3^*(6) = \int_0^\infty \frac{\sin(6\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0236079 | $\Psi_3(6) = \Psi_3^*(6) + \frac{1}{12\pi}$ | 0.05013372.. | 0.04402303... |
| $\Psi_3^*(\frac{1}{2}) = \int_0^\infty \frac{\sin(\frac{\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0112676 | $\Psi_3(\frac{1}{2}) = \Psi_3^*(\frac{1}{2}) + \frac{1}{\pi}$ | 0.32957748.. | 1.14817681... |
| $\Psi_3^*(\frac{1}{4}) = \int_0^\infty \frac{\sin(\frac{\pi x}{4})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0062498 | $\Psi_3(\frac{1}{4}) = \Psi_3^*(\frac{1}{4}) + \frac{2}{\pi}$ | 0.64286962.. | 2.76739849... |
| $\Psi_3^*(\frac{1}{6}) = \int_0^\infty \frac{\sin(\frac{\pi x}{6})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0042697 | $\Psi_3(\frac{1}{6}) = \Psi_3^*(\frac{1}{6}) + \frac{3}{\pi}$ | 0.95919942.. | 4.60397754... |
| $\Psi_3^*(\frac{2}{3}) = \int_0^\infty \frac{\sin(\frac{2\pi x}{3})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0138583 | $\Psi_3(\frac{2}{3}) = \Psi_3^*(\frac{2}{3}) + \frac{3}{4\pi}$ | 0.25259071.. | 0.79327000... |
| $\Psi_3^*(\frac{3}{2}) = \int_0^\infty \frac{\sin(\frac{3\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0209448 | $\Psi_3(\frac{3}{2}) = \Psi_3^*(\frac{3}{2}) + \frac{1}{3\pi}$ | 0.12704809.. | 0.27545208... |
| $\Psi_3^*(\frac{2}{5}) = \int_0^\infty \frac{\sin(\frac{2\pi x}{5})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0236442 | $\Psi_3(\frac{2}{5}) = \Psi_3^*(\frac{2}{5}) + \frac{5}{4\pi}$ | 0.40731895.. | 1.52651727... |
| $\Psi_3^*(\frac{5}{2}) = \int_0^\infty \frac{\sin(\frac{5\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0094316 | $\Psi_3(\frac{5}{2}) = \Psi_3^*(\frac{5}{2}) + \frac{1}{5\pi}$ | 0.08730617.. | 0.14029143... |
| $\Psi_3^*(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\sin(\frac{\sqrt{5}\pi x}{\sqrt{3}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0198071 | $\Psi_3(\frac{\sqrt{5}}{\sqrt{3}}) = \Psi_3^*(\frac{\sqrt{5}}{\sqrt{3}}) + \frac{\sqrt{3}}{2\pi\sqrt{5}}$ | 0.14308798.. | 0.33551023... |
| $\Psi_3^*(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{\sin(\frac{\sqrt{3}\pi x}{\sqrt{5}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0152607 | $\Psi_3(\frac{\sqrt{3}}{\sqrt{5}}) = \Psi_3^*(\frac{\sqrt{3}}{\sqrt{5}}) + \frac{\sqrt{5}}{2\pi\sqrt{3}}$ | 0.22072884.. | 0.65334364... |
| Table 4.2 of Theorem III. | | | | |

*Table 4.2* contains numerical values of Ramanujan's integral $\Psi_3(n) = \int_0^\infty \frac{\sin(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx + \frac{1}{2n\pi}$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (1.13).

$\Phi_3(n)$ of Ramanujan's theorem III , represented by the graphical Method:

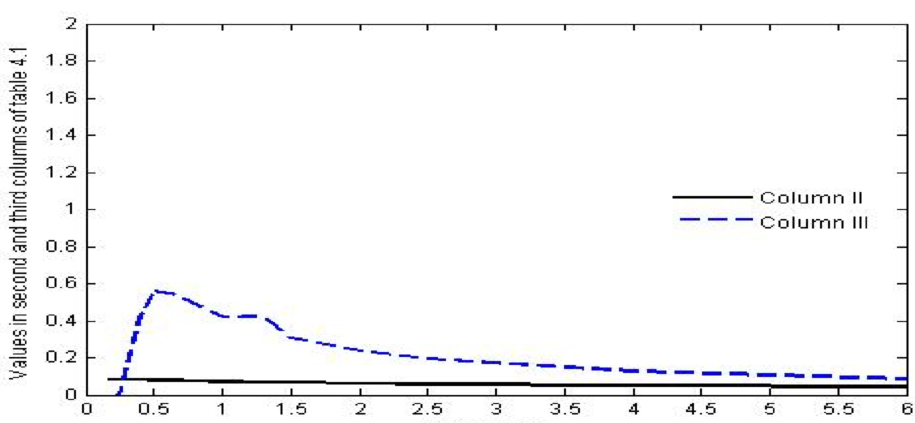


**Figure 5.** Graph of the table 5

$\Psi_3(n)$ of Ramanujan's theorem III, represented by the graphical Method:

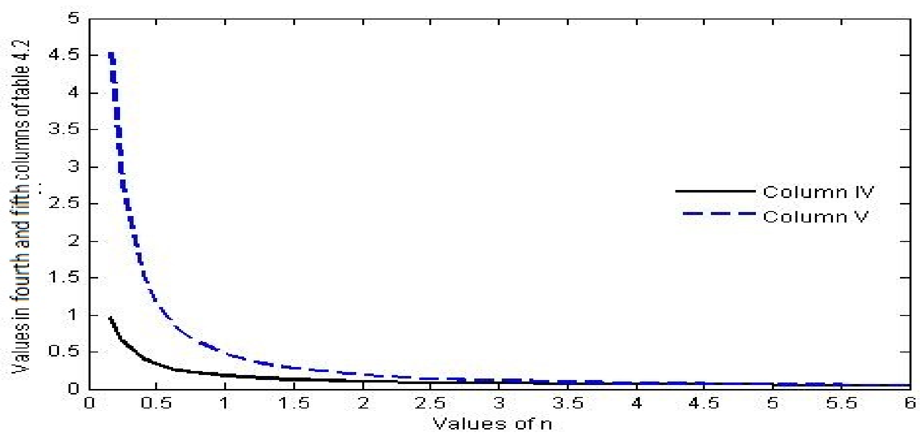


**Figure 6.** Graph of the table 6

## 5 Three modified theorems of Ramanujan [3, AMS.Chelsea Publ.1962]

No analytical proofs of the following three theorems were given in [3].
**Theorem IV**: [3, p.59, eq.(3)and $(\acute{3})$]
If

$$\Phi_1(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{\cosh(\pi x)} dx, \tag{5.1}$$

and

$$\Psi_1(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{\cosh(\pi x)} dx, \tag{5.2}$$

then

$$\Phi_1(n) = \sqrt{\left(\frac{2}{n}\right)}\, \Psi_1\left(\frac{1}{n}\right) + \Psi_1(n), \tag{5.3}$$

and

$$\Psi_1(n) = \sqrt{\left(\frac{2}{n}\right)}\, \Phi_1\left(\frac{1}{n}\right) - \Phi_1(n), \tag{5.4}$$

where $n \in \mathbb{R}_>$ .

**Theorem V**:[3, p.60, eq.(6) and $(\acute{6})$]
If

$$\Phi_2(n) = \int_0^\infty \frac{\cos(\pi n x^2)}{\{1 + 2\cosh\left(2\pi x/\sqrt{3}\right)\}} dx, \tag{5.5}$$

and

$$\Psi_2(n) = \int_0^\infty \frac{\sin(\pi n x^2)}{\{1 + 2\cosh\left(2\pi x/\sqrt{3}\right)\}} dx, \tag{5.6}$$

then

$$\Phi_2(n) = \sqrt{\left(\frac{2}{n}\right)}\, \Psi_2\left(\frac{1}{n}\right) + \Psi_2(n), \tag{5.7}$$

and

$$\Psi_2(n) = \sqrt{\left(\frac{2}{n}\right)}\, \Phi_2\left(\frac{1}{n}\right) - \Phi_2(n), \tag{5.8}$$

where $n \in \mathbb{R}_>$ .

**Theorem VI**: [3, p.60, eq.(10)and $(10^{'})$; p.66, after eq.(44)], see also [1]
If

$$\Phi_3(n) = \int_0^\infty \frac{\cos(\pi n x)}{\{-1 + \exp\left(2\pi\sqrt{x}\right)\}} dx, \tag{5.9}$$

and

$$\Psi_3(n) = \frac{1}{2\pi n} + \int_0^\infty \frac{\sin(\pi n x)}{\{-1 + \exp\left(2\pi\sqrt{x}\right)\}} dx, \tag{5.10}$$

then

$$\Phi_3(n) = \frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\, \Psi_3\left(\frac{1}{n}\right) - \Psi_3(n), \tag{5.11}$$

and

$$\Psi_3(n) = \frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\, \Phi_3\left(\frac{1}{n}\right) + \Phi_3(n), \tag{5.12}$$

where $n \in \mathbb{R}_>$ .

In a paper of Ramanujan [4], it is said that $n$ is a rational number and in IV notebook of Berndt [1], it is said that $n$ is any real number. Now by observing the tables (6.1),(6.2),(7.1),(7.2), (8.1),(8.2), we decided that $n$ is any positive rational and irrational numbers associated with the theorems of Ramanujan.

## 6 Verification of Ramanujan theorem IV ([3]) via Wolfram Mathematica software

| Comparative study of $\Phi_1(n)=\sqrt{\left(\frac{2}{n}\right)}\,\Psi_1\left(\frac{1}{n}\right)+\Psi_1(n)$... (5.3) | | |
|---|---|---|
| *Ramanujan's integral* $\Phi_1(n)=\int_0^\infty \frac{\cos(\pi n x^2)}{\cosh(\pi x)}dx$ | Numerical values of the integral $\Phi_1(n)$ by using Wolfram Mathematica software | Numerical values of $\Phi_1(n)$ from (5.3) by using the values of $\Psi_1(n)$,$\Psi_1(\frac{1}{n})$ given in second column of the table (6.2) |
| $\Phi_1(1)=\int_0^\infty \frac{\cos(\pi x^2)}{\cosh(\pi x)}dx$ | 0.353553 | 0.3535543 ... |
| $\Phi_1(2)=\int_0^\infty \frac{\cos(2\pi x^2)}{\cosh(\pi x)}dx$ | 0.270598 | 0.2705980... |
| $\Phi_1(4)=\int_0^\infty \frac{\cos(4\pi x^2)}{\cosh(\pi x)}dx$ | 0.19352 | 0.193520... |
| $\Phi_1(6)=\int_0^\infty \frac{\cos(6\pi x^2)}{\cosh(\pi x)}dx$ | 0.15623 | 0.156230... |
| $\Phi_1(\frac{1}{2})=\int_0^\infty \frac{\cos(\frac{\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.42388 | 0.423935... |
| $\Phi_1(\frac{1}{4})=\int_0^\infty \frac{\cos(\frac{\pi x^2}{4})}{\cosh(\pi x)}dx$ | 0.468769 | 0.4687697... |
| $\Phi_1(\frac{1}{6})=\int_0^\infty \frac{\cos(\frac{\pi x^2}{6})}{\cosh(\pi x)}dx$ | 0.483342 | 0.4833427... |
| $\Phi_1(\frac{2}{3})=\int_0^\infty \frac{\cos(\frac{2\pi x^2}{3})}{\cosh(\pi x)}dx$ | 0.397336 | 0.3973365... |
| $\Phi_1(\frac{3}{2})=\int_0^\infty \frac{\cos(\frac{3\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.305333 | 0.3053337... |
| $\Phi_1(\frac{2}{5})=\int_0^\infty \frac{\cos(\frac{2\pi x^2}{5})}{\cosh(\pi x)}dx$ | 0.441331 | 0.4413314... |
| $\Phi_1(\frac{5}{2})=\int_0^\infty \frac{\cos(\frac{5\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.244361 | 0.2443608... |
| $\Phi_1(\frac{\sqrt{5}}{\sqrt{3}})=\int_0^\infty \frac{\cos(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{\cosh(\pi x)}dx$ | 0.323448 | 0.3234476... |
| $\Phi_1(\frac{\sqrt{3}}{\sqrt{5}})=\int_0^\infty \frac{\cos(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{\cosh(\pi x)}dx$ | 0.381883 | 0.3818821... |

Table 6.1 of Theorem IV.

*Table 6.1* contains numerical values of Ramanujan's integral $\Phi_1(n)=\int_0^\infty \frac{\cos(\pi n x^2)}{\cosh(\pi x)}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (5.3).

| Comparative study of $\Psi_1(n)=\sqrt{\left(\frac{2}{n}\right)}\,\Phi_1\left(\frac{1}{n}\right)-\Phi_1(n)$...(5.4) | | |
|---|---|---|
| *Ramanujan's integral* $\Psi_1(n)=\int_0^\infty \frac{\sin(\pi n x^2)}{\cosh(\pi x)}dx$ | Numerical values of the integral $\Psi_1(n)$ by using the Wolfram Mathematica software | Numerical values of $\Psi_1(n)$ from (5.4) by using the values of $\Phi_1(n)$,$\Phi_1(\frac{1}{n})$ given in second column of the table (6.1) |
| $\Psi_1(1)=\int_0^\infty \frac{\sin(\pi x^2)}{\cosh(\pi x)}dx$ | 0.146447 | 0.1464464... |
| $\Psi_1(2)=\int_0^\infty \frac{\sin(2\pi x^2)}{\cosh(\pi x)}dx$ | 0.153281 | 0.1532820... |
| $\Psi_1(4)=\int_0^\infty \frac{\sin(4\pi x^2)}{\cosh(\pi x)}dx$ | 0.13795 | 0.137949... |
| $\Psi_1(6)=\int_0^\infty \frac{\sin(6\pi x^2)}{\cosh(\pi x)}dx$ | 0.122828 | 0.1228276... |
| $\Psi_1(\frac{1}{2})=\int_0^\infty \frac{\sin(\frac{\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.117317 | 0.1173160... |
| $\Psi_1(\frac{1}{4})=\int_0^\infty \frac{\sin(\frac{\pi x^2}{4})}{\cosh(\pi x)}dx$ | 0.0785882 | 0.07858821... |
| $\Psi_1(\frac{1}{6})=\int_0^\infty \frac{\sin(\frac{\pi x^2}{6})}{\cosh(\pi x)}dx$ | 0.0578541 | 0.05785459... |
| $\Psi_1(\frac{2}{3})=\int_0^\infty \frac{\sin(\frac{2\pi x^2}{3})}{\cosh(\pi x)}dx$ | 0.131517 | 0.1315162... |

| | | |
|---|---|---|
| $\Psi_1(\frac{3}{2})=\int_0^\infty \frac{\sin(\frac{3\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.153471 | 0.1534710... |
| $\Psi_1(\frac{2}{5})=\int_0^\infty \frac{\sin(\frac{2\pi x^2}{5})}{\cosh(\pi x)}dx$ | 0.105076 | 0.1050768... |
| $\Psi_1(\frac{5}{2})=\int_0^\infty \frac{\sin(\frac{5\pi x^2}{2})}{\cosh(\pi x)}dx$ | 0.150378 | 0.1503774... |
| $\Psi_1(\frac{\sqrt{5}}{\sqrt{3}})=\int_0^\infty \frac{\sin(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{\cosh(\pi x)}dx$ | 0.151868 | 0.1518687... |
| $\Psi_1(\frac{\sqrt{3}}{\sqrt{5}})=\int_0^\infty \frac{\sin(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{\cosh(\pi x)}dx$ | 0.137852 | 0.1378516... |

Table 6.2 of Theorem IV.

*Table 6.2* contains numerical values of Ramanujan's integral $\Psi_1(n)=\int_0^\infty \frac{\sin(\pi n x^2)}{\cosh(\pi x)}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (5.4).

## 7 Verification of Ramanujan theorem V (see [3]) via Wolfram Mathematica software

| Comparative study of $\Phi_2(n)=\sqrt{\left(\frac{2}{n}\right)}\,\Psi_2\left(\frac{1}{n}\right)+\Psi_2(n)$...(5.7) | | |
|---|---|---|
| $Ramanujan's\ integral$ $\Phi_2(n)=\int_0^\infty \frac{\cos(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | Numerical values of the integral $\Phi_2(n)$ by using Wolfram Mathematica software | Numerical values of $\Phi_2(n)$ from (5.7) by using the values of $\Psi_2(n)$,$\Psi_2(\frac{1}{n})$ given in the second column of table (7.2) |
| $\Phi_2(1)=\int_0^\infty \frac{\cos(\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.12059 | 0.120590 ... |
| $\Phi_2(2)=\int_0^\infty \frac{\cos(2\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0915064 | 0.09150630... |
| $\Phi_2(4)=\int_0^\infty \frac{\cos(4\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0647048 | 0.06470477... |
| $\Phi_2(6)=\int_0^\infty \frac{\cos(6\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right))}dx$ | 0.0520162 | 0.05201628... |
| $\Phi_2(\frac{1}{2})=\int_0^\infty \frac{\cos(\frac{\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.144338 | 0.1443375... |
| $\Phi_2(\frac{1}{4})=\int_0^\infty \frac{\cos(\frac{\pi x^2}{4})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.158248 | 0.1582483... |
| $\Phi_2(\frac{1}{6})=\int_0^\infty \frac{\cos(\frac{\pi x^2}{6})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.162372 | 0.1623722... |
| $\Phi_2(\frac{2}{3})=\int_0^\infty \frac{\cos(\frac{2\pi x^2}{3})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.135566 | 0.1355661 ... |
| $\Phi_2(\frac{3}{2})=\int_0^\infty \frac{\cos(\frac{3\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.10372 | 0.103719... |
| $\Phi_2(\frac{2}{5})=\int_0^\infty \frac{\cos(\frac{2\pi x^2}{5})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.149914 | 0.1499141... |
| $\Phi_2(\frac{5}{2})=\int_0^\infty \frac{\cos(\frac{5\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0823142 | 0.08231429... |
| $\Phi_2(\frac{\sqrt{5}}{\sqrt{3}})=\int_0^\infty \frac{\cos(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.110082 | 0.1100819... |
| $\Phi_2(\frac{\sqrt{3}}{\sqrt{5}})=\int_0^\infty \frac{\cos(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.130338 | 0.1303382... |

Table 7.1 of Theorem V.

*Table 7.1* contains numerical values of Ramanujan's integral $\Phi_2(n)=\int_0^\infty \frac{\cos(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (5.7).

| Comparative study of $\Psi_2(n)=\sqrt{\left(\frac{2}{n}\right)}\,\Phi_2\left(\frac{1}{n}\right)-\Phi_2(n)$...(5.8) | | |
|---|---|---|
| $Ramanujan's\ integral$ $\Psi_2(n)=\int_0^\infty \frac{\sin(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | Numerical values of the integral $\Psi_2(n)$ by using Wolfram Mathematica software | Numerical values of $\Psi_2(n)$ from (5.8) by using the values of $\Phi_2(n)$,$\Phi_2(\frac{1}{n})$ given in the second column of table (7.1) |
| $\Psi_2(1)=\int_0^\infty \frac{\sin(\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0499502 | 0.04995001... |
| $\Psi_2(2)=\int_0^\infty \frac{\sin(2\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0528312 | 0.05283160... |
| $\Psi_2(4)=\int_0^\infty \frac{\sin(4\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0471937 | 0.04719343... |
| $\Psi_2(6)=\int_0^\infty \frac{\sin(6\pi x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0417294 | 0.04172931... |
| $\Psi_2(\frac{1}{2})=\int_0^\infty \frac{\sin(\frac{\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0386751 | 0.03867480... |
| $\Psi_2(\frac{1}{4})=\int_0^\infty \frac{\sin(\frac{\pi x^2}{4})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0247644 | 0.02476481 ... |
| $\Psi_2(\frac{1}{6})=\int_0^\infty \frac{\sin(\frac{\pi x^2}{6})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0178174 | 0.01781740... |
| $\Psi_2(\frac{2}{3})=\int_0^\infty \frac{\sin(\frac{2\pi x^2}{3})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0440817 | 0.04408231... |
| $\Psi_2(\frac{3}{2})=\int_0^\infty \frac{\sin(\frac{3\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0528186 | 0.05281813... |
| $\Psi_2(\frac{2}{5})=\int_0^\infty \frac{\sin(\frac{2\pi x^2}{5})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0341462 | 0.03414614... |
| $\Psi_2(\frac{5}{2})=\int_0^\infty \frac{\sin(\frac{5\pi x^2}{2})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.051773 | 0.0517729... |
| $\Psi_2(\frac{\sqrt{5}}{\sqrt{3}})=\int_0^\infty \frac{\sin(\frac{\sqrt{5}\pi x^2}{\sqrt{3}})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0521457 | 0.05214527... |
| $\Psi_2(\frac{\sqrt{3}}{\sqrt{5}})=\int_0^\infty \frac{\sin(\frac{\sqrt{3}\pi x^2}{\sqrt{5}})}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$ | 0.0465476 | 0.04654801... |

Table 7.2 of Theorem V.

*Table 7.2* contains numerical values of Ramanujan's integral $\Psi_2(n)=\int_0^\infty \frac{\sin(\pi n x^2)}{1+2\cosh\left(\frac{2\pi x}{\sqrt{3}}\right)}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (5.8).

## 8 Verification of Ramanujan theorem VI (see [3]) via Wolfram Mathematica software

| Comparative study of $\Phi_3(n)=\frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\Psi_3\left(\frac{1}{n}\right)-\Psi_3(n)$...(5.11) | | |
|---|---|---|
| $Ramanujan's\ integral$ $\Phi_3(n)=\int_0^\infty \frac{\cos(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx$ | Numerical values of the integral $\Phi_3(n)$ by using Wolfram Mathematica software | Numerical values of $\Phi_3(n)$ from (5.11) by using the values of $\Psi_3(n)$,$\Psi_3(\frac{1}{n})$ given in fourth column of the table (8.2) |
| $\Phi_3(1)=\int_0^\infty \frac{\cos(\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0732233 | 0.07322332... |
| $\Phi_3(2)=\int_0^\infty \frac{\cos(2\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0625 | 0.06249... |
| $\Phi_3(4)=\int_0^\infty \frac{\cos(4\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0495558 | 0.04955583... |
| $\Phi_3(6)=\int_0^\infty \frac{\cos(6\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0421653 | 0.04216528... |

| | | |
|---|---|---|
| $\Phi_3(\frac{1}{2}) = \int_0^\infty \frac{\cos(\frac{\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0795775 | 0.07957760... |
| $\Phi_3(\frac{1}{4}) = \int_0^\infty \frac{\cos(\frac{\pi x}{4})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0822095 | 0.08220939... |
| $\Phi_3(\frac{1}{6}) = \int_0^\infty \frac{\cos(\frac{\pi x}{6})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0828111 | 0.08281045... |
| $\Phi_3(\frac{2}{3}) = \int_0^\infty \frac{\cos(\frac{2\pi x}{3})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0774899 | 0.07748991... |
| $\Phi_3(\frac{3}{2}) = \int_0^\infty \frac{\cos(\frac{3\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0673963 | 0.06739632... |
| $\Phi_3(\frac{2}{5}) = \int_0^\infty \frac{\cos(\frac{2\pi x}{5})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0807373 | 0.08073740... |
| $\Phi_3(\frac{5}{2}) = \int_0^\infty \frac{\cos(\frac{5\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0584207 | 0.05842068... |
| $\Phi_3(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{5}\pi x}{\sqrt{3}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0697198 | 0.06971980... |
| $\Phi_3(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{\cos(\frac{\sqrt{3}\pi x}{\sqrt{5}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.076099 | 0.07609906... |
| Table 8.1 of Theorem VI. | | |

*Table 8.1* contains numerical values of Ramanujan's integral $\Phi_3(n) = \int_0^\infty \frac{\cos(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (5.11).

| Comparative study of $\Psi_3(n) = \frac{1}{n}\sqrt{\left(\frac{2}{n}\right)}\,\Phi_3\left(\frac{1}{n}\right) + \Phi_3(n)$ ...(5.12) | | | | |
|---|---|---|---|---|
| Auxiliary integral $\Psi_3^*(n) = \int_0^\infty \frac{sin(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx$ | Numerical values of the auxiliary integral $\Psi_3^*(n)$ by using Wolfram Mathematica software | *Ramanujan's integral* $\Psi_3(n) = \Psi_3^*(n) + \frac{1}{2\pi n}$ | Numerical values of the Ramanujan integral $\Psi_3(n)$ with the help of auxiliary integral $\Psi_3^*(n)$ | Numerical values of $\Psi_3(n)$ from (5.12) by using the values of $\Phi_3(n)$,$\Phi_3(\frac{1}{n})$ given in second column of the table (8.1) |
| $\Psi_3^*(1) = \int_0^\infty \frac{\sin(\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0176218 | $\Psi_3(1) = \Psi_3^*(1) + \frac{1}{2\pi}$ | 0.17677674... | 0.17677668... |
| $\Psi_3^*(2) = \int_0^\infty \frac{\sin(2\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0227113 | $\Psi_3(2) = \Psi_3^*(2) + \frac{1}{4\pi}$ | 0.10228877... | 0.10228875... |
| $\Psi_3^*(4) = \int_0^\infty \frac{\sin(4\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0242998 | $\Psi_3(4) = \Psi_3^*(4) + \frac{1}{8\pi}$ | 0.06408853... | 0.06408852... |
| $\Psi_3^*(6) = \int_0^\infty \frac{\sin(6\pi x)}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0236079 | $\Psi_3(6) = \Psi_3^*(6) + \frac{1}{12\pi}$ | 0.05013372... | 0.05013380... |
| $\Psi_3^*(\frac{1}{2}) = \int_0^\infty \frac{\sin(\frac{\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0112676 | $\Psi_3(\frac{1}{2}) = \Psi_3^*(\frac{1}{2}) + \frac{1}{\pi}$ | 0.32957748... | 0.32957750... |
| $\Psi_3^*(\frac{1}{4}) = \int_0^\infty \frac{\sin(\frac{\pi x}{4})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0062498 | $\Psi_3(\frac{1}{4}) = \Psi_3^*(\frac{1}{4}) + \frac{2}{\pi}$ | 0.64286962... | 0.64286937... |
| $\Psi_3^*(\frac{1}{6}) = \int_0^\infty \frac{\sin(\frac{\pi x}{6})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0042697 | $\Psi_3(\frac{1}{6}) = \Psi_3^*(\frac{1}{6}) + \frac{3}{\pi}$ | 0.95919942... | 0.95920040... |
| $\Psi_3^*(\frac{2}{3}) = \int_0^\infty \frac{\sin(\frac{2\pi x}{3})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0138583 | $\Psi_3(\frac{2}{3}) = \Psi_3^*(\frac{2}{3}) + \frac{3}{4\pi}$ | 0.25259071... | 0.25259062... |
| $\Psi_3^*(\frac{3}{2}) = \int_0^\infty \frac{\sin(\frac{3\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0209448 | $\Psi_3(\frac{3}{2}) = \Psi_3^*(\frac{3}{2}) + \frac{1}{3\pi}$ | 0.12704809... | 0.12704805... |
| $\Psi_3^*(\frac{2}{5}) = \int_0^\infty \frac{\sin(\frac{2\pi x}{5})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0094316 | $\Psi_3(\frac{2}{5}) = \Psi_3^*(\frac{2}{5}) + \frac{5}{4\pi}$ | 0.40731895... | 0.40731894... |
| $\Psi_3^*(\frac{5}{2}) = \int_0^\infty \frac{\sin(\frac{5\pi x}{2})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0236442 | $\Psi_3(\frac{5}{2}) = \Psi_3^*(\frac{5}{2}) + \frac{1}{5\pi}$ | 0.08730617... | 0.08730615... |
| $\Psi_3^*(\frac{\sqrt{5}}{\sqrt{3}}) = \int_0^\infty \frac{\sin(\frac{\sqrt{5}\pi x}{\sqrt{3}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0198071 | $\Psi_3(\frac{\sqrt{5}}{\sqrt{3}}) = \Psi_3^*(\frac{\sqrt{5}}{\sqrt{3}}) + \frac{\sqrt{3}}{2\pi\sqrt{5}}$ | 0.14308798... | 0.14308791... |
| $\Psi_3^*(\frac{\sqrt{3}}{\sqrt{5}}) = \int_0^\infty \frac{\sin(\frac{\sqrt{3}\pi x}{\sqrt{5}})}{-1+e^{2\pi\sqrt{x}}}dx$ | 0.0152607 | $\Psi_3(\frac{\sqrt{3}}{\sqrt{5}}) = \Psi_3^*(\frac{\sqrt{3}}{\sqrt{5}}) + \frac{\sqrt{5}}{2\pi\sqrt{3}}$ | 0.22072884... | 0.22072876... |
| Table 8.2 of Theorem VI. | | | | |

*Table 8.2* contains numerical values of Ramanujan's integral $\Psi_3(n) = \int_0^\infty \frac{\sin(\pi n x)}{-1+e^{2\pi\sqrt{x}}}dx + \frac{1}{2n\pi}$, using Wolfram Mathematica software and calculated by Ramanujan's theorem (5.12).

## 9 Conclusion

Here, we have analysed the three Ramanujan's theorems. The tables established above may be of significant in nature. We conclude our investigation by remarking that the theorems (I, II, III) are inaccurate (see table 2.1, table 2.2; table 3.1, table 3.2; table 4.1, table 4.2) with the help of Wolfram Mathematica software. Also, we have found the accuracy of the theorems (IV,V,VI) (see table 6.1, table 6.2; table 7.1, table 7.2; table 8.1, table 8.2) by using Wolfram Mathematica software.

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