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AN INVESTIGATION OF COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS

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Abstract In this paper, we evaluate three different expressions for the composition of two fractional integral operators whose kernals involve the product of generalized extended Mittag-Leffler function and S_U^V polynomial. Due to general nature of the operators in study we can consider them as extension of some simpler fractional integral operators previously studied by several authors. By specializing certain coefficients and parameters of these functions and taking different values for $f(t)$, we can obtain a number of interesting applications for the composition of fractional integral operators out of which we have stated a few.

1 Introduction

The importance of Mittag-Leffler function and its generalizations were understood when it was observed that Mittag-Leffler function naturally arises in the solution of integral and fractional order differential equations as well as solution of general problem of the theory of analytic functions. Mittag-Leffler type functions have considerably developed in last two decades showing vast potential of applications in stochastic systems theory, dynamical system theory, statistical distribution theory, disordered and chaotic systems, etc. These functions are amenable to fractional calculus techniques studied by Srivastava *et al.* [17, 22], Gorenflo *et al.* [2], Samko *et al.* [13] , Saxena-Saigo [14], Kiryakova [4], Kumar and Saxena [7], etc.
The one-parametric Mittag-Leffler function of the form $E_\delta(z)$ was introduced by Gosta Mittag-Leffler [8] in 1903 and was defined by the power series of $z \in \mathbb{C}$ as follows

$$E_\delta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)} \qquad (\delta \in \mathbb{C}, \operatorname{Re}(\delta) > 0), \tag{1.1}$$

which for $\delta = 1$ is a generalization of the exponential function. The very first generalization of $E_\delta(z)$ was given by Wiman [25] in 1905 as the two-parametric Mittag-Leffler function also known as *Wiman's function or generalized Mittag-Leffler function* given by the series of $z \in \mathbb{C}$ as

$$E_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + \kappa)} \qquad (\delta, \kappa \in \mathbb{C},\ \operatorname{Re}(\delta) > 0, \operatorname{Re}(\kappa) > 0). \tag{1.2}$$

In 1971, three-parametric generalisation of Mittag-Leffler function was introduced in a paper written by Prabhakar [11] defined as,

$$E_{\delta,\kappa}^{\vartheta}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\delta n + \kappa)} \frac{z^n}{n!} \tag{1.3}$$

$$(\kappa, \delta, \vartheta \in \mathbb{C},\ \operatorname{Re}(\delta) > 0, \operatorname{Re}(\kappa) > 0, \operatorname{Re}(\vartheta) > 0),$$

where $(\vartheta)_n$ defines the usual Pochhammer symbol (see for details, [12], [19] and [20]):

$$(\vartheta)_0 = 1, \qquad (\vartheta)_n = \vartheta(\vartheta + 1)(\vartheta + 2)\cdots(\vartheta + n - 1). \tag{1.4}$$

Further, extension of $E_\delta(z)$ upto four parameters was introduced and studied by Shukla and Prajapati [15] defined as follows

$$E_{\delta,\kappa}^{\vartheta,r}(z)=\sum_{n=0}^{\infty}\frac{(\vartheta)_{rn}}{\Gamma(\delta n+\kappa)}\frac{z^n}{n!} \tag{1.5}$$

$$(\vartheta,\delta,\kappa\in\mathbb{C},\ \mathrm{Re}(\delta)>0,\mathrm{Re}(\kappa)>0,\mathrm{Re}(\vartheta)>0\ \text{and}\ r\in(0,1)\cup\mathbb{N}),$$

(1.5) was later on further investigated by Srivastava and Tomovski [24] by taking

$$E_{\delta,\kappa}^{\vartheta,\xi}(z)=\sum_{n=0}^{\infty}\frac{(\vartheta)_{\xi n}}{\Gamma(\delta n+\kappa)}\frac{z^n}{n!} \tag{1.6}$$

$$(z,\kappa,\vartheta\in\mathbb{C},\ \mathrm{Re}(\delta)>\max(0,\mathrm{Re}(\xi)-1),\mathrm{Re}(\xi)>0).$$

The most recent generalization of Mittag-Leffler function is introduced by Özarslan and Yilmaz [9] as *extended Mittag-Leffler function* and is defined in the following manner

$$E_{\delta,\kappa}^{(\vartheta;d)}(z;q)=\sum_{n=0}^{\infty}\frac{B_q(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{(d)_n}{\Gamma(\delta n+\kappa)}\frac{z^n}{n!}\quad(q\geq0,\mathrm{Re}(d)>\mathrm{Re}(\vartheta)>0), \tag{1.7}$$

$$\text{where}\qquad B_q(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{\frac{-q}{t(1-t)}}\,dt \tag{1.8}$$

$$(\mathrm{Re}(q)\geq0,\mathrm{Re}(x)>0,\mathrm{Re}(y)>0).$$

The generalization of extended Mittag-Leffler function [1] is defined as follows:

$$E_{\delta,\kappa}^{\vartheta;d}(z;q,\rho,\zeta)=\sum_{n=0}^{\infty}\frac{B_q^{(\rho,\zeta)}(\vartheta+n,d-\vartheta)}{B(\vartheta,d-\vartheta)}\frac{(d)_n}{\Gamma(\delta n+\kappa)}\frac{z^n}{n!} \tag{1.9}$$

$$(q\geq0,\ \mathrm{Re}(d)>\mathrm{Re}(\vartheta)>0,\ \mathrm{Re}(\delta)>0,\ \mathrm{Re}(\kappa)>0),$$

where $B_q^{(\rho,\zeta)}(x,y)$ denotes generalized Beta type function as (see, for details, [18, p. 348, Eq.(1.2)]; see also [10, p. 32, Chapter 4])

$$B_q^{(\rho,\zeta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\rho;\zeta;\frac{-q}{t(1-t)}\right)\,dt \tag{1.10}$$

$$(\mathrm{Re}(q)\geq0,\min(\mathrm{Re}(x),\mathrm{Re}(y),\mathrm{Re}(\zeta),\mathrm{Re}(\rho))>0).$$

S_V^U Polynomial

The S_V^U polynomial was introduced and investigated by Srivastava [16] and is represented in the following manner :

$$S_V^U[x]=\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}A_{V,R}}{R!}x^R,\qquad V=0,1,2,.... \tag{1.11}$$

where U is an arbitrary positive integer, the coefficients $A_{V,R}$ are constants, real or complex.

Multivariable H-Function

The Multivariable H-Function is defined as follows [21, p. 251, Eqs. (C.1-C.3)]:

$$H^{0,B:A_1,B_1;...;A_r,B_r}_{C,D:C_1,D_1;...;C_r,D_r}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,C} : (c_j^{(1)},\gamma_j^{(1)})_{1,C_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,C_r} \\ . & \\ . & \\ . & \\ z_r & (b_j;\beta_j^{(1)},...,\beta_j^{(r)})_{1,D} : (d_j^{(1)},\delta_j^{(1)})_{1,D_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,D_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{\mathfrak{L}_1}\int_{\mathfrak{L}_2}...\int_{\mathfrak{L}_r}\Phi(\xi_1,\xi_2,...,\xi_r)\prod_{i=1}^{r}\Theta_i(\xi_i)z^{\xi_i}\,d\xi_1 d\xi_2,...,d\xi_r, \tag{1.12}$$

where $\omega = \sqrt{-1}$,

$$\Phi(\xi_1,\xi_2,...,\xi_r) = \frac{\prod_{j=1}^{B}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=1}^{D}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i)\prod_{j=B+1}^{C}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}$$

$$\Theta_i(\xi_i) = \frac{\prod_{j=1}^{A_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{B_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=B_i+1}^{C_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)\prod_{j=A_i+1}^{D_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)}(i=1,2,...,r), \tag{1.13}$$

All the Greek letters occurring on the left hand side of (1.12) are assumed to be positive real numbers for standardization purposes. The definition of the multivariable H-function will however be meaningful even if some of these quantities are zero. The details about the nature of contour $\mathfrak{L}_1,...,\mathfrak{L}_r$, conditions of convergence of the integral given by (1.12). Throughout the paper it is assumed that this function always satisfies its appropriate conditions of convergence [21, p. 251, Eqs. (C.4-C.6)].

Fractional Integral Operators

A number of different extensions of several Fractional integral operators have been previously studied by Nisar *et al.*[5], Nisar [6], Choi [3] and so forth. Considering aforementioned Fractional integral operators as motivation, we propose to study two fractional integral operators whose kernals involve the product of generalised extended Mittag-Leffler function and S_U^V polynomial defined and represented in the following manner:

$$I_x^{\eta,\rho}\{f(t)\} = x^{-\eta-\rho-1}\int_0^x t^{\eta}(x-t)^{\rho}E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{t}{x}\right)^{\rho_0};q,\sigma,\zeta\right) \cdot S_V^U\left[y\left(\frac{t}{x}\right)^{\eta_1}\left(1-\frac{t}{x}\right)^{\rho_1}\right]f(t)dt \tag{1.14}$$

and

$$J_t^{\eta',\rho'}\{f(z)\} = t^{\eta'}\int_t^{\infty} z^{-\eta'-\rho'-1}(z-t)^{\rho'}E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{z}\right)^{\rho'_0};q',\sigma',\zeta'\right) \cdot S_{V'}^{U'}\left[y'\left(\frac{t}{z}\right)^{\eta'_1}\left(1-\frac{t}{z}\right)^{\rho'_1}\right]f(z)dz \tag{1.15}$$

where $f(t) \in \Lambda$, Λ denotes the class of function for which

$$f(t) := \begin{cases} O\{|t|^{\varsigma}\}; & Max\{|t|\} \to 0 \\ O\{|t|^{w_1} e^{-w_2|t|}\}; & Min\{|t|\} \to \infty \end{cases} \tag{1.16}$$

provided that the conditions given by (1.17) are satisfied

$$\min \Re\,(\eta + \varsigma + 1, \rho + 1) > 0 \quad \text{and} \quad \min(\rho_0, \eta_1, \rho_1) > 0 \tag{1.17}$$

and conditions (1.18) of operator (1.15) are satisfied

$$\begin{aligned} \Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \min \Re\,(\eta' - w') > 0; \\ \Re(\rho' + 1) > 0, \min(\rho'_0, \eta'_1, \rho'_1) \geq 0 \end{aligned} \tag{1.18}$$

2 Main Results

In the present section we provide three interesting results which are as follows:
Result 1

$$I_x^{\eta',\rho'}\left[J_t^{\eta,\rho}\{f(z)\}\right] = \frac{1}{x}\int_0^x F\left(\frac{z}{x}\right) f(z)dz + \int_x^\infty \frac{1}{z} F^*\left(\frac{x}{z}\right) f(z)dz \tag{2.1}$$

where

$$F(t) = \frac{\Gamma(\zeta)\Gamma(\zeta')\Gamma(\eta+\eta'+\eta_1 R+\eta'_1 R'+1)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')} \sum_{R=0}^{[V/U]} \sum_{R'=0}^{[V'/U']} \frac{(-V)_{UR}(-V')_{U'R'} A_{V,R} A_{V',R'}}{R!R'!}$$

$$\cdot y^R (y')^{R'} t^{\eta+\eta_1 R} (1-t)^{\rho+\rho'+\rho_1 R+\rho'_1 R'+1} H^{0,4:1,0;1,1;1,1;1,2;1,2}_{4,4:0,1;1,2;1,2;3,1;3,1}\left[\begin{array}{c|c} \begin{array}{c} -t \\ -w(1-t)^{\rho_0} \\ -w'(1-t)^{\rho'_0} \\ \frac{1}{q} \\ \frac{1}{q'} \end{array} & \begin{array}{c} A^* : \quad C^* \\ \\ B^* : \quad D^* \end{array} \end{array}\right] \tag{2.2}$$

where

$$\left.\begin{aligned} A^* = (-1-\eta-\eta'-\rho-\rho'-(\rho'_1+\eta'_1)R'-(\rho_1+\eta_1)R; 1, \rho_0, \rho'_0, 0, 0), \\ (-\rho'-\rho'_1 R'; 1, 0, \rho'_0, 0, 0) \\ (1-\vartheta; 0, 1, 0, 1, 0), (1-\vartheta'; 0, 0, 1, 0, 1) \\ B^* = (-1-\eta-\eta'-\rho-\rho'-(\rho'_1+\eta'_1)R'-(\rho_1+\eta_1)R; 0, \rho_0, \rho'_0, 0, 0), \\ (-1-\eta-\eta'-\rho'-\eta_1 R-(\rho'_1+\eta'_1)R'; 1, 0, \rho'_0, 0, 0)(1-d; 0, 1, 0, 2, 0), \\ (1-d'; 0, 0, 1, 0, 2) \\ C^* = -; (1-d, 1); (1-d', 1); (1, 1), (1-d+\vartheta, 1), (\zeta, 1); (1, 1), (1-d'+\vartheta', 1), (\zeta', 1) \\ D^* = (0, 1); (0, 1), (1-\kappa, \delta); (0, 1), (1-\kappa', \delta'); (\sigma, 1); (\sigma', 1) \end{aligned}\right\} \tag{2.3}$$

and $F^*(t)$ can be obtained from F(t) by interchanging the parameters with dashes with those without dashes and following conditions are satisfied

$$\left.\begin{aligned} \text{where} \quad f(t) \in \Lambda \\ \Re(\eta'+\eta+\varsigma) > -2, \Re(\rho+\rho'+\rho'_0+\rho_0) > -2 \\ \Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \Re\,(\eta'-w') > 0 \end{aligned}\right\} \tag{2.4}$$

Result 2

$$I_x^{\eta',\rho'}[I_t^{\eta,\rho}\{f(z)\}] = \frac{1}{x}\int_0^x G\left(\frac{z}{x}\right) f(z)dz \tag{2.5}$$

where

$$G(x) = \frac{\Gamma(\zeta)\Gamma(\zeta')\Gamma(\eta+\eta'+\eta_1 R+\eta_1' R'+1)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')} \sum_{R=0}^{[V/U]} \sum_{R'=0}^{[V'/U']} \frac{(-V)_{UR}(-V')_{U'R'}A_{V,R}A_{V',R'}}{R!R'!}$$

$$\cdot y^R (y')^{R'} x^{\eta+\eta_1 R}(1-x)^{\rho+\rho'+\eta_1 R+\eta_1' R'+1} H^{0,5:1,0;1,1;1,1;1,2;1,2}_{5,4:0,1;1,2;1,2;3,1;3,1}\left[\begin{array}{c|c} & A^{**}: \quad C^{**} \\ -(1-x) & \\ -w(1-x)^{\rho_0} & \\ -w'(1-x)^{\rho_0'} & \\ \frac{1}{q} & \\ \frac{1}{q'} & \\ & B^{**}: \quad D^{**} \end{array}\right] \tag{2.6}$$

where

$$\left.\begin{array}{r} A^{**} = (-\eta+\eta'-\rho-(\rho_1+\eta_1)R+\eta_1' R';1,\rho_0,0,0,0),(-\rho-\eta_1 R;0,\rho_0,0,0,0) \\ (-\rho'-\eta_1' R';1,0,0,0,,\rho_0'),(1-\vartheta;0,1,0,1,0),(1-\vartheta';0,0,1,0,1) \\ B^{**} = (1-\rho-\rho'-\rho_0' R'-\rho_0 R;1,\rho_0,\rho_0',0,0), \\ (-\eta-\eta'-\rho-\eta_1' R'-(\rho_1+\eta_1)R;0,\rho,0,0,0)(1-d;0,2,0,1,0), \\ (1-d';0,0,2,0,1) \\ C^{**} = -;(1-d,1);(1-d',1);(1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1) \\ D^{**} = (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(\sigma,1);(\sigma',1) \end{array}\right\} \tag{2.7}$$

and following conditions are satisfied

$$\left.\begin{array}{r} \text{where} \quad f(t)\in\Lambda \\ \Re(\eta'+\eta+\varsigma) > -2, \Re(\rho+\rho'+\rho_0'+\rho_0) > -2 \\ \min\{\eta_1,\eta_1',\rho_1,\rho_1'\} \geq 0 \end{array}\right\} \tag{2.8}$$

Result 3

$$J_x^{\eta',\rho'}[J_t^{\eta,\rho}\{f(z)\}] = \int_x^\infty \frac{1}{z} G^*\left(\frac{x}{z}\right) f(z)dz \tag{2.9}$$

where $G(x)$ is given by (2.6), $f(t)\in\Lambda$ exists and following conditions are satisfied

$$\left.\begin{array}{r} \Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \Re(\eta+\eta'-w') > 0; \\ \Re(\rho+\rho'+\rho_0'+\rho_0) > -2, \min\{\eta_1,\eta_1',\rho_1,\rho_1'\} \geq 0 \end{array}\right\}$$

Proof of (2.1),(2.9) & (2.5): In order to prove **Result 1**, we proceed by expressing I- and J-operators in the left hand side of (2.1) in their integral forms with the help of (1.14) and (1.15), we obtain:

$$I_x^{\eta',\rho'}[J_t^{\eta,\rho}\{f(z)\}] = x^{\eta-\rho-1}\int_0^x t^\eta (x-t)^\rho E^{\vartheta;d}_{\delta,\kappa}\left(w\left(1-\frac{t}{x}\right)^{\rho_0};q,\sigma,\zeta\right)$$

$$\cdot S_V^U\left[y\left(\frac{t}{x}\right)^{\eta_1}\left(1-\frac{t}{x}\right)^{\rho_1}\right] t^{\eta'}\int_t^\infty z^{-\eta'-\rho'-1}(z-t)^{\rho'} E^{\vartheta';d'}_{\delta',\kappa'}\left(w'\left(1-\frac{t}{z}\right)^{\rho_0'};q',\sigma',\zeta'\right)$$

$$\cdot S_{V'}^{U'}\left[y'\left(\frac{t}{z}\right)^{\eta_1'}\left(1-\frac{t}{z}\right)^{\rho_1'}\right] f(z)dzdt \tag{2.10}$$

Next, we exchange the order of t- and z- integrals (which is permissible under the conditions stated) and obtain the following after a little simplification:

$$I_x^{\eta',\rho'}[J_t^{\eta,\rho}\{f(z)\}] = \int_0^x f(z)\int_0^z g(x,z,t)dtdz + \int_x^\infty f(z)\int_0^x g(x,z,t)dtdz$$
$$= \int_0^x f(z)I_1dz + \int_x^\infty f(z)I_2dz \qquad \text{(say)} \tag{2.11}$$

where

$$g(x,z,t) = x^{-\eta-\rho-1}t^{\eta+\eta'}(x-t)^{\rho}(z-t)^{\rho'}z^{-\eta'-\rho'-1}E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{t}{x}\right)^{\rho_0};q,\sigma,\zeta\right)$$
$$\cdot S_V^U\left[y\left(\frac{t}{x}\right)^{\eta_1}\left(1-\frac{t}{x}\right)^{\rho_1}\right]E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{z}\right)^{\rho_0'};q',\sigma',\zeta'\right)$$
$$\cdot S_{V'}^{U'}\left[y'\left(\frac{t}{z}\right)^{\eta_1'}\left(1-\frac{t}{z}\right)^{\rho_1'}\right] \tag{2.12}$$

and

$$I_1 = \int_0^z g(x,z,t)dt \qquad \text{and} \qquad I_2 = \int_0^x g(x,z,t)dt$$

To evaluate I_1, we first express both the generalised extended Mittag-Leffler functions in terms of their respective contour integral forms using (1.9). Next, express both the S_U^V polynomials in terms of their respective series with the help of (1.11). Further, on exchanging the order of summations and contour integral, we get:

$$I_1 = \left(\frac{1}{2\pi\omega}\right)^4 \frac{\Gamma(\zeta)\Gamma(\zeta')}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')}\sum_{R=0}^{[V/U]}\sum_{R'=0}^{[V'/U']}\frac{(-V)_{UR}(-V')_{U'R'}A_{V,R}A_{V',R'}}{R!R'!}$$
$$\cdot y^R(y')^{R'}\int_{\mathfrak{L}_1}\cdots\int_{\mathfrak{L}_4}\int_0^z t^{\eta+\eta'+\eta_1R+\eta_1'R'}x^{-\eta-\rho-(\eta_1+\rho_1)R-\rho_0\xi_2-1}(x-t)^{\rho+\rho_1R+\rho_0\xi_2}$$
$$\cdot (z-t)^{\rho'+\rho_1'R'+\rho_0'\xi_4}z^{-\eta-\rho-(\eta_1'+\rho_1')R'-\rho_0'\xi_4-1}\frac{\Gamma(-\xi_1)\Gamma(\xi_2)\Gamma(\sigma-\xi_2)}{\Gamma(\zeta-\xi_2)}$$
$$\cdot\frac{\Gamma(\vartheta+\xi_1+\xi_2)\Gamma(d-\vartheta+\xi_2)\Gamma(d+\xi_1)}{\Gamma(d+\xi_1+2\xi_2)\Gamma(\kappa+\delta\xi_1)}q^{-\xi_2}(-\omega)^{\xi_1}\frac{\Gamma(-\xi_3)\Gamma(\xi_4)\Gamma(\sigma'-\xi_4)}{\Gamma(\zeta'-\xi_4)}$$
$$\cdot\frac{\Gamma(\vartheta'+\xi_4+\xi_3)\Gamma(d'-\vartheta'+\xi_4)\Gamma(d'+\xi_3)}{\Gamma(d'+\xi_3+2\xi_4)\Gamma(\kappa'+\delta'\xi_3)}(q')^{-\xi_4}(-\omega')^{\xi_3}dtd\xi_1\cdots d\xi_4 \tag{2.13}$$

Now, we substitute $t = uz$ in (2.13) and evaluate the u-integral using the known result[12, p. 47, Eq.(16)]. Finally, re-interpreting the result in terms of the Multivariable H-function we obtain I_1.
In order to evaluate I_2, we proceed on similar lines as mentioned above substituting $t = ux$. On substituting the values of I_1 and I_2 in (2.11), we get the required result.
Similarly, we can prove the results (2.5) and (2.9) with the help of the results given by [12, p.60, Eq.(5)], so we omit the details.

3 Applications

In this section, we provide the applications of **result 1** and **result 2**.
Firstly, in **result 1**, we consider $f(z) = (1-\gamma z)^l$ and both S_U^V polynomials equal to unity, we

arrive at the following expression after a little simplification:

$$x^{-\eta-1}\int_0^x t^{\eta+l}\left(1-\frac{t}{x}\right)^{\rho}\left(1-\frac{1}{\gamma t}\right)^{\rho'+l+1} H^{0,4:1,1;1,1;1,0;1,2;1,2}_{4,3:1,2;3,1;0,1;3,1;3,1}\left[\begin{array}{c|cc} -w\left(1-\frac{t}{x}\right)^{\rho_0} & I^*: & K^* \\ -w'\left(1-\frac{1}{\gamma t}\right)^{\rho_0'} & & \\ \frac{-1}{\gamma t} & & \\ \frac{1}{q} & & \\ \frac{1}{q'} & J^*: & L^* \end{array}\right] dt$$

$$= (1-\gamma x)^{\rho+\rho'+l+2}\left\{ H^{0,6:1,1;1,1;1,1;1,0;1,2;1,2}_{6,6:1,1;1,2;1,2;0,1;3,1;3,1}\left[\begin{array}{c|cc} -1 & P^*: & R^* \\ -w(1-\gamma x)^{\rho_0} & & \\ -w'(1-\gamma x)^{\rho_0'} & & \\ -\gamma x & & \\ \frac{1}{q} & & \\ \frac{1}{q'} & Q^*: & S^* \end{array}\right]\right.$$

$$\left. + \left(\frac{-1}{\gamma x}\right)^{\rho'+\rho+2} H^{0,6:1,1;1,1;1,1;1,0;1,2;1,2}_{6,6:1,1;1,2;1,2;0,1;3,1;3,1}\left[\begin{array}{c|cc} -1 & E^*: & G^* \\ -w\left(1-\frac{1}{\gamma x}\right)^{\rho_0} & & \\ -w'\left(1-\frac{1}{\gamma x}\right)^{\rho_0'} & & \\ -\frac{1}{\gamma x} & & \\ \frac{1}{q} & & \\ \frac{1}{q'} & F^*: & H^* \end{array}\right]\right\} \tag{3.1}$$

where

$$\left.\begin{array}{r} I^* = (1-\vartheta;1,0,0,1,0),(1-\vartheta';0,1,0,0,1), \\ (-\rho';0,\rho_0',1,0,0),(-\rho'-\eta';0,\rho_0',1,0,0) \\ J^* = (1-d;1,0,0,2,0),(1-d';0,1,2,0,0),(l-\eta'-\rho';0,\rho_0',1,0,0) \\ K^* = (1-d,1);(1-d',1);-; \\ (1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1) \\ L^* = (0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta'),(1-\rho-\eta'-l,\rho_0');(0,1);(\sigma,1);(\sigma',1) \end{array}\right\}$$

$$\left.\begin{array}{r} P^* = (-1-\eta-\eta'-\rho-\rho';1,\rho_0,\rho_0',0,0,0),(-\rho';1,0,\rho_0',0,0,0), \\ (1-\vartheta;0,1,0,0,1,0),(1-\vartheta';0,0,1,0,0,1), \\ (1-\rho-\rho'-\eta-l;1,\rho_0,\rho_0',0,0,0),(-1-\rho-\rho';0,\rho_0,\rho_0',1,0,0) \\ Q^* = (-1-\eta-\eta'-\rho-\rho';0,\rho_0,\rho_0',0,0,0),(-1-\eta-\eta'-\rho';1,0,\rho_0',0,0,0), \\ (1-d;0,1,0,0,2,0),(1-d';0,0,1,0,0,2), \\ (-1-\rho-\rho'-\eta-l;1,\rho_0,\rho_0',0,0,0),(-1-\eta-\rho-\rho';1,\rho_0,\rho_0',0,0,0) \\ R^* = (1-\eta,1);(1-d,1);(1-d',1);-; \\ (1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1) \\ S^* = (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(0,1);(\sigma,1);(\sigma',1) \end{array}\right\}$$

$$\left.\begin{array}{r} E^* = (-1-\eta-\eta'-\rho-\rho';1,0,\rho_0,\rho_0',0,0),(-\rho;1,0,\rho_0,0,0,0),\\ (1-\vartheta;0,0,1,0,0,1),(1-\vartheta';0,1,0,0,1,0),(-1-\rho-\rho'-\eta';1,0,\rho_0,\rho_0',0,0),\\ (-1-\rho-\rho';0,1,\rho_0,\rho_0',0,0)\\ F^* = (-1-\eta-\eta'-\rho-\rho';0,0,\rho_0,\rho_0',0,0),(-1-\eta-\eta'-\rho;1,0,\rho_0,0,0,0),\\ (1-d;0,0,1,0,0,2),\\ (1-d';0,0,0,1,2,0),(-1-\rho-\rho'-\eta';1,0,\rho_0,\rho_0',0,0),\\ (-1-\eta'-\rho-\rho'-l;1,0,\rho_0,\rho_0',0,0)\\ G^* = (1-\eta'-l,1);(1-d,1);(1-d',1);-;\\ (1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1)\\ H^* = (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(0,1);(\sigma,1);(\sigma',1) \end{array}\right\}$$

provided the conditions obtainable from **result 1** are satisfied.
Further, in **result 2**, if we take $f(z) = {}_2F_1\left(a,b;c;\left(\alpha-\frac{z}{x}\right)^m\right)$, we obtain the following expression after a little simplification:

$$x^{-\eta'-\rho'-1}\int_0^x\int_0^t t^{\eta'-\eta-\rho-1}(x-t)^{\rho'}(t-z)^{\rho}z^{\eta}E_{\delta,\kappa}^{\vartheta;d}\left(w\left(1-\frac{z}{t}\right)^{\rho_0};q,\sigma,\zeta\right)S_V^U\left[y\left(\frac{z}{t}\right)^{\eta_1}\left(1-\frac{z}{t}\right)^{\rho_1}\right]$$

$$\cdot E_{\delta',\kappa'}^{\vartheta';d'}\left(w'\left(1-\frac{t}{x}\right)^{\rho_0'};q',\sigma',\zeta'\right)S_{V'}^{U'}\left[y'\left(\frac{t}{x}\right)^{\eta_1'}\left(1-\frac{t}{x}\right)^{\rho_1'}\right]{}_2F_1\left(a,b;c;\left(\alpha-\frac{z}{x}\right)^m\right)dzdt$$

$$=\frac{\Gamma(\zeta)\Gamma(\zeta')\Gamma(\eta+\eta_1R+1)\Gamma(c)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d-\vartheta)\Gamma(d'-\vartheta')\Gamma(\vartheta)\Gamma(\vartheta')\Gamma(a)\Gamma(b)}\sum_{R=0}^{[V/U]}\sum_{R'=0}^{[V'/U']}\frac{(-V)_{UR}(-V')_{U'R'}A_{V,R}A_{V',R'}}{R!R'!}y^R(y')^{R'}$$

$$\cdot\left(1-\frac{1}{\alpha}\right)^{\rho+\rho'+\eta_1R+\eta_1'R'+2}H_{7,6:0,1;1,2;1,2;3,1;3,1;2,2;0,1}^{0,7:1,0;1,1;1,1;1,2;1,2;1,2;1,0}\left[\begin{array}{c|c} -(1-x)\left(1-\frac{1}{\alpha}\right) & M^*:\quad O^*\\ -w(1-x)^{\rho_0}\left(1-\frac{1}{\alpha}\right)^{\rho_0} & \\ -w'(1-x)^{\rho_0'}\left(1-\frac{1}{\alpha}\right)^{\rho_0'} & \\ \frac{1}{q} & \\ \frac{1}{q'} & \\ -(\alpha-1)^m & \\ \frac{1}{\alpha} & N^*:\quad P^* \end{array}\right] \tag{3.2}$$

where

$$\left.\begin{array}{r} M^* = (-\eta+\eta'-\rho-(\rho_1+\eta_1)R+\eta_1'R';1,\rho_0,0,0,0,0,0),(-\rho-\eta_1R;0,\rho_0,0,0,0,0,0),\\ (-\rho'-\eta_1'R';1,0,0,0,,\rho_0',0,0),(1-\vartheta;0,1,0,1,0,0,0),\\ (1-\vartheta';0,0,1,0,1,0,0),(-2-\rho-\rho'-\eta-2\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,m,1),\\ (-1-\rho-\rho'-\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,0,1)\\ N^* = (1-\rho-\rho'-\rho_0'R'-\rho_0R;1,\rho_0,\rho_0',0,0,0,0),\\ (-\eta-\eta'-\rho-\eta_1'R'-(\rho_1+\eta_1)R;0,\rho,0,0,0,0,0),\\ (1-d;0,2,0,1,0,0,0),(1-d';0,0,2,0,1,0,0),\\ (-1-\rho-\rho'-\eta-2\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,0,0),\\ (-2-\rho-\rho'-\eta-2\eta_1R-\eta_1'R';1,\rho_0,\rho_0',0,0,0,1)\\ O^* = -;(1-d,1);(1-d',1);(1,1),(1-d+\vartheta,1),(\zeta,1);(1,1),(1-d'+\vartheta',1),(\zeta',1);\\ (1-a,1),(1-b,1),(0,1);-\\ P^* = (0,1);(0,1),(1-\kappa,\delta);(0,1),(1-\kappa',\delta');(\sigma,1);(\sigma',1);(1-c,1);(0,1) \end{array}\right\}$$

provided the conditions obtainable from **result 2** are satisfied.

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