AN INVESTIGATION OF COMPOSITION FORMULAE FOR FRACTIONAL INTEGRAL OPERATORS

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Abstract In this paper, we evaluate three different expressions for the composition of two fractional integral operators whose kernels involve the product of generalized extended Mittag-Leffler function and \( S_U^V \) polynomial. Due to general nature of the operators in study we can consider them as extension of some simpler fractional integral operators previously studied by several authors. By specializing certain coefficients and parameters of these functions and taking different values for \( f(t) \), we can obtain a number of interesting applications for the composition of fractional integral operators out of which we have stated a few.

1 Introduction

The importance of Mittag-Leffler function and its generalizations were understood when it was observed that Mittag-Leffler function naturally arises in the solution of integral and fractional order differential equations as well as solution of general problem of the theory of analytic functions. Mittag-Leffler type functions have considerably developed in last two decades showing vast potential of applications in stochastic systems theory, dynamical system theory, statistical distribution theory, disordered and chaotic systems, etc. These functions are amenable to fractional calculus techniques studied by Srivastava et al. [17, 22], Gorenflo et al. [2], Samko et al. [13], Saxena-Saigo [14], Kiryakova [4], Kumar and Saxena [7], etc.

The one-parametric Mittag-Leffler function of the form \( E_{\delta}(z) \) was introduced by Gosta Mittag-Leffler [8] in 1903 and was defined by the power series of \( z \in \mathbb{C} \) as follows

\[
E_{\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + 1)} \quad (\delta \in \mathbb{C}, \Re(\delta) > 0),
\]  

(1.1)

which for \( \delta = 1 \) is a generalization of the exponential function. The very first generalization of \( E_{\delta}(z) \) was given by Wiman [25] in 1905 as the two-parametric Mittag-Leffler function also known as Wiman’s function or generalized Mittag-Leffler function given by the series of \( z \in \mathbb{C} \) as

\[
E_{\delta,\kappa}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\delta n + \kappa)} \quad (\delta, \kappa \in \mathbb{C}, \Re(\delta) > 0, \Re(\kappa) > 0).
\]  

(1.2)

In 1971, three-parametric generalisation of Mittag-Leffler function was introduced in a paper written by Prabhakar [11] defined as,

\[
E_{\delta,\kappa}^{0}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\delta n + \kappa)} \frac{z^n}{n!} \quad (\kappa, \delta, \vartheta \in \mathbb{C}, \Re(\delta) > 0, \Re(\kappa) > 0, \Re(\vartheta) > 0),
\]  

(1.3)

where \((\vartheta)_n\) defines the usual Pochhammer symbol (see for details, [12], [19] and [20]):

\[
(\vartheta)_0 = 1, \quad (\vartheta)_n = \vartheta(\vartheta + 1)(\vartheta + 2) \cdots (\vartheta + n - 1).
\]  

(1.4)
Further, extension of $E_{\delta}(z)$ up to four parameters was introduced and studied by Shukla and Prajapati [15] as follows:

$$E_{\delta,\kappa}^{\vartheta}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\delta n + \kappa)} \frac{z^n}{n!}$$  \hspace{1cm} (1.5)

$(\vartheta, \delta, \kappa \in \mathbb{C}, \Re(\delta) > 0, \Re(\kappa) > 0, \Re(\vartheta) > 0$ and $r \in (0, 1) \cup \mathbb{N})$.

(1.5) was later on further investigated by Srivastava and Tomovski [24] by taking

$$E_{\delta,\kappa}^{\vartheta}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_{\xi n}}{\Gamma(\delta n + \kappa)} \frac{z^n}{n!}$$  \hspace{1cm} (1.6)

$(z, \kappa, \vartheta \in \mathbb{C}, \Re(\delta) > \max(0, \Re(\xi) - 1), \Re(\xi) > 0)$.

The most recent generalization of Mittag-Leffler function is introduced by Özarslan and Yılmaz [9] as extended Mittag-Leffler function and is defined in the following manner:

$$E_{\delta,\kappa}^{(\rho;\vartheta)}(z; q) = \sum_{n=0}^{\infty} \frac{B_q(\vartheta + n, d - \vartheta)}{B(\vartheta, d - \vartheta)} \frac{(d)_n}{\Gamma(\delta n + \kappa)} \frac{z^n}{n!} \hspace{1cm} (q \geq 0, \Re(d) > \Re(\vartheta) > 0),$$  \hspace{1cm} (1.7)

where

$$B_q(x, y) = \int_{0}^{1} t^{x-1}(1 - t)^{y-1} e^{\frac{-nt}{1-t}} \, dt$$  \hspace{1cm} (1.8)

$(\Re(q) \geq 0, \Re(x) > 0, \Re(y) > 0)$.

The generalization of extended Mittag-Leffler function [1] is defined as follows:

$$E_{\delta,\kappa}^{(\rho;\vartheta)}(z; q, \rho, \zeta) = \sum_{n=0}^{\infty} \frac{B_q^{(\rho;\vartheta)}(\vartheta + n, d - \vartheta)}{B(\vartheta, d - \vartheta)} \frac{(d)_n}{\Gamma(\delta n + \kappa)} \frac{z^n}{n!}$$  \hspace{1cm} (1.9)

$(q \geq 0, \Re(d) > \Re(\vartheta) > 0, \Re(\delta) > 0, \Re(\kappa) > 0)$,

where $B_q^{(\rho;\vartheta)}(x, y)$ denotes generalized Beta type function as (see, for details, [18, p. 348, Eq.(1.2)]; see also [10, p. 32, Chapter 4])

$$B_q^{(\rho;\vartheta)}(x, y) = \int_{0}^{1} t^{x-1}(1 - t)^{y-1} F_1(\rho; \zeta; \frac{t}{1-t}) \, dt$$  \hspace{1cm} (1.10)

$(\Re(q) \geq 0, \min(\Re(x), \Re(y), \Re(\zeta), \Re(\rho)) > 0)$.

$S^U_V$ Polynomial

The $S^U_V$ polynomial was introduced and investigated by Srivastava [16] and is represented in the following manner:

$$S^U_V[x] = \sum_{R=0}^{[V/U]} \frac{(-V)_UR A_{V,R}}{R!} x^R, \hspace{1cm} V = 0, 1, 2, \ldots$$  \hspace{1cm} (1.11)

where $U$ is an arbitrary positive integer, the coefficients $A_{V,R}$ are constants, real or complex.
Multivariable H-Function

The Multivariable H-Function is defined as follows [21, p. 251, Eqs. (C.1-C.3)]:

\[
\frac{1}{(2\pi)^r} \int_{\mathbb{L}_1} \cdots \int_{\mathbb{L}_r} \Phi(\xi_1, \xi_2, \ldots, \xi_r) \prod_{i=1}^{r} \Theta_i(\xi_i) z^{\xi_1} d\xi_1 d\xi_2, \ldots, d\xi_r, \tag{1.12}
\]

where \( \omega = \sqrt{-1} \).

\[
\Phi(\xi_1, \xi_2, \ldots, \xi_r) = \frac{\prod_{j=1}^{B} \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha^{(i)}_j \xi_i)}{\prod_{j=1}^{C} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta^{(i)}_j \xi_i) \prod_{j=B+1}^{D} \Gamma(a_j - \sum_{i=1}^{r} \alpha^{(i)}_j \xi_i)}
\]

\[
\Theta_i(\xi_i) = \frac{\prod_{j=1}^{A} \Gamma(d^{(i)}_j - \delta^{(i)}_j \xi_i)}{\prod_{j=1}^{B} \Gamma(c^{(i)}_j - \gamma^{(i)}_j \xi_i) \prod_{j=B+1}^{D} \Gamma(1 - d^{(i)}_j + \delta^{(i)}_j \xi_i)} \quad (i = 1, 2, \ldots, r), \tag{1.13}
\]

All the Greek letters occurring on the left hand side of (1.12) are assumed to be positive real numbers for standardization purposes. The definition of the multivariable H-function will however be meaningful even if some of these quantities are zero. The details about the nature of contour \( \mathbb{L}_1, \ldots, \mathbb{L}_r \), conditions of convergence of the integral given by (1.12). Throughout the paper it is assumed that this function always satisfies its appropriate conditions of convergence [21, p. 251, Eqs. (C.4-C.6)].

Fractional Integral Operators

A number of different extensions of several Fractional integral operators have been previously studied by Nisar et al. [5], Nisar [6], Choi [3] and so forth. Considering aforementioned Fractional integral operators as motivation, we propose to study two fractional integral operators whose kernals involve the product of generalised extended Mittag-Leffler function and fractional integral operators as motivation, we propose to study two fractional integral operators.

\[
I^{\eta, \rho}_x \{ f(t) \} = x^{-\eta-\rho-1} \int_0^x t^{\eta}(x-t)^{\rho} E_{\delta, \kappa} \left( w \left( 1 - \frac{t}{x} \right) \rho_{\delta} q, \sigma, \zeta \right) \cdot S^U_V \left[ y \left( \frac{t}{x} \right) \eta_{\delta} \left( 1 - \frac{t}{x} \right)^{\rho_{\delta}} f(t) \right] dt \tag{1.14}
\]

and

\[
J^{\rho', \eta'}_t \{ f(z) \} = t^{\eta'} \int_z^\infty z^{-\eta'-\rho'-1}(z-t)^{\rho'} E_{\delta', \kappa'} \left( w' \left( 1 - \frac{t}{z} \right) \rho_{\delta'} q', \sigma', \zeta' \right) \cdot S^U_V \left[ y' \left( \frac{t}{z} \right) \eta_{\delta'} \left( 1 - \frac{t}{z} \right)^{\rho_{\delta'}} f(z) \right] dz \tag{1.15}
\]
where \( f(t) \in \Lambda, \Lambda \) denotes the class of function for which
\[
f(t) = \begin{cases} O(|t|); & \max{|t|} \to 0 \\ O(|t|^{\infty} e^{-\infty |t|}); & \min{|t|} \to \infty \end{cases}
\]

provided that the conditions given by (1.17) are satisfied
\[
\min \Re (\eta + \varsigma + 1, \rho + 1) > 0 \quad \text{and} \quad \min (\rho_0, \eta_1, \rho_1) > 0
\]

and conditions (1.18) of operator (1.15) are satisfied
\[
\Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \min \Re(\eta' - w') > 0;
\]
\[
\Re(\rho' + 1) > 0, \min(\rho_0', \eta_1', \rho_1') \geq 0
\]

2 Main Results

In the present section we provide three interesting results which are as follows:

Result 1
\[
I^T_{\rho}J^\eta R^\epsilon [J_1^{\rho_0}(f(z))] = \frac{1}{x} \int_0^x F \left( \frac{z}{x} \right) f(z) dz + \int_x^\infty \frac{1}{z} F^* \left( \frac{z}{x} \right) f(z) dz
\]

where
\[
F(t) = \frac{\Gamma(\zeta)\Gamma(\zeta')\Gamma(\eta + \eta' + \eta_1 R + \eta_1' R' + 1)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(-\sigma + \eta - \eta' + 1) \sum_{R=0}^{[V]} \sum_{R'=0}^{[U']}} \left[ \begin{matrix} -t \\ -w(1-t)^{\rho_0} \\ -w'(1-t)^{\rho_0'} \\ \frac{1}{\eta} \\ \frac{1}{\eta'} \end{matrix} \right]
\]

\[
A^* = \begin{bmatrix} A^* \\ C^* \\ B^* \\ D^* \end{bmatrix}
\]

\[
A^* = (-1 - \eta - \eta' - \rho - \rho'; -\rho_1 + \eta_1)R' - (\rho_1 + \eta_1)R; 1, \rho_0, \rho_0, 0, 0, (1 - b; 0, 1, 0, 1, 0, 1), (1 - b; 0, 0, 1, 0, 1)
\]

\[
B^* = (-1 - \eta - \eta' - \rho - \rho'; -\rho_1 + \eta_1)R' - (\rho_1 + \eta_1)R; 0, \rho_0, \rho_0, 0, 0, (1 - d; 1, 0, 1, 0, 1)
\]

\[
C^* = -; (1 - d; 1); (1 - d', 1); (1, 1); (1 - d + \theta, 1), (\zeta, 1); (1, 1); (1 - d' + \delta, 1), (\zeta', 1)
\]

\[
D^* = (0, 1); (0, 1), (1 - \kappa, \delta); (0, 1); (1 - \kappa', \delta'); (\sigma, 1); (\sigma', 1)
\]

and \( F^*(t) \) can be obtained from \( F(t) \) by interchanging the parameters with dashes with those without dashes and following conditions are satisfied
\[
\min \Re(\eta' + \eta + \varsigma) > -2, \min(\rho + \rho' + \rho_0 + \rho_0) > -2
\]

\[
\Re(w_2) > 0 \quad \text{or} \quad \Re(w_2) = 0 \quad \text{and} \quad \Re(\eta' - w') > 0
\]
Result 2

\[ I_x^\rho \cdot \gamma^\eta [J_t^\eta \cdot \gamma^\rho \{ f(z) \}] = \frac{1}{x} \int_0^x G \left( \frac{z}{x} \right) f(z) dz \]  
\hspace{1cm} (2.5)

where

\[ G(x) = \frac{\Gamma(\zeta)\Gamma(\zeta)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d - \vartheta)\Gamma(d' - \vartheta')\Gamma(\theta)\Gamma(\vartheta')} \sum_{R=0}^{[V/U]} \sum_{R'=0}^{[V'/U']} (-1)^{V-R} \frac{\eta R'(\eta + \eta') + \eta \gamma R + \gamma R' + 1}{R! R'} \left[ \begin{array}{c} -(1 - x) \\ -w(1 - x) \rho_0 \\ -w' \end{array} \right] \]

\[ \left[ \begin{array}{c} \frac{1}{q} \\ \frac{1}{q} \\ \frac{1}{q} \end{array} \right] \]

\[ \left[ \begin{array}{c} A^{**} \\ C^{**} \\ B^{**} \\ D^{**} \end{array} \right] \]

where

\[ A^{**} = \left( -\eta + \eta' - \rho - (\rho_1 + \eta_1) R + \gamma_1 R'; 1, \rho_0, 0, 0, 0 \right) \]
\[ B^{**} = \left( -\rho' - \eta_1 R'; 1, 0, 0, 0, 0 \right) \]
\[ C^{**} = \left( -\rho - \eta_1 R' - (\eta_1 + \eta) R; 0, \rho_0, 0, 0, 0 \right) \]
\[ D^{**} = \left( -\rho - \eta_1 R' - (\eta_1 + \eta) R; 0, 0, 0, 0, 0 \right) \]

and following conditions are satisfied

\[ \Re(\eta' + \eta + \zeta) > -2, \Re(\rho + \rho' + \rho_0 + \rho_0) > -2 \]
\[ \min\{\eta_1, \eta_1', \rho_1, \rho_1'\} \geq 0 \]
\hspace{1cm} (2.8)

Result 3

\[ J_x^{\rho \cdot \gamma\eta} [J_t^{\eta \cdot \gamma\rho} \{ f(z) \}] = \int_0^\infty G^* \left( \frac{x}{z} \right) f(z) dz \]  
\hspace{1cm} (2.9)

where \( G(x) \) is given by (2.6), \( f(t) \in \Lambda \) exists and following conditions are satisfied

\[ \Re(\eta_2) > 0 \text{ or } \Re(\eta_2) = 0 \text{ and } \Re(\eta + \eta' - w) > 0 \]
\[ \Re(\rho + \rho' + \rho_0 + \rho_0) > -2, \min\{\eta, \eta_1, \rho_1, \rho_1'\} \geq 0 \]
\hspace{1cm} (2.10)

Proof of \((2.1), (2.9) \& (2.5)\): In order to prove Result 1, we proceed by expressing I- and J-operators in the left hand side of (2.1) in their integral forms with the help of (1.14) and (1.15), we obtain:

\[ I_x^{\rho \cdot \gamma\eta} [J_t^{\eta \cdot \gamma\rho} \{ f(z) \}] = x^{\eta - \rho - 1} \int_0^x t^n(x - t)\rho E_{\eta, \rho} \left( w \left( 1 - \frac{t}{x} \right)^{\rho_0} ; q, \sigma, \zeta \right) \]
\[ \cdot S_{\eta}^{\eta} \left[ y \left( \frac{t}{x} \right) \eta \left( 1 - \frac{t}{x} \right)^{\rho_1} \right] t^{\rho_1} \int_t^\infty z^{\eta' - \rho' - 1}(z - t)\rho' E_{\eta', \rho'} \left( w' \left( 1 - \frac{t}{z} \right)^{\rho_0'} ; q', \sigma', \zeta' \right) \]
\[ \cdot S_{\eta'}^{\eta'} \left[ y' \left( \frac{t}{z} \right) \eta' \left( 1 - \frac{t}{z} \right)^{\rho_1'} \right] f(z) dz dt \]  
\hspace{1cm} (2.10)
Next, we exchange the order of t- and z-integrals (which is permissible under the conditions stated) and obtain the following after a little simplification:

\[
I_x^{\eta,\rho_1}[J_x^{\eta,\rho_2}\{f(z)\}] = \int_0^x f(z)\int_0^z g(x, z, t)dt\,dz + \int_x^\infty f(z)\int_0^z g(x, z, t)dt\,dz
\]

\[= \int_x^z f(z)I_1dz + \int_f(z)I_2dz \quad \text{(say)} \quad (2.11)\]

where

\[g(x, z, t) = x^{-\sigma - \rho_1 - \delta rt + \delta z} (x - t)^{\sigma_\rho_1 - \rho_1 - 1} E^{\rho_1,\rho_2}_{\delta,\kappa} (w (1 - \frac{t}{x})^{\rho_2}; q, \sigma, \zeta) \]

\[= \cdot S_V^{U} \left[ y \left( \frac{t}{x} \right)^{\eta_1} \left( 1 - \frac{t}{x} \right)^{\rho_2} \right] \]

\[= \cdot S_V^{U} \left[ y \left( \frac{t}{x} \right)^{\eta_1} \left( 1 - \frac{t}{x} \right)^{\rho_2} \right] \quad (2.12)\]

and

\[I_1 = \int_0^z g(x, z, t)dt \quad \text{and} \quad I_2 = \int_0^x g(x, z, t)dt\]

To evaluate \(I_1\), we first express both the generalised extended Mittag-Leffler functions in terms of their respective contour integral forms using (1.9). Next, express both the \(S^{U}_{V}\) polynomials in terms of their respective series with the help of (1.11). Further, on exchanging the order of summations and contour integral, we get:

\[I_1 = \left( \frac{1}{2\pi i} \right)^4 \frac{\Gamma(\zeta_1)\Gamma(\zeta_2)}{\Gamma(\sigma)\Gamma(\sigma')\Gamma(d - \sigma)\Gamma(d' - \sigma')} \sum_{R=0}^{V/U} \sum_{R'=0}^{V'/U'} \left( -V \right)^{U}_R \left( -V' \right)^{U'}_R A_{V,R}A_{V',R'} \frac{V_{U,R}}{R!R'!} \]

\[\cdot \int \int \frac{y^{\eta_1 + \eta_1 R_R + \eta_1 R'_{R'}}}{\xi_1} \int^\zeta \int^\zeta \int^z \int^z (x - t)^{\rho_1 + 1} R + R + R + R \]

\[\cdot \frac{\Gamma(\delta + \xi_1 + \xi_2)\Gamma(d - \delta + \xi_2)\Gamma(d + \xi_1)}{\Gamma(d + \xi_1 + 2\xi_2)} (q' - \xi_1 - \xi_2 - \xi_3) dt d\xi_1 \cdots d\xi_4 \quad (2.13)\]

Now, we substitute \(t = uz\) in (2.13) and evaluate the u-integral using the known result[12, p. 47, Eq.(16)]. Finally, re-interpreting the result in terms of the Multivariable H-function we obtain \(I_1\).

In order to evaluate \(I_2\), we proceed on similar lines as mentioned above substituting \(t = uz\). On substituting the values of \(I_1\) and \(I_2\) in (2.11), we get the required result.

Similarly, we can prove the results (2.5) and (2.9) with the help of the results given by [12, p.60, Eq.(5)], so we omit the details.

3 Applications

In this section, we provide the applications of result 1 and result 2.

Firstly, in result 1, we consider \(f(z) = (1 - \gamma z)^\beta\) and both \(S^{U}_{V}\) polynomials equal to unity, we
arrive at the following expression after a little simplification:

\[
\begin{align*}
x^{-\eta-1} & \int_0^x t^{\eta+t} \left( 1 - \frac{t}{x} \right)^{\rho'} \left( 1 - \frac{1}{\gamma t} \right)^{\rho' + l + 1} H_{\rho',\gamma}^{\rho,1;1,1;1,0,1,2,1,2} [ -w \left( 1 - \frac{\frac{1}{x^l}}{\frac{1}{\gamma}} \right) \rho' ] \, dt \\
= (1 - \gamma x)^{\rho' + \rho' + l + 2} & \left\{ H_{\rho',\gamma}^{\rho,1;1,1;1,1,1,0,1,2,1,2} \left[ -1 \right], \right.
\left. P^\ast : R^\ast \right\}
\end{align*}
\]

\[
\begin{align*}
+ \left( \frac{-1}{\gamma x} \right)^{\rho' + \rho' + 2} & H_{\rho',\gamma}^{\rho,1;1,1;1,1,1,1,0,1,2,1,2} \left[ \frac{-1}{\gamma^2}, \frac{-w(1 - \gamma x)^{\rho' \ast}}{\gamma^2}, \frac{-w'((1 - \gamma x)^{\rho'} \ast)}{\gamma^2} \right], \right.
\left. Q^\ast : S^\ast \right\}
\end{align*}
\]

\[
\begin{align*}
E^\ast : G^\ast & \left[ \frac{-1}{\gamma^2}, \frac{-w(1 - \gamma x)^{\rho' \ast}}{\gamma^2}, \frac{-w'((1 - \gamma x)^{\rho'} \ast)}{\gamma^2 \gamma^2} \right],
\left. F^\ast : H^\ast \right\}
\end{align*}
\]

where

\[
\begin{align*}
I^\ast & = (1 - \vartheta; 1, 0, 0, 1, 0), (1 - \vartheta'; 0, 1, 0, 0, 1), \\
J^\ast & = (1 - d; 1, 0, 0, 2, 0), (1 - \vartheta; 0, 1, 2, 0, 0), (1 - \eta' - \rho' \ast; 0, 0, 1, 0, 0), \\
K^\ast & = (1 - d; 1, 1), (1 - d'; 1), (1 - \eta' - \rho' \ast; 0, 0, 1, 0, 0), \\
L^\ast & = (0, 1), (1 - \kappa, \delta); (0, 1), (1 - \kappa'; \delta'), (1 - \rho - \eta' - l, \rho' \ast); (0, 1); (\sigma, 1); (\sigma', 1).
\end{align*}
\]

\[
\begin{align*}
P^\ast & = (-1 - \eta - \eta' - \rho - \rho' \ast; 1, \rho_0, \rho' \ast, 0, 0, 0), (-1 - \rho' \ast; 1, 0, \rho' \ast, 0, 0), \\
(1 - \rho - \rho' - \eta - l; 1, \rho_0, \rho' \ast, 0, 0, 0), (1 - \rho - \rho' \ast; 0, \rho_0, \rho' \ast, 1, 0, 0), \\
Q^\ast & = (-1 - \eta - \eta' - \rho - \rho' \ast; 0, \rho_0, \rho' \ast, 0, 0, 0), (-1 - \eta - \eta' - \rho' \ast; 1, 0, \rho_0, 0, 0, 0), \\
(1 - d; 0, 1, 0, 2, 0, 0), (1 - d'; 0, 0, 1, 0, 0, 2), \\
(1 - \rho - \rho' - \eta - l; 1, \rho_0, \rho' \ast, 0, 0, 0), (1 - \eta - \rho - \rho' \ast; 1, \rho_0, \rho_0, 0, 0, 0), \\
R^\ast & = (1 - \eta, 1); (1 - d, 1); (1 - d'; 1), (1 - \eta - \rho - \rho' \ast; 1, \rho_0, \rho_0, 0, 0, 0), \\
S^\ast & = (0, 1); (0, 1), (1 - \kappa, \delta); (0, 1), (1 - \kappa', \delta'); (0, 1); (\sigma, 1); (\sigma', 1).
\end{align*}
\]
provided the conditions obtainable from result 1 are satisfied.

Further, in result 2, if we take $f(z) = 2F_1(a, b; c; (\alpha - \frac{z}{x})^m)$, we obtain the following expression after a little simplification:

\[
E^* = \left\{ \begin{array}{l}
(1 - \eta - \eta' - \rho - \rho'; 1, 0, \rho_0, \rho_0', 0, 0), (-\rho; 1, 0, \rho_0, 0, 0, 0),

(1 - \vartheta; 0, 0, 1, 0, 0, 1), (1 - \vartheta'; 0, 1, 0, 0, 1, 0), (-1 - \rho - \rho' - \eta'; 1, 0, \rho_0, \rho_0', 0, 0),

(-1 - \rho - \rho'; 0, 1, \rho_0, \rho_0', 0, 0)
\end{array} \right. \\
F^* = \left\{ \begin{array}{l}
(1 - \eta - \eta' - \rho - \rho'; 0, 0, \rho_0, \rho_0', 0, 0), (-1 - \eta - \eta' - \rho; 1, 0, \rho_0, 0, 0, 0),

(1 - d; 0, 0, 1, 0, 0, 2),

(1 - d'; 0, 0, 1, 0, 2, 0), (-1 - \rho - \rho' - \eta'; 1, 0, \rho_0, \rho_0', 0, 0),

(-1 - \rho - \rho' - l; 1, 0, \rho_0, \rho_0', 0, 0)
\end{array} \right. \\
G^* = \left\{ \begin{array}{l}
(1 - \eta - l; 1), (1 - d, 1), (1 - d', 1);

(1, 1), (1 - d + \vartheta, 1), (\zeta, 1); (1, 1), (1 - d' + \vartheta', 1), (\zeta', 1)
\end{array} \right. \\
H^* = \left\{ \begin{array}{l}
(0, 1); (0, 1), (1 - \kappa, \delta); (0, 1), (1 - \kappa', \delta'); (0, 1); (\sigma, 1); (\sigma', 1)
\end{array} \right. \\
\]

provided the conditions obtainable from result 2 are satisfied.
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