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A CLASSIFICATION OF IDEMPOTENT ELEMENTS OF COMPLEX CLIFFORD ALGEBRAS

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Abstract In this paper we will give a classification of idempotent elements of complex Clifford algebras $\mathbb{C}l(p,q)$ via the trace operation. Using isomorphisms between Clifford algebras $\mathbb{C}l(p,q)$ and appropriate matrix algebras, a geometric and algebraic description of each class is studied.

1 Introduction

In mathematics, a Clifford algebra is an algebra generated by a vector space with a quadratic form, and is a unital associative algebra. As $\mathbb{K}$-algebras, they generalize the real numbers, complex numbers, quaternions and several other hypercomplex number systems [4]. The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. It is well known that Clifford algebras are defined by symmetric bilinear forms [10, 3]: Given a symmetric bilinear form B on a vector space E, one defines the Clifford algebra $Cl(E,B)$ to be the associative algebra with unity e generated by the elements of E , with relations $vu+uv=2B(u,v)e,\ u,v\in E$ [7]. The Clifford algebra $Cl(E,B)$ can be regarded as the factor algebra $\otimes E/J$, where $\otimes E$ denotes the tensor algebra of E and J is the ideal of $\otimes E$ generated by the elements $u\otimes v+v\otimes u-2B(u,v)e,\ u,v\in E$ [3]. Clifford algebras and spinors have been used to describe electromagnetic fields, supersymmetry, and quantum mechanics. A problem that is frequently arises is how to project onto a specific invariant irreducible subspace of the spinor space. This problem is related to constructing and classifying idempotents of the corresponding Clifford algebras. Discrete families of primitive idempotents have been described by Chevalley [3] and are traditionally used to generate spinor representations of Clifford algebras [14]. Our goal here is to classify idempotents for arbitrary Clifford algebras $\mathbb{C}l(p,q)$, we can use its matrix representations and operation trace. The present paper is organized as follows. After introduction, section 2 is concerned with recalling some facts about the Clifford algebras $\mathbb{C}l(p,q)$. In section 3, we will give a classification of idempotents of $\mathbb{C}l(p,q)$ via the operation trace, some examples are given. A geometric description of idempotent classes is given in section 4. Finally in section 5, using a Hermitian product on the Clifford algebras $\mathbb{C}l(p,q)$, we will give an application of idempotents to make a matrix representations of $\mathbb{C}l(p,q)$.

2 Basic results of the Clifford algebra $\mathbb{C}l(p,q)$

Given p,q and n a non-negative integers such that $n=p+q$. We denote $Cl(p,q)$ the Clifford algebra of the quadratic space $\mathbb{R}^{p,q}$ and $\mathbb{C}l(p,q):=\mathbb{C}\otimes Cl(p,q)$ the complexifed algebra of $Cl(p,q)$. If $e_i, 1\le i\le n$ is an orthonormal basis of $\mathbb{R}^{p,q}$, then $\mathbb{C}l(p,q)$ is generated by e_i with relations

$$e_ie_j+e_je_i=2g_{ij}e,\ 1\le i,j\le n,$$

where e is the unitary element of $\mathbb{C}l(p,q)$, $g_{ii}=1$ if $1\le i\le p$, $g_{ii}=-1$ if $p+1\le i\le n$ and $g_{ij}=0$ if $i\ne j$. It has basis

$$e,e_i,e_{i_1}e_{i_2},...,e_1...e_n,\ 1\le i_1<i_2<...\le n, \tag{2.1}$$

with $i, i_1, ...$ are indexes from 1 to n. Thus $\mathbb{C}l(p,q)$ is 2^n-dimensional complex vector space [7]. Throughout this paper e_I will denote the product element $e_{i_1} \dots e_{i_k}$ of $\mathbb{C}l(p,q)$ for any $I = \{i_1, \dots, i_k\}$ with $1 \le i_1 < i_k \dots < i_k \le n$, and $e_\emptyset := e$. So, any Clifford algebra element $X \in \mathbb{C}l(p,q)$ can be written in the following form

$$X = xe + \sum_{I \neq \emptyset} \lambda_I e_I, \tag{2.2}$$

where x, λ_I are complex constants.

Complex Clifford algebras $\mathbb{C}l(p,q)$ of dimension n and different signatures (p,q), $p+q=n$ are isomorphic. Clifford algebras $\mathbb{C}l(p,q)$ are isomorphic to the matrix algebras of complex matrices. In the case of even n, these matrices are of order $2^{\frac{n}{2}}$. In the case of odd n, these matrices are block diagonal of order $2^{\frac{n+1}{2}}$ with 2 blocks of order $2^{\frac{n-1}{2}}$ [14, 7]. Precisely, we have the following well-known matrix-representations of complex Clifford algebras (of minimal dimension)

$$Cl(\mathbb{C}^n) \cong \mathbb{C}l(p,q) \cong \begin{cases} Mat(2^{\frac{n}{2}}, \mathbb{C}) & \text{if } n \text{ is even,} \\ Mat(2^{\frac{n-1}{2}}, \mathbb{C}) \oplus Mat(2^{\frac{n-1}{2}}, \mathbb{C}) & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, these Clifford algebras $\mathbb{C}l(p,q)$ are a simple algebras (if n is even) or a semi-simple algebras (if n is odd).

For any element $t \in \mathbb{C}l(p,q)$, the set $S(t) := \{Xt / X \in \mathbb{C}l(p,q)\}$ is a left ideal of $\mathbb{C}l(p,q)$ (generated by t). A left ideal that does not contain other left ideals except itself and the trivial ideal $\{0\}$, is called a minimal left ideal of $\mathbb{C}l(p,q)$.

3 Classification of idempotent elements of $\mathbb{C}l(p,q)$

Let us recall some facts about idempotent elements of $\mathbb{C}l(p,q)$.

(i) An element t of $\mathbb{C}l(p,q)$ is said to be idempotent if $t^2 = t$.

(ii) Two idempotents $t, t' \in \mathbb{C}l(p,q)$ are said to be conjugate if there exists an invertible element $x \in \mathbb{C}l(p,q)$ such that $t' = xtx^{-1}$.

(iii) Two idempotents $t, t' \in \mathbb{C}l(p,q)$ are said to be orthogonal if $tt' = t't = 0$.

(iv) An idempotent $t \in \mathbb{C}l(p,q)$ is said to be primitive if it is not a sum of two nonzero orthogonal idempotents.

(v) An idempotent is said to be minimal if it is minimal element in the set of all nonzero idempotents with order relation $t \le t'$ if and only if $tt' = t = t't$.

These last two properties of an idempotent t are equivalent. More precisely, we have the following proposition.

Proposition 3.1. *Let t be a nonzero idempotent of $\mathbb{C}l(p,q)$. Then the following properties are equivalent.*

(i) t is primitive.

(ii) t is minimal.

(iii) t is the only nonzero idempotent in the subalgebra $t\mathbb{C}l(p,q)t$, where

$$t\mathbb{C}l(p,q)t := \{txt / \, x \in \mathbb{C}l(p,q)\}.$$

Proof. Assume that t is primitive. And given a nonzero idempotent t' of $\mathbb{C}l(p,q)$ such that $t' \le t$ then $t = t' + (e - t')t$ with $(e-t')t$ is an idempotent orthogonal with t. Since t is primitive then $(e-t')t = 0$ and so t is minimal. Thus (1) implies (2).
Assume that t is minimal. Set $f \in t\mathbb{C}l(p,q)t$ be a nonzero idempotent element, then $f = tf = ft$ and so $f \le t$. By minimality of t we deduce that $t = f$. So, t is the only nonzero idempotent element of $t\mathbb{C}l(p,q)t$. Thus, (2) implies (3).

Assume that t is the only nonzero idempotent element of $t\mathbb{C}l(p,q)t$. Set f and h two orthogonal idempotents such that $t = f + h$, then f, h are two idempotents of $t\mathbb{C}l(p,q)t$ and so f or h is zero. So, (3) implies (1). □

To classify idempotents of $\mathbb{C}l(p,q)$, we introduce the operation of trace of Clifford algebra element $X \in \mathbb{C}l(p,q)$ as the following operation of projection onto subspace $\mathbb{C}.e$: for arbitrary element $X \in \mathbb{C}l(p,q)$ in the form (2.2) we have $tr(X = xe + \sum \lambda_I e_I) = x$.
It is easy to see that the trace map $tr : \mathbb{C}l(p,q) \longrightarrow \mathbb{C}.e$ has the following properties:

$$tr(\lambda X + Y) = \lambda tr(X) + tr(Y) \textit{ and } tr(XY) = tr(YX),$$

$$\forall X, Y \in \mathbb{C}l(p,q),\ \forall \lambda \in \mathbb{C}.$$

There is a relationship between operation trace tr of Clifford algebra element $X \in \mathbb{C}l(p,q)$ and operation trace Tr of quadratic matrix. The following lemma gives this relation.

Lemma 3.2. $tr(X) = \frac{1}{2^{[\frac{n+1}{2}]}} Tr(\rho(X))$*, for all* $X \in \mathbb{C}l(p,q)$*, where* ρ *is any arbitrary matrix-representation of* $\mathbb{C}l(p,q)$ *(of minimal dimension).*

Proof. By Pauli's Theorem (see [15]). The trace $Tr(\rho(X))$ is doesn't depend on the choice of matrix representation. So, let us consider the recurrent matrix representation $\rho : e_I \longmapsto \beta_I$ given by (Theorem 4.1 [16]). We have

$$tr(e_I) = Tr(\beta_I) = 0, \text{ where } I\text{- any multi-index except empty.}$$

The only exception is identity element e, which corresponds to the identity matrix. In this case, we have $Tr(\rho(e)) = 2^{[\frac{n+1}{2}]}$. Further, we use linearity of trace and obtain

$$Tr(\rho(X)) = Tr(x\rho(e)) = Tr(\rho(e))x = 2^{[\frac{n+1}{2}]}x = 2^{[\frac{n+1}{2}]}tr(X).$$

Hence, we get the result. □

Remark 3.3. For even n, we can show the previous lemma using the following formula $g : \rho(\mathbb{C}l(p,q)) \longrightarrow \mathbb{C}$, $A \longmapsto tr(\rho^{-1}(A))$ which is a linear map satisfies $g(AB) = g(BA)$, then there exists $\lambda \in \mathbb{C}$ such that $g = \lambda Tr$. By a simple calculation, we find $\lambda = 2^{[\frac{n+1}{2}]}$.

The set that we denote $\mathcal{D}$, of nonzero idempotent elements of $\mathbb{C}l(p,q)$ can be provided with the equivalence relation $\mathcal{R}$ given by $t\mathcal{R}t'$ if and only if $tr(t) = tr(t')$. For any element t of $\mathcal{D}$ let us denote $r(t) = Tr(\rho(t)) = 2^{[\frac{n+1}{2}]}tr(t)$ and $\mathcal{D}_{r(t)}$ the equivalence class of t with respect to $\mathcal{R}$.

When $n = 2d$ is even we have the following theorem.

Theorem 3.4. *Let* t *and* t' *tow elements in* $\mathcal{D}$*, when* $n = 2d$ *is even, we have*

(i) $t\mathcal{R}t'$ *if and only if* t *and* t' *are conjugated.*

(ii) $r(t)$ *is an integer between 1 and* $2^{[\frac{n+1}{2}]}$.

(iii) t *is minimal if and only if* $r(t) = 1$.

(iv) $\dim(S(t)) = 2^{[\frac{n}{2}]}r(t)$.

(v) $S(t)$ *is minimal if and only if* t *is minimal.*

The proof of this Theorem requires the following lemma.

Lemma 3.5. *Let* X, Y *two idempotent matrices of* $M(N,\mathbb{C})$*, where* N *is a nonzero integer. Then, the following statements are equivalent.*

(i) $Tr(X) = Tr(Y)$.

(ii) X *and* Y *are similar. That is, there exists* $Z \in GL(N,\mathbb{C})$ *such that* $Y = ZXZ^{-1}$.

Proof. Since X is an idempotent matrix. By means of a similarity transformation, it can be transformed into its Jordan form. Consequently, the idempotency implies that the Jordan form of X must be (up to basis vector transposition) of the form $J_r = \text{diag}(\underbrace{1, \cdots, 1}_{r}, \underbrace{0, \cdots, 0}_{N-r})$, where $r = 0, \cdots, N$ denotes the rank of X. Therefore, X has the form

$$X = SJ_rS^{-1}, \text{ where } S \in GL(N, \mathbb{C}).$$

To finish the proof, just pick up that $Tr(X)$ is equal to the rank of X. □

Proof. Of the above theorem.

(i) This follows easily from the previous Lemma.

(ii) Again by the previous Lemma, we have $r(t) = Tr(\rho(t)) = rank(\rho(t))$, so we get the result.

(iii) If $t = f + h$ where f and h are two nonzero orthogonal idempotent elements, then $r(t) = r(f) + r(h) \geq 2$. It follows that, $r(t) = 1$ implies that t is minimal.
Conversely: By the previous Lemma, one can assume that $\rho(t) = J_{r(t)} = J_1 + (J_{r(t)} - J_1)$, then $t = \rho^{-1}(J_1) + \rho^{-1}\left(J_{r(t)} - J_1\right)$. Therefore $\rho^{-1}(J_1)$ and $\rho^{-1}\left(J_{r(t)} - J_1\right)$ are two orthogonal idempotent elements. It follows that, if t is minimal then $r(t) = 1$.

(iv) Notice that $\dim(S(t)) = \dim(M(2^d, \mathbb{C})J_{r(t)})$, where $M(2^d, \mathbb{C})J_{r(t)}$ denotes the left ideal of $M(2^d, \mathbb{C})$ generated by $J_{r(t)}$. So just determine $\dim(M(2^d, \mathbb{C})J_{r(t)})$. Let denote E_{ij} the canonical basis elements of $M(2^d, \mathbb{C})$. We have $E_{ij}J_{r(t)} = \begin{cases} E_{ij} & \textit{if} \quad j \leq r(t) \\ 0 & \textit{if} \quad j > r(t) \end{cases}$. So $\dim(M(2^d, \mathbb{C})J_{r(t)}) = 2^d r(t)$.

(v) Since any nonzero minimal left ideal of a complex simple algebra is generated by a nonzero minimal idempotent element (see [2]), if $S(t)$ is minimal, then there is a nonzero minimal idempotent element t' such that $S(t) = S(t')$. According to the fourth point of this theorem, we have $r(t) = r(t') = 1$. Hence, t is minimal.
If $S(t)$ is not minimal then, it contains strictly a nonzero minimal left ideal $S(t')$ where t' is a nonzero minimal idempotent element. Hence, $\dim(S(t)) > \dim(S(t'))$ and so $r(t) > r(t') = 1$. Consequently, t is not minimal, by the third point above. □

Remark 3.6. (i) Except for the first one, the properties of the previous theorem remain true when n is odd.

(ii) If $t = f + h$ where f and h are tow orthogonal idempotents, then $S(t) = S(f) \oplus S(h)$.

(iii) When t is a minimal idempotent element, then any non-zero idempotent of $S(t)$ is also minimal.

(iv) If $t_0, \cdots, t_m$ are a commuting idempotent elements of $\mathbb{C}l(p, q)$, then their product $\prod t_k$ is also idempotent.

(v) There are $2^{[\frac{n+1}{2}]}$ classes (types) of idempotent in the Clifford algebra $\mathbb{C}l(p, q)$.

Example 3.7. Set $t_0 := \frac{1}{2}(e - e_1)$ and $t_k := \frac{1}{2}(e - i^{b_k}e_{2k}e_{2k+1})$ for $1 \leq k \leq m$, where $b_k = 0$ for $2k = p$, $b_k = 1$ for $2k \neq p$ and $m = [\frac{n-1}{2}]$.
They are commuting idempotent elements of $\mathbb{C}l(p, q)$ with $r(t_k) = 2^{[\frac{n-1}{2}]}$ for all $0 \leq k \leq m$. Furthermore their product:

$$t := \prod_{k=0}^{m} t_k \tag{3.1}$$

is a minimal idempotent element of $\mathbb{C}l(p, q)$.
Indeed: Firstly all terms in the previous product commute. Thus, t is idempotent.
On the other hand, we can write the idempotent t in the form

$$t = 2^{-(m+1)}e + \sum_{r=1}^{n} \sum_{i_1 < i_2 < \cdots < i_r} \lambda_{i_1 \ldots i_r} e_{i_1} \ldots e_{i_r}, \tag{3.2}$$

where $\lambda_{i_1 \dots i_r} \in \mathbb{C}$ and every term $e_{i_1} \dots e_{i_r}$ contains, at least, one generator e_i with odd index (for even n the idempotent t doesn't contain the generator e_n). Hence, $tr(t) = 2^{-(m+1)}$ and so $r(t) = 2^{[\frac{n+1}{2}]-(m+1)} = 1$. Consequently, by the above theorem, t is a minimal idempotent element, and so the left ideal $S(t)$ is minimal with $2^{[\frac{n}{2}]}$-dimensional.

The following proposition gives a standard form for a minimal idempotent of $\mathbb{C}l(p,q)$ for even n.

Proposition 3.8. *Any minimal idempotent element of $\mathbb{C}l(p,q)$ for even n is necessarily of the form given by Formula (3.1), for some generators γ_i of $\mathbb{C}l(p,q)$ (instead of the e_i) satisfying the relation*

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}e,\ 1 \le i,j \le n\ (*).$$

Proof. We denote by t the minimal idempotent element of $\mathbb{C}l(p,q)$ given by Formula 3.1 in the example above. Let t' be a minimal idempotent element of $\mathbb{C}l(p,q)$. By Theorem 3.4, there is an invertible element T of $\mathbb{C}l(p,q)$ such that $t' = T^{-1}tT$. The family $\gamma_i := T^{-1}e_iT$, $1 \le i \le n$ satisfies the relationship $(*)$. Hence, by Pauli's Theorem (see [15]), γ_i are a generator elements of $\mathbb{C}l(p,q)$ and T is the unique (up to nonzero constant) invertible element of $\mathbb{C}l(p,q)$ satisfying $\gamma_i = T^{-1}e_iT$, $1 \le i \le n$. Replacing the generators e_i by the γ_i, Formula (3.1) gives t'. □

4 Geometric study of idempotent equivalence classes

Using isomorphisms between Clifford algebras $\mathbb{C}l(p,q)$ and appropriate matrix algebras, and the conjugation action of the Clifford group $\mathbb{C}l^\times(p,q)$ on the set $\mathcal{D}$ of nonzero idempotent elements of $\mathbb{C}l(p,q)$, it possible to give a geometrical description for the orbits of this action. These are smooth manifolds in the natural topology.

The case $n = 2d$ even. Set t an idempotent element of ID, and $r = r(t)$. By Theorem 3.4 the t-orbit of the $\mathbb{C}l^\times(p,q)$-action is none other than $\mathcal{D}_r = \{h \in \mathcal{D}/tRh\}$. i.e. the equivalence R-class of t. Let denote G_r the isotropy group at t; that is, the set of all $x \in \mathbb{C}l^\times(p,q)$ such that $xtx^{-1} = t$. G_r is a closed subgroup of the Lie group $\mathbb{C}l^\times(p,q)$. The coset space $\mathbb{C}l^\times(p,q)/G_r$ has a manifold structure [8]. Moreover the mapping

$$\mathbb{C}l^\times(p,q)/G_r \longrightarrow \mathcal{D}_r,\ \overline{x} \longmapsto x.t := xtx^{-1} \tag{4.1}$$

is bijective (see[8]). Consequently, D_r has a manifold structure. More precisely, we give the following result.

Theorem 4.1. *$\mathcal{D}_r$ has a real manifold structure of dimension $4r(2^d - r)$. Precisely, $\mathcal{D}_r$ is homeomorphic to the coset space*

$$\frac{GL(2^d,\mathbb{C})}{GL(r,\mathbb{C}) \times GL(2^d - r,\mathbb{C})}.$$

Proof. let us start with $\mathbb{C}l(p,q) \approx M(2^d,\mathbb{C})$ and so, $D_r \approx \{SJ_rS^{-1}/S \in GL(2^d,\mathbb{C})\}$ - i.e. J_r-orbit of the $GL(2^d,\mathbb{C})$-action- and $G_r \approx \{S \in GL(2^d,\mathbb{C})/SJ_rS^{-1} = J_r\}$; the stability group of J_r. Since any matrix S of G_r is of the form $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$, where $S_1 \in GL(r,\mathbb{C})$ and $S_2 \in GL(2^d - r,\mathbb{C})$, hence $G_r \approx GL(r,\mathbb{C}) \times GL(2^d - r,\mathbb{C})$. Formula (4.1) completes the proof of the Theorem. □

The case $n = 2d+1$ odd. In this case $\mathbb{C}l(p,q) \approx M(2^d,\mathbb{C}) \oplus M(2^d,\mathbb{C})$. Taking into account Lemma 3.5, we can diagonalize an idempotent $t \in \mathcal{D}$ in this case to the form $J_{r,s} = \mathrm{diag}(J_r, J_s)$, $r,s = 0,\dots,2^d$. By the same argument as the previous Theorem, we have the following result.

Theorem 4.2. *The orbit of t (under the $\mathbb{C}l^\times(p,q)$ group action) is homeomorphic to the coset space*

$$\frac{GL(2^d,\mathbb{C})}{GL(r,\mathbb{C}) \times GL(2^d - r,\mathbb{C})} \times \frac{GL(2^d,\mathbb{C})}{GL(s,\mathbb{C}) \times GL(2^d - s,\mathbb{C})}.$$

5 Hermitian product on $\mathbb{C}l(p,q)$

Consider the following operation of conjugation on $\mathbb{C}l(p,q)$:

$$x=\sum \lambda_I e_I \in \mathbb{C}l(p,q) \longmapsto x^* := \sum \overline{\lambda}_I (e_I)^{-1},$$

where $\overline{\lambda}_I$ is the complex conjugation of λ. For example, $e_k^* = (e_k)^{-1} = g_{kk}e_k$. The conjugation operation has the following properties:

$$(x^*)^* = x,\ (\lambda x + y)^* = \overline{\lambda}x^* + y^* \text{ and } (xy)^* = y^*x^*,\ \forall x,y \in \mathbb{C}l(p,q),\ \forall \lambda \in \mathbb{C}.$$

Proposition 5.1. *The operation $x,y \in \mathbb{C}l(p,q) \longrightarrow \langle x,y\rangle := tr(x^*y)$ defines an hermitian product on $\mathbb{C}l(p,q)$.*

Proof. Let $x,y,z \in \mathbb{C}l(p,q)$ and $\lambda \in \mathbb{C}$. It easy to see that

$$\langle \lambda x, y\rangle = \overline{\lambda}\langle x,y\rangle,\ \langle y,x\rangle = \overline{\langle x,y\rangle} \text{ and } \langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle.$$

Furthermore, by Formula (2.2), we can write $x=\sum \lambda_I e_I$ then
$\langle x,x\rangle = \sum |\lambda_I|^2 \geq 0$ with $\langle x,x\rangle = 0$ if and only if $x=0$. So we get the result. □

Remark 5.2. (i) The basis (2.1) of $\mathbb{C}l(p,q)$ is orthonormal with respect to this hermitian product $\langle .,.\rangle$; That is,

$$\langle e_I, e_J\rangle = \delta_{IJ} := \begin{cases} 1 & \text{if} \quad I=J \\ 0 & \text{if} \quad I \neq J \end{cases}$$

(ii) $\langle xz, y\rangle = \langle z, x^*y\rangle$ for all $x,y,z \in \mathbb{C}l(p,q)$.

Let S be a left ideal of $\mathbb{C}l(p,q)$. S is a vector space. The Hermitian scalar product $x,y \in S \longrightarrow \langle x,y\rangle$, gives us the structure of unitary space on S. Let us take an orthonormal basis $\sigma_1,\dots,\sigma_N$ of S, where $N=\dim(S)$. By definition of left ideal, we have for any $x \in \mathbb{C}l(p,q)$ and any $k=1,\dots,N$, $x\sigma_k \in S$, hence we can write $x\sigma_k = \sum_{l=1}^N x_{lk}\sigma_l$, where $x_{lk} = \langle \sigma_l, x\sigma_k\rangle$. Let us define the map

$$\pi : \mathbb{C}l(p,q) \longrightarrow M(N,\mathbb{C}),\ x \longmapsto \pi(x) := (x_{lk}). \tag{5.1}$$

Theorem 5.3. *π is a matrix representation of $\mathbb{C}l(p,q)$ satisfy*

$$\pi(x^*) = (\pi(x))^*,$$

where $(\pi(x))^*$ denotes the Hermitian conjugated matrix. This representation is faithful when n is even.

Proof. It is easy to see that π is a linear map. Furthermore, for any $x,y \in \mathbb{C}l(p,q)$, we have $(xy)_{lk} = \langle \sigma_l, xy\sigma_k\rangle = \langle \sigma_l, x\sum_j y_{jk}\sigma_j\rangle = \sum_j y_{jk}\langle \sigma_l, x\sigma_j\rangle = \sum_j y_{jk}x_{lj}$. Hence $\pi(xy) = \pi(x)\pi(y)$. Therefore, we have
$x^*_{lk} = \langle \sigma_l, x^*\sigma_k\rangle = \langle x\sigma_l, \sigma_k\rangle = \overline{\langle \sigma_k, x\sigma_l\rangle}$. Thus $\pi(x^*) = (\pi(x))^*$.
On the other hand, if n is even then $\mathbb{C}l(p,q)$ is a simple algebra, and so $\ker(\pi) = \{0\}$, since $\ker(\pi)$ is a bi-ideal of $\mathbb{C}l(p,q)$, which completes the proof of the Theorem. □

Corollary 5.4. *If $n=2d$ is even and $N=2^d$ then π is an isomorphism. In particular π is an irreducible matrix representation.*

Proof. By the previous Theorem, π is injective and because of dimension it is bijective, thus it is an irreducible representation. □

Consider the set of Clifford algebra elements
$U\mathbb{C}l(p,q) := \{x \in \mathbb{C}l(p,q)/x^*x = e\}$. This set is a Lie group (closed subgroup of $\mathbb{C}l^{\times}(p,q)$),

which is called the unitary group of Clifford algebra.
The properties $x^*x = e$ and $\pi(x^*) = (\pi(x))^*$ lead to the property $(\pi(x))^*(\pi(x)) = 1$, where 1, is the identity matrix of $M(N,\mathbb{C})$. That means $\pi(x)$ is a unitary matrix. Consequently, $\pi(U\mathbb{C}l(p,q)) \subset U(N)$, where $U(N)$ is the group of unitary matrices of dimension N.
For even n, the previous Corollary establishes the isomorphism

$$U\mathbb{C}l(p,q) \cong U(2^{\frac{n}{2}}).$$

For odd n, we can establish the isomorphism $U\mathbb{C}l(p,q) \cong U(2^{\frac{n-1}{2}}) \oplus U(2^{\frac{n-1}{2}})$, where $U(2^{\frac{n-1}{2}}) \oplus U(2^{\frac{n-1}{2}})$ is the set of block-diagonal matrices diag(X,Y) with $X, Y \in U(2^{\frac{n-1}{2}})$.

Example 5.5. Let us consider $S(t)$ the minimal left ideal generated by the minimal idempotent t given by Formula (3.1). For even $n = 2d$, set $\sigma_I = \sqrt{2}^d e_{2I}t$, $I \subset \{1,\dots,d\}$, 2^d-elements of $S(t)$, where $2I := \{2k/k \in I\}$. Using Formula (3.2), we have $\langle \sigma_I, \sigma_J \rangle = \delta_{IJ}$. We deduce that σ_I is an orthonormal family and hence an orthonormal basis of $S(t)$, since dim$(S(t)) = 2^d$ (see Example 3.7).
By Corollary 5.4, the resulting representation is an isomorphism of algebras. It is hence an irreducible matrix representation of $\mathbb{C}l(p,q)$.
In the special case $(p,q) = (1,3)$, we obtain
$\sigma_\emptyset = 2t = \frac{1}{2}(e - e_1 - ie_2e_3 + ie_1e_2e_3)$, $\sigma_1 = 2e_2t = \frac{1}{2}(e_2 + e_1e_2 + ie_3 + ie_1e_3)$, $\sigma_2 = \frac{1}{2}(e_4 + e_1e_4 - i2e_3e_4 - ie_1e_2e_3e_4)$ and $\sigma_{12} := \sigma_{\{1,2\}} = 2e_2e_4t = \frac{1}{2}(e_2e_4 + ie_3e_4 - e_1e_2e_4 - ie_2e_3e_4)$.

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