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On weakly 2-irreducible ideals of commutative rings

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Abstract All rings are commutative with $1 \neq 0$. The purpose of this paper is to investigate the concept of weakly 2-irreducible ideals generalizing weakly irreducible ideals and strongly 2-irreducible ideals. We say that a proper ideal I of a ring R is weakly 2-irreducible provided that for each ideals J, K and L of R, $J \cap K \cap L \subseteq I$ implies that either $J \cap K \subseteq \sqrt{I}$ or $J \cap L \subseteq \sqrt{I}$ or $K \cap L \subseteq \sqrt{I}$. A number of results concerning weakly 2-irreducible ideals are given. For instance, the relationships between the notions weakly 2-irreducible, 2-absorbing, 2-absorbing primary and 2-absorbing quasi-primary in different rings, has been given.

1 Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let R be a commutative ring. An ideal I of R is irreducible if $I = J \cap K$ for some ideals J and K of R implies that either $I = J$ or $I = K$. A proper ideal I of R is said to be strongly irreducible if for each ideals J, K of R, $J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$ (see [3] and [11]). In this paper, we study weakly 2-irreducible ideals, which are a generalization of weakly irreducible ideals. Recall that 2-irreducible (resp.; n-irreducible) and strongly 2-irreducible (resp.; strongly n-irreducible) ideals, which are a generalization of irreducible ideals, and strongly irreducible ideals were introduced and investigated in [13] and [16] respectively. As usual, if I is a proper ideal of R, then $\sqrt{I}$ denotes the radical ideal of I. Recall from [14] that a proper ideal I of a ring R is said to be a weakly irreducible ideal of R if for each pair of ideals J, K of R, $J \cap K \subseteq I$ implies that either $J \subseteq \sqrt{I}$ or $K \subseteq \sqrt{I}$. Also recall from [13] that an ideal I is called 2-irreducible (resp.; strongly 2-irreducible) if whenever $I = J \cap K \cap L$ (resp.; $J \cap K \cap L \subseteq I$) for ideals J, K and L of R then either $I = J \cap K$ or $I = J \cap L$ or $I = K \cap L$ (resp.; $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$). Obviously, any irreducible ideal (resp.; strongly irreducible ideal) is a 2-irreducible ideal (resp.; strongly 2-irreducible ideal). Now, we recall some definitions which are the motivation of our work. The notion of 2-absorbing ideal, which is a generalization of prime ideal, was introduced by [4] and investigated in [2, 5, 6, 9, 8]. Also the notion of 2-absorbing primary ideal, which is a generalization of primary ideal, was introduced by Badawi, Tekir and Yetkin in [6]. A proper ideal I of R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This concept has a generalization, called weakly 2-absorbing ideals, which has studied in [5]. Also, a proper ideal I of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Note that a 2-absorbing ideal of a commutative ring R is a 2-absorbing primary ideal of R. In [15], Tekir et al. introduced the notion of 2-absorbing quasi-primary ideal which is a generalization of quasi-primary ideal. A proper ideal I is called a 2-absorbing quasi-primary ideal of R if $\sqrt{I}$ is a 2-absorbing ideal of R. It is clear that every 2-absorbing primary ideal of a ring R is a 2-absorbing quasi-primary ideal of R from [6, Theorem 2.2]. However, the converse is not true; for this see [6, Example 2.9].

Motivated by these concepts, in this paper, we introduce the notion of weakly 2-irreducible ideals. A proper ideal I of a ring R is called weakly 2-irreducible ideal if whenever $J \cap K \cap L \subseteq I$ for ideals J, K and L of R, then either $J \cap K \subseteq \sqrt{I}$ or $J \cap L \subseteq \sqrt{I}$ or $K \cap L \subseteq \sqrt{I}$. Clearly, any weakly irreducible ideal is a weakly 2-irreducible ideal. Various properties of weakly 2-irreducible ideals of a ring R are considered.

In Section 2, we give some basic properties of weakly 2-irreducible ideals. For example, we show that if $\sqrt{I}$ strongly 2-irreducible ideal of a ring R, then I is a weakly 2-irreducible ideal of R (Proposition 2.2). We show in Theorem 2.4 that a proper ideal I of a ring R is weakly 2-irreducible if and only if for every elements x, y, z of R, $(Rx+Ry)\cap(Rx+Rz)\cap(Ry+Rz)\subseteq I$ implies that either $(Rx+Ry)\cap(Rx+Rz)\subseteq\sqrt{I}$ or $(Rx+Ry)\cap(Ry+Rz)\subseteq\sqrt{I}$ or $(Rx+Rz)\cap(Ry+Rz)\subseteq\sqrt{I}$. In Theorem 2.6 and Proposition 2.7, we discuss the relationship between weakly 2-irreducible ideals, 2-absorbing ideals and 2-absorbing quasi-primary ideals. After this, we study weakly 2-irreducible ideals in several classes of commutative rings.

In Section 3, we study the stability of weakly 2-irreducible ideals with respect to various ring-theoretic constructions such as localization, factor rings, and idealization. In particular, we show that if I is an ideal of a ring R and X is an indeterminate. Then (I, X) is a weakly 2-irreducible ideal of $R[X]$ if and only if I is a weakly 2-irreducible ideal of R (Theorem 3.4). Moreover, we discuss the relationship between primary ideals and weakly 2-irreducible ideals (Proposition 3.8). Also, we determine the weakly 2-irreducible ideals in the direct product of a finite number of rings and in integral domains built with $D+M$ constructions [7] (Proposition 3.5 and Theorem 3.9).

We next summarize some notations and conventions that are used below. Let R be a ring. Then $Spec(R)$ denotes the set of prime ideals of R, $Max(R)$ denotes the set of maximal ideals of R, $qf(R)$ denotes the quotient field of R when R is an integral domain. As usual, $\mathbb{N}, \mathbb{Z}$ will denote the positive integers and integers, respectively. We will use $\subset$ to denote proper inclusion. For any undefined concepts or terminology, see [10].

2 Basic properties of weakly 2-irreducible ideals

Definition 2.1. We say that a proper ideal I of a ring R is weakly 2-irreducible provided that for each ideals J, K and L of R, $J\cap K\cap L\subseteq I$ implies that either $J\cap K\subseteq\sqrt{I}$ or $J\cap L\subseteq\sqrt{I}$ or $K\cap L\subseteq\sqrt{I}$.

We start with the following two trivial propositions that we omit its proofs.

Proposition 2.2. *Let I be a proper ideal of a ring R. If $\sqrt{I}$ is strongly 2-irreducible, then I is weakly 2-irreducible.*

Proposition 2.3. *If P_1 and P_2 are two weakly irreducible ideals of a commutative ring R, then $P_1\cap P_2$ is a weakly 2-irreducible ideal of R.*

It is clear that every strongly 2-irreducible ideal of a ring R is a weakly 2-irreducible ideal of R. But the converse is not true in general. For example, let $R=\mathbb{Z}[X, Y, Z]$ and let $I=(XYZ, Y^3, X^3)R$. According to [6, Example 2.7], I is not a 2-absorbing primary ideal of R, and since R is Noetherian, then [13, Corollary 4] ensures that I is not strongly 2-irreducible. On the other hand, $\sqrt{I}=XR\cap YR$ is an intersection of two prime ideals. Hence, by [13, Proposition 3], $\sqrt{I}$ is a strongly 2-irreducible ideal of R. Therefore, I is weakly 2-irreducible by Proposition 2.2.

Theorem 2.4. *Let I be a proper ideal of a ring R. Then the following conditions are equivalent:*

(i) I is weakly 2-irreducible;

(ii) For every elements x, y, z of R, $(Rx+Ry)\cap(Rx+Rz)\cap(Ry+Rz)\subseteq I$ implies that either $(Rx+Ry)\cap(Rx+Rz)\subseteq\sqrt{I}$ or $(Rx+Ry)\cap(Ry+Rz)\subseteq\sqrt{I}$ or $(Rx+Rz)\cap(Ry+Rz)\subseteq\sqrt{I}$.

Proof. $(1)\Rightarrow(2)$ This follows by the definition.

$(2)\Rightarrow(1)$ Suppose that J, K and L are ideals of R such that $J\cap K\cap L\subseteq I$ and neither $J\cap K\subseteq\sqrt{I}$ nor $J\cap L\subseteq\sqrt{I}$ nor $K\cap L\subseteq\sqrt{I}$. Then there exist elements x, y and z of R such that $x\in(J\cap K)\setminus\sqrt{I}$, $y\in(J\cap L)\setminus\sqrt{I}$ and $z\in(K\cap L)\setminus\sqrt{I}$. On the other hand $(Rx+Ry)\cap(Rx+Rz)\cap(Ry+Rz)\subseteq(Rx+Ry)\subseteq J$, $(Rx+Ry)\cap(Rx+Rz)\cap(Ry+Rz)\subseteq(Rx+Rz)\subseteq K$ and $(Rx+Ry)\cap(Rx+Rz)\cap(Ry+Rz)\subseteq(Ry+Rz)\subseteq L$. Hence, $(Rx+Ry)\cap(Rx+Rz)\cap(Ry+Rz)\subseteq I$, and so by hypothesis either $(Rx+Ry)\cap(Rx+Rz)\subseteq\sqrt{I}$ or $(Rx+Ry)\cap(Ry+Rz)\subseteq\sqrt{I}$ or $(Rx+Rz)\cap(Ry+Rz)\subseteq\sqrt{I}$. Therefore, either $x\in\sqrt{I}$ or $y\in\sqrt{I}$ or $z\in\sqrt{I}$. which any of these cases has a contradiction. Consequently I is a weakly 2-irreducible ideal of R. □

In the next, we study the relations between weakly 2-irreducible ideals, 2-absorbing ideals, 2-absorbing primary ideals and 2-absorbing quasi-primary ideals of a ring R.

Proposition 2.5. *If I is a 2-absorbing primary ideal of R, then I is a weakly 2-irreducible ideal of R.*

Proof. Since I is a 2-absorbing primary ideal of R, then according to [6, Theorem 2.3], either $\sqrt{I}$ is a prime ideal of R or $\sqrt{I}$ is exactly the intersection of two primes ideals of R. If $\sqrt{I}$ is a prime ideal, hence $\sqrt{I}$ is a strongly irreducible ideal by [11, Lemma 2.2 (2)] and so $\sqrt{I}$ is strongly 2-irreducible. If $\sqrt{I}$ is the intersection of two prime ideals, then $\sqrt{I}$ is a strongly 2-irreducible ideal by [13, Proposition 3]. Hence, I is a weakly 2-irreducible ideal of R by Proposition 2.2. □

Theorem 2.6. *Let I be a proper ideal of a ring R. Then the following are equivalent:*

(i) $\sqrt{I}$ is a weakly 2-irreducible ideal of R;

(ii) $\sqrt{I}$ is a 2-absorbing ideal of R;

(iii) I is a 2-absorbing quasi-primary ideal of R.

Proof. Suppose I is a proper ideal of R. $(2) \Leftrightarrow (3)$ follows from [15, Definition 2.4].

$(1) \Rightarrow (2)$ Suppose that $\sqrt{I}$ is a weakly 2-irreducible ideal of R. Let J, K and L be ideals of R such that $JKL \subseteq \sqrt{I}$. Hence, $J \cap K \cap L \subseteq \sqrt{J \cap K \cap L} \subseteq \sqrt{\sqrt{I}} = \sqrt{I}$. So, either $J \cap K \subseteq \sqrt{I}$ or $J \cap L \subseteq \sqrt{I}$ or $K \cap L \subseteq \sqrt{I}$. Then, either $JK \subseteq \sqrt{I}$ or $JL \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$. Consequently, $\sqrt{I}$ is a 2-absorbing ideal of R by [4, Theorem 2.13].

$(3) \Rightarrow (1)$ If I is a 2-absorbing quasi-primary ideal of R, then either $\sqrt{I}$ is a prime ideal or is an intersection of exactly two prime ideals by [15, Theorem 2.15]. Since any prime ideal is strongly irreducible by [11, Lemma 2.2(2)]. It follows from [13, Proposition 3] that $\sqrt{I}$ is a strongly 2-irreducible ideal of R, and hence $\sqrt{I}$ is a weakly 2-irreducible ideal of R. □

A commutative ring R is called a von Neumann regular ring (or an absolutely flat ring) if for any $a \in R$ there exists an $x \in R$ with $a^2x = a$, equivalently, $I^2 = I$ for each ideal I of R. Note that every von Neumann regular ring is a Boolean ring. Since a ring R is von Neumann regular if and only if $\sqrt{I} = I$ for every proper ideal I of R, we have the following result.

Proposition 2.7. *Let R be a von Neumann regular ring and I be a proper ideal of R. Then the following statements are equivalent:*

(i) I is a weakly 2-irreducible ideal of R;

(ii) I is a 2-absorbing ideal of R;

(iii) I is a 2-absorbing quasi-primary ideal of R.

Recall that a ring R is said to be a Laskerian ring, if every proper ideal of R has a primary decomposition. We know that every Noetherian ring is a Laskerian ring.

Proposition 2.8. *In Laskerian ring R, every weakly 2-irreducible ideal is 2-absorbing quasi-primary.*

Proof. Let I be a weakly 2-irreducible ideal of R and $\bigcap_{i=1}^{n} Q_i$ be a minimal primary decomposition of I. Then there are $1 \leq r, s \leq n$ such that $Q_r \cap Q_s \subseteq \sqrt{I} = \sqrt{\bigcap_{i=1}^{n} Q_i} \subseteq \sqrt{Q_r \cap Q_s}$, and hence $Q_r \cap Q_s = \sqrt{Q_r \cap Q_s} = \sqrt{I}$, namely, $\sqrt{I}$ is an intersection of two prime ideals. Thus, $\sqrt{I}$ is a 2-absorbing ideal, that is, I is a 2-absorbing quasi-primary ideal. □

We recall from [1] that an integral domain R is called a GCD-domain if any two nonzero elements of R have a greatest common divisor (GCD), equivalently, any two nonzero elements of R have a least common multiple (LCM). Unique factorization domains (UFD's) are well-known examples of GCD-domains. Let R be a GCD-domain, we denote the least common multiple of every two elements $x, y \in R$ by $[x, y]$. Notice that for every elements x, y of R, $Rx \cap Ry = R[x, y]$. Moreover, for every elements $x, y, z \in R$, we have $[[x, y], z] = [x, [y, z]]$. So we denote $[[x, y], z]$ by $[x, y, z]$.

Theorem 2.9. *Let I be a proper ideal of a ring R.*

(i) If R is a GCD domain, then I is weakly 2-irreducible if and only if for each x, y, z of R, $[x, y, z] \in I$ implies that either $[x, y] \in \sqrt{I}$ or $[x, z] \in \sqrt{I}$ or $[y, z] \in \sqrt{I}$.

(ii) If R is a UFD domain, then I is weakly 2-irreducible if and only if $p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \in I$, where p_i are distinct prime elements of R and n_i are natural numbers, implies that $p_r^{n_r} p_s^{n_s} \in \sqrt{I}$ for some $1 \leq r, s \leq k$.

Proof. (1) Let I be a weakly 2-irreducible ideal of R and for x, y, z of R, $[x, y, z] \in I$. If $[x, y, z] = c$, then $Rx \cap Ry \cap Rz \subseteq I$. Hence, either $Rx \cap Ry \subseteq \sqrt{I}$ or $Rx \cap Rz \subseteq \sqrt{I}$ or $Ry \cap Rz \subseteq \sqrt{I}$. Moreover, since for every elements x, y of R we have $Rx \cap Ry = R[x, y]$, it follows that either $[x, y] \subseteq \sqrt{I}$ or $[x, z] \subseteq \sqrt{I}$ or $[y, z] \subseteq \sqrt{I}$.

Conversely, if $Rx \cap Ry \cap Rz \subseteq I$ for $x, y, z \in R$, then $[x, y, z] \in Rx \cap Ry \cap Rz \subseteq I$. Hence, by our assumption, either $[x, y] \in \sqrt{I}$ or $[x, z] \in \sqrt{I}$ or $[y, z] \in \sqrt{I}$.

(2) A slight modification of the proof of [13, Theorem 10(2)]. Suppose that I is weakly 2-irreducible and $p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \in I$ in which p_i's are distinct prime elements of R and n_i's are natural numbers. Then $[p_1^{n_1}, p_2^{n_2}, \ldots, p_k^{n_k}] = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} \in I$. Hence, by part (1), there are $1 \leq r, s \leq k$ such that $[p_r^{n_r}, p_s^{n_s}] = p_r^{n_r} p_s^{n_s} \in \sqrt{I}$.

Conversely, let $[x, y, z] \in I$ for $x, y, z \in R \setminus \{0\}$, and

$$x = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$$

$$y = p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_k^{\gamma_k} r_1^{\delta_1} r_2^{\delta_2} \ldots r_u^{\delta_u}$$

$$z = p_1^{\epsilon_1} p_2^{\epsilon_2} \ldots p_{k'}^{\epsilon_{k'}} q_1^{\lambda_1} q_2^{\lambda_2} \ldots q_{s'}^{\lambda_{s'}} r_1^{\mu_1} r_2^{\mu_2} \ldots r_{u'}^{\mu_{u'}} s_1^{\kappa_1} s_2^{\kappa_2} \ldots s_v^{\kappa_v}$$

be prime decompositions for x, y and z, respectively. Therefore,

$$[x, y, z] = p_1^{\nu_1} p_2^{\nu_2} \ldots p_{k'}^{\nu_{k'}} p_{k'+1}^{\omega_{k'+1}} \ldots p_k^{\omega_k} q_1^{\rho_1} q_2^{\rho_2} \ldots q_{s'}^{\rho_{s'}} q_{s'+1}^{\beta_{s'+1}} \ldots q_s^{\beta_s}$$

$$r_1^{\sigma_1} r_2^{\sigma_2} \ldots r_{u'}^{\sigma_{u'}} r_{u'+1}^{\delta_{u'+1}} \ldots r_u^{\delta_u} s_1^{\kappa_1} s_2^{\kappa_2} \ldots s_v^{\kappa_v} \in I,$$

where $\nu_i = \max\{\alpha_i, \gamma_i, \varepsilon_i\}$ for every $1 \leq i \leq k'$, $\omega_j = \max\{\alpha_j, \gamma_j\}$ for every $k' \leq j \leq k$; $\rho_i = \max\{\beta_i, \lambda_i\}$ for every $1 \leq i \leq s'$; $\sigma_i = \max\{\delta_i, \mu_i\}$ for every $1 \leq i \leq u'$.
By part (1), we have twenty one cases. For example we investigate the following two cases. The other cases can be verified in a similar way.
Case 1. For some $1 \leq i, j \leq k'$, $p_i^{\nu_i} p_j^{\nu_j} \in \sqrt{I}$. If $\nu_i = \alpha_i$ and $\nu_j = \alpha_j$, then $x \in \sqrt{I}$ and hence $[x, y] \in \sqrt{I}$. If $\nu_i = \alpha_i$ and $\nu_j = \gamma_j$, then $p_i^{\alpha_i} p_j^{\gamma_j} | [x, y]$ and thus $[x, y] \in \sqrt{I}$. If $\nu_i = \alpha_i$ and $\nu_j = \epsilon_j$, then $p_i^{\alpha_i} p_j^{\epsilon_j} | [x, z]$ and thus $[x, z] \in \sqrt{I}$.
Case 2. Let $p_i^{\nu_i} p_j^{\omega_j} \in \sqrt{I}$ for some $1 \leq i \leq k'$ and $k' + 1 \leq j \leq k$. If $\nu_i = \alpha_i$ and $\omega_j = \alpha_j$, then $x \in \sqrt{I}$ and hence $[x, y] \in \sqrt{I}$. If $\nu_i = \epsilon_i$ and $\omega_j = \gamma_j$, then $[y, z] \in \sqrt{I}$. Hence, I is a weakly 2-irreducible ideal of R. □

3 Extensions of weakly 2-irreducible ideals

In this section, we investigate the stability of weakly 2-irreducible ideals in various ring-theoretic constructions.

Theorem 3.1. *Let $f : R \longrightarrow S$ be a surjective homomorphism of commutative rings, and let I be an ideal of R containing* $\ker(f)$*. Then, I is a weakly 2-irreducible ideal of R if and only if $f(I)$ is a weakly 2-irreducible ideal of S. In particular, this holds if f is an isomorphism.*

Proof. Since f is surjective, then $f(I' \cap R) = I'$ for every ideal I' of S. Moreover, $f(K \cap L) = f(K) \cap f(L)$, $f(K) \cap R = K$ for every ideals K, L of R with contain $\ker(f)$.

Suppose that I is a weakly 2-irreducible ideal of R. then $I = f(I) \cap R = R$, which is a contradiction. Let J', K' and L' be ideals of S such that $J' \cap K' \cap L' \subseteq f(I)$. Then, $f^{-1}(J' \cap K' \cap L') = f^{-1}(J') \cap f^{-1}(K') \cap f^{-1}(L') \subseteq f^{-1}(f(I)) = I$. Hence, either $f^{-1}(J') \cap f^{-1}(K') \subseteq \sqrt{I}$ or $f^{-1}(J') \cap f^{-1}(L') \subseteq \sqrt{I}$ or $f^{-1}(K') \cap f^{-1}(L') \subseteq \sqrt{I}$. So, either $J' \cap K' \subseteq f(\sqrt{I}) \subseteq \sqrt{f(I)}$

or $J' \cap L' \subseteq f(\sqrt{I}) \subseteq \sqrt{f(I)}$ or $K' \cap L' \subseteq f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Consequently, $f(I)$ is a weakly 2-irreducible ideal of S.

Conversely, let $f(I)$ be a weakly 2-irreducible ideal of S, and let J, K and L be ideals of R such that $J \cap K \cap L \subseteq I$. Then, $f(J \cap K \cap L) = f(J) \cap f(K) \cap f(L) \subseteq f(I)$. Hence, either $f(J) \cap f(K) \subseteq \sqrt{f(I)}$ or $f(J) \cap f(L) \subseteq \sqrt{f(I)}$ or $f(K) \cap f(L) \subseteq \sqrt{f(I)}$. We may assume that $f(J) \cap f(K) \subseteq \sqrt{f(I)}$. Therefore, $f^{-1}(f(J) \cap f(K)) = J \cap K \subseteq f^{-1}(\sqrt{f(I)}) \subseteq \sqrt{f^{-1}f(I)} = \sqrt{I}$. Consequently, I is weakly 2-irreducible. □

Corollary 3.2. *Let $f : R \longrightarrow S$ be a surjective homomorphism of commutative rings. There is a one-to-one correspondence between the weakly 2-irreducible ideals of R which contain* $\ker(f)$ *and weakly 2-irreducible ideals of S.*

Corollary 3.3. *Let $I \subseteq J$ be ideals of a ring R. Then, I is a weakly 2-irreducible ideal of R, then J/I is a weakly 2-irreducible ideal of R/I.*

Proof. Let $\pi : R \longrightarrow R/I$ be the natural homomorphism. Note that $\ker(\pi) = I \subseteq J$. By Theorem 3.1, I is a weakly 2-irreducible ideal of R if and only if $f(I) = J/I$ is a weakly 2-irreducible ideal of R/J. □

We next briefly consider extensions of weakly 2-irreducible ideals of R in the polynomial ring $R[X]$.

Theorem 3.4. *Let I be an ideal of a ring R. Then, (I, X) is a weakly 2-irreducible ideal of $R[X]$ if and only if I is a weakly 2-irreducible ideal of R.*

Proof. This follows directly from Corollary 3.3 since $(I, X)/(X) \cong I$ in $R[X]/(X) \cong R$. □

We also investigate weakly 2-irreducible ideals for the "$D+M$" construction. Let $T = K+M$ be an integral domain, where K is a field which is a subring of T and M is a nonzero maximal ideal of T, and let D be a subring of K. Then $R = D+M$ is a subring of T with $qf(R) = qf(T)$. This construction has proved very useful for constructing examples [7].

Proposition 3.5. *Let $T = K + M$ be an integral domain, where K is a field which is a subring of T and M is a nonzero maximal ideal of T. Let D be a subring of K and $R = D + M$. Let I be an ideal of D. Then $I + M$ is a weakly 2-irreducible ideal of R if and only if I is a weakly 2-irreducible ideal of D.*

Proof. This follows directly from Corollary 3.3(b) since $(I + M)/M \cong I$ in $R/M \cong D$. □

Let S be a multiplicatively closed subset of a ring R. In the next theorem, consider the natural homomorphism $f : R \to S^{-1}R$ defined by $f(x) = x/1$. For each ideal I of the ring $S^{-1}R$, we consider $I^c = \{x \in R |\ x/1 \in I\} = I \cap R$ and $C = \{I^c |\ I \text{ is an ideal of } S^{-1}R\}$.

Lemma 3.6. *Let I be a proper ideal of a ring R and S be a multiplicatively closed set in R. If I is a weakly 2-irreducible ideal of $S^{-1}R$, then I^c is a weakly 2-irreducible ideal of R.*

Proof. Assume I is a weakly 2-irreducible ideal of $S^{-1}R$. Let J, K, L be ideals of R such that $J \cap K \cap L \subseteq I^c$, then $S^{-1}J \cap S^{-1}K \cap S^{-1}L \subseteq S^{-1}I^c = I$. Hence, either $S^{-1}J \cap S^{-1}K \subseteq \sqrt{I}$ or $S^{-1}J \cap S^{-1}L \subseteq \sqrt{I}$ or $S^{-1}K \cap S^{-1}L \subseteq \sqrt{I}$ since I is weakly 2-irreducible. Then, either $J \cap K \subseteq (\sqrt{I})^c = \sqrt{I^c}$ or $J \cap L \subseteq (\sqrt{I})^c = \sqrt{I^c}$ or $K \cap L \subseteq (\sqrt{I})^c = \sqrt{I^c}$. Thus I^c is a weakly 2-irreducible ideal of R. □

Theorem 3.7. *Let R be a ring and S be a multiplicatively closed set of R. Then there is a one-to-one correspondence between the weakly 2-irreducible ideals of $S^{-1}R$ and weakly 2-irreducible ideals of R contained in C which do not meet S.*

Proof. Let I be a weakly 2-irreducible ideal of $S^{-1}R$. Obviously, $I^c \neq R$, $I^c \in C$ and $I^c \cap S = \emptyset$. By Lemma 3.6, I^c is a weakly 2-irreducible ideal of R. Conversely, let I be a weakly 2-irreducible ideal of R, $I \cap S = \emptyset$ and $I \in C$. Since, $I \cap S = \emptyset$, then $S^{-1}I \neq S^{-1}R$. Let J, K and L be ideals of $S^{-1}R$ such that $J \cap K \cap L \subseteq S^{-1}I$. Then $J^c \cap K^c \cap L^c = (J \cap K \cap L)^c \subseteq (S^{-1}I)^c = I$ since $I \in C$. It follows that either $J^c \cap K^c \subseteq \sqrt{I}$ or $J^c \cap L^c \subseteq \sqrt{I}$ or $K^c \cap L^c \subseteq \sqrt{I}$. Therefore, $J \cap K = (S^{-1}(J \cap K))^c \subseteq S^{-1}(\sqrt{I}) \subseteq \sqrt{S^{-1}I}$ or $J \cap L \subseteq \sqrt{S^{-1}I}$ or $K \cap L \subseteq \sqrt{S^{-1}I}$. Then, $S^{-1}I$ is a weakly 2-irreducible ideal of $S^{-1}R$. □

We now consider the relationship between weakly 2-irreducible ideals and primary ideals.

Proposition 3.8. *Let R be a ring. Then the following conditions are equivalent:*

(i) Every primary ideal of R is a weakly 2-irreducible ideal;

(ii) For any prime ideal P of R, every primary ideal of R_P is a weakly 2-irreducible ideal;

(iii) For any maximal ideal M of R, every primary ideal of R_M is a weakly 2-irreducible ideal.

Proof. $(1) \Rightarrow (2)$. Let I be a primary ideal of R_P. We know that I^c is a primary ideal of R, $I^c \cap (R \setminus P) = \emptyset$, $I^c \in C$ and by the assumption, I^c is a weakly 2-irreducible ideal of R. Now, by Theorem 3.7, $I = (I^c)_P$ is a weakly 2-irreducible ideal of R_P.

$(2) \Rightarrow (3)$. The proof is clear.

$(3) \Rightarrow (1)$ Let I be a primary ideal of R and let M be a maximal ideal of R containing I. Then, I_M is a primary ideal of R_M and so, by our assumption, I_M is a weakly 2-irreducible ideal of R_M. Now by Lemma 3.6, $(I_M)^c$ is a weakly 2-irreducible ideal of R, and since I is a primary ideal of R, $(I_M)^c = I$, that is, I is a weakly 2-irreducible ideal of R. □

We next determine the weakly 2-irreducible ideals in the product of two, and hence any finite number of rings. Recall that an ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for ideals I_i of R_i.

Theorem 3.9. *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with $1 \neq 0$. Let J be a proper ideal of R. Then the following statements are equivalent:*

(i) J is a weakly 2-irreducible ideal of R;

(ii) Either $J = I_1 \times R_2$ for some weakly 2-irreducible ideal I_1 of R_1 or $J = R_1 \times I_2$ for some weakly 2-irreducible ideal I_2 of R_2 or $J = I_1 \times I_2$ for some weakly irreducible ideal I_1 of R_1 and some weakly irreducible ideal I_2 of R_2.

Proof. A slight modification of the proof of [13, Theorem 8]. $(1) \Rightarrow (2)$ Assume that J is a weakly 2-irreducible ideal of R. Then $J = I_1 \times I_2$ for some ideals I_1 of R_1 and I_2 of R_2. Suppose that $I_2 = R_2$. Since J is a proper ideal of R, $I_1 \neq R_1$. Let $R' = \frac{R}{\{0\} \times R_2}$. Then $J' = \frac{J}{\{0\} \times R_2}$ is a weakly 2-irreducible ideal of R' by Corollary 3.3(b). Since R' ring-isomorphic to R_1 and $I_1 \simeq J'$, I_1 is a weakly 2-irreducible of R_1. If $I_1 = R_1$. By a similar argument as in the previous case, we prove that I_2 is a weakly 2-irreducible of R_2. Hence assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Suppose that I_1 is not a weakly irreducible ideal of R_1. Then there are K_1, L_1 be ideals of R_1 such that $R_1K_1 \cap R_1L_1 \subseteq I_1$ and neither $K_1 \subseteq \sqrt{I_1}$ nor $L_1 \subseteq \sqrt{I_1}$. Notice that $(R_1K_1 \times R_2) \cap (R_1 \times \{0\}) \cap (R_1L_1 \times R_2) = (R_1K_1 \cap R_1L_1) \times \{0\} \subseteq J$, but neither $(R_1K_1 \times R_2) \cap (R_1 \times \{0\}) = R_1K_1 \times \{0\} \subseteq \sqrt{J}$ nor $(R_1K_1 \times R_2) \cap (R_1L_1 \times R_2) = (R_1K_1 \cap R_1L_1) \times R_2 \subseteq \sqrt{J}$ nor $(R_1 \times \{0\}) \cap (R_1L_1 \times R_2) = R_1L_1 \times \{0\} \subseteq \sqrt{J}$, which is a contradiction. Thus I_1 is a weakly irreducible ideal of R_1. By a similar argument we show that I_2 is a weakly irreducible ideal of R_2.

$(2) \Rightarrow (1)$ If $J = I_1 \times R_2$ for some weakly 2-irreducible ideal I_1 of R_1 or $J = R_1 \times I_2$ for some weakly 2-irreducible ideal I_2 of R_2, then it is clear that J is a weakly 2- irreducible ideal of R. Hence assume that $J = I_1 \times I_2$ for some weakly irreducible ideal I_1 of R_1 and some weakly irreducible ideal I_2 of R_2. Then $I'_1 = I_1 \times R_2$ and $I'_2 = R_1 \times I_2$ are weakly irreducible ideals of R. Hence, $I'_1 \cap I'_2 = I_1 \times I_2 = J$ is a weakly 2-irreducible ideal of R by Proposition 2.3. □

Corollary 3.10. *Let $R = R_1 \times \ldots \times R_n$, where $2 \leq n \leq \infty$, and $R_1, \ldots, R_n$ are a rings. Let J be a proper ideal of R. Then the following conditions are equivalent:*

(i) J is a weakly 2-irreducible ideal of R;

(ii) Either $J = \times_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, \ldots, n\}$, I_k is a weakly 2-irreducible ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $J = \times_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, \ldots, n\}$ such that I_k is a weakly irreducible ideal of R_k, I_m is a weakly irreducible ideal of R_m, and $I_t = R_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k, m\}$.

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