CERTAIN RESULTS ON $M$—PROJECTIVE CURVATURE TENSOR ON $(k, \mu)$—CONTACT SPACE FORMS

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Abstract The object of the present paper is to study $(k, \mu)$—contact space forms satisfying certain curvature tensor. We also study $\xi - M$—projectively flat, $M$—projectively flat and $(k, \mu)$—contact space forms satisfying $\tilde{F}.S = 0$ and $Q.\tilde{F} = 0$. Also we study $\phi - M$—projectively semi-symmetric $(k, \mu)$—contact space form.

1 Introduction

The notion of $(k, \mu)$—contact metric manifold was introduced by Blair, Koufogiorgos and Papanioniou [4]. A class of contact metric manifolds with contact metric structure $(\phi, \xi, \eta, g)$ in which the curvature tensor $R$ satisfies the condition

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y),$$

for all $X, Y \in TM$ is called $(k, \mu)$—contact metric manifolds.

The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector $X$ orthogonal to $\xi$ is called a $\phi$—sectional curvature. If the $(k, \mu)$—contact metric manifold $M$ has constant $\phi$—sectional curvature $c$, then it is called a $(k, \mu)$—contact space form and is denoted by $M(c)$.

$(k, \mu)$—contact space forms have been studied by K. Arslan, R. Ezentas, I. Mihai, C. Murthan [18] and Özgür, C. [2] and A. Akbar and A. Sarkar [1] and many others.

The $M$—projective curvature tensor is important tensor from the differential geometric point of view. Let $M$ be a $(2n + 1)$—dimensional Riemannian manifold. $M$ is said to be locally $M$—projectively flat for $n \geq 1$, if and only if the $M$—projective curvature tensor $\tilde{F}$ vanishes, which is defined by

$$\tilde{F}(X, Y)Z = R(X, Y)Z - \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY],$$

(1.1)

for all $X, Y, Z \in TM$, where $R$ is the curvature tensor and $S$ is the Ricci tensor.

Let $M$ be an almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$. Since at each point $p \in M$ the tangent space $T_pM$ can be decomposed into direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1—dimensional linear subspace of $T_pM$ generated by $\{\xi_p\}$, the conformal curvature tensor $C$ is a map

$$C : T_pM \times T_pM \times T_pM \rightarrow \phi(T_pM) \oplus \{\xi_p\}, p \in M.$$

It may be natural to consider the following particular cases:

1. the projection of the image of $C$ in $\phi(T_pM)$ is zero;
2. the projection of the image of $C$ in $\{\xi_p\}$ is zero;
3. the projection of the image of $C|_{\phi(T_pM) \times \phi(T_pM) \times \phi(T_pM)}$ in $\phi(T_pM)$ is zero.

An almost contact metric manifold satisfying the case (1), (2), and (3) is said to be conformally symmetric [18], $\xi$—conformally flat [19], and $\phi$—conformally flat[7] respectively. In an analogous way, we define $\xi - M$—projectively flat $(k, \mu)$—contact space forms.
Definition A contact metric manifold is called \( M - \text{projectively flat} \) if the manifold satisfies 
\[ \tilde{F}(X, Y)\xi = 0 \]
for all vector fields \( X, Y \).

As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor satisfies 
\[ R(X, Y).R = 0, \]
where \( R(X, Y)Z \) is considered as a field of linear operators acting on \( R \).

Motivated by the above studied, in this paper we characterize \((k, \mu)\)-contact space forms satisfying certain curvature conditions on the \( M - \text{projectively curvature tensor} \). The paper is organized as follows:

In section 2, we give necessary details about \((k, \mu)\)-contact space forms. In section 3, we study \( M - \text{Projectively flat} \) \((k, \mu)\)-contact space forms. Section 4 deals with the study of \((k, \mu)\)-contact space forms satisfying \( \tilde{F}.S = 0 \). In section 5, \( \xi - M - \text{projectively flat} \) \((k, \mu)\)-contact space forms have been studied. Section 6, we study \( \phi - M - \text{projectively semisymmetric} \) \((k, \mu)\)-contact space form.

2 Preliminaries

A \((2n + 1)\)-dimensional differential manifold \( M \) is called an almost contact manifold [3] if there is an almost contact structure \((\phi, \xi, \eta)\) consisting of a \((1,1)\) tensor field \( \phi \), a vector field \( \xi \), a 1-form \( \eta \) satisfying

\[ \phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0 \tag{2.1} \]

An almost contact structure is said to be normal if the induced almost complex structure \( J \) on the product manifold \( M \times \mathbb{R} \) defined by

\[ J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt}) \tag{2.2} \]

is integrable where \( X \) is tangent to \( M \), \( t \) is the coordinate of \( \mathbb{R} \) and \( f \) is a smooth function on \( M \times \mathbb{R} \).

The condition for being normal is equivalent to vanishing of the torsion tensor \([\phi, \phi] + 2d\eta \otimes \xi\), where \([\phi, \phi]\) is the Nijenhuis tensor of \( \phi \).

Let \( g \) be a compatible Riemannian metric with \((\phi, \xi, \eta)\), that is,

\[ g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.3} \]

or equivalently,

\[ g(X, \xi) = \eta(X), g(\phi X, Y) = -g(X, \phi Y), \tag{2.4} \]

for all \( X, Y \in TM \).

An almost contact metric structure becomes a contact metric structure if

\[ g(X, \phi Y) = d\eta(X, Y), \tag{2.5} \]

for all \( X, Y \in TM \).

Given a contact metric manifold \( M(\phi, \xi, \eta, g) \), we define a \((1,1)\) tensor field \( h \) by \( h = \frac{1}{2}L\xi\phi \) where \( L \) denotes the Lie differentiation. Then \( h \) is symmetric and satisfies

\[ h\xi = 0, h\phi + \phi h = 0, \tag{2.6} \]

\[ \nabla\xi = -\phi - \phi h, \text{trace}(h) = \text{trace}(\phi h) = 0, \tag{2.7} \]

where \( \nabla \) is the Levi-Civita connection.
A contact metric manifold is said to be an $\eta$–Einstein manifold if
\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]
where $a$, $b$ are smooth functions and $X, Y \in TM$, $S$ is the Ricci tensor.

A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if
\[
(\nabla_X \phi) = g(X, Y)\xi - \eta(Y)X.
\]

On a Sasakian manifold the following relation holds
\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y,
\]
for all $X, Y \in TM$.

Blair, Koufogiorgos and Papantoniou [4] considered the $(k, \mu)$–nullity condition and gave several reasons for studying it. The $(k, \mu)$–nullity distribution $N(k, \mu)$ [4] of a contact metric manifold $M$ is defined by
\[
N(k, \mu) : p \mapsto N_p(k, \mu) = \{U \in T_pM \mid R(X, Y)U = (kI + \mu h)(g(Y, U)X - g(X, U)Y)\},
\]
for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2(Y)$.

A Contact metric manifold $M$ with $\xi \in N(k, \mu)$ is called a $(k, \mu)$–contact metric manifold. Then we have
\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],
\]
for all $X, Y \in TM$. For $(k, \mu)$–contact metric manifolds, it follows that $h^2 = (k - 1)\phi^2$. This class contains Sasakian manifolds for $k = 1$ and $h = 0$. In fact, for a $(k, \mu)$–contact metric manifold, the condition of being Sasakian manifold, $K$–contact manifold, $k = 1$ and $h = 0$ are equivalent. If $\mu = 0$, then the $(k, \mu)$–nullity distribution $N(k, \mu)$ is reduced to $k$–nullity distribution $N(k)$ [12]. If $\xi \in N(k)$, then we call a contact metric manifold $M$ an $N(k)$–contact metric manifold.

The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector $X$ orthogonal to $\xi$ is called a $\phi$–sectional curvature. If the $(k, \mu)$–contact metric manifold $M$ has constant $\phi$–sectional curvature $c$, then it is called a $(k, \mu)$–contact space form and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [14]
\[
R(X, Y)Z = \frac{c + 3}{4}[g(Y, Z)X - g(X, Z)Y] + \frac{3 - c}{4}[\eta(Y)\phi X + \phi Z(\phi Y)X - g(Y, Z)\phi X] + \frac{1}{2}[g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hX - g(\phi hY, Z)\phi hY] + \frac{k}{2}[g(hY, Z)hY - g(hX, Z)hY] + \frac{\mu}{\eta(Y)\eta(Z)hX - g(hY, Z)\phi X] + \frac{1}{2}[g(hY, Z)hY - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hY] + \frac{1}{2}[g(hY, Z)hY - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hY] + \frac{\mu}{\eta(Y)\eta(Z)hX],
\]
for all $X, Y, Z \in T(M)$, where $c + 2k = -1 = k - \mu$ if $k < 1$. 
From (2.12), we obtain for \((k, \mu)\)–contact space forms:

\[
R(X, Y)\phi Z = \frac{c + 3}{4} [g(Y, \phi Z)X - g(X, \phi Z)Y] + \frac{c - 1}{4} [-2g(X, \phi Y)Z + 2g(X, \phi Y)\eta(Z)\xi - g(X, Z)\phi Y + \eta(Z)\eta(X)\phi Y + g(Y, Z)\phi X - \eta(Y)\eta(Z)\phi X] + \frac{c + 3 - 4k}{4} [g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi] + \frac{1}{2} [g(hY, \phi Z)hX - g(hX, \phi Z)hY + g(hX, Z)\phi hY - g(hY, Z)\phi hX - g(\phi X, Z)hY + g(\phi Y, Z)hX + g(\phi_X, Z)\phi hY - g(hX, \phi Z)\eta(Y)\xi - g(hY, \phi Z)\eta(X)\xi - g(hX, \phi Z)\eta(Y)\xi], \tag{2.13}
\]

\[
\phi R(X, Y)Z = \frac{c + 3}{4} [g(Y, Z)\phi X - g(X, Z)\phi Y] + \frac{c - 1}{4} [-2g(X, \phi Y)Z + 2g(X, \phi Y)\eta(Z)\xi - g(X, \phi Z)Y + g(X, \phi Z)\eta(Y)\xi + g(Y, \phi Z)X - g(Y, \phi Z)\eta(X)\xi] + \frac{c + 3 - 4k}{4} [\eta(Z)\eta(X)\phi Y - \eta(Y)\eta(Z)\phi X] + \frac{1}{2} [g(hY, Z)\phi hX - g(hX, Z)\phi hY - g(\phi X, Z)hX + g(\phi Y, Z)\phi hX - g(\phi X, Z)\phi hY + g(\phi Y, Z)hX + g(\phi_X, Z)\phi hY - g(hX, \phi Z)\eta(Y)\xi - g(hY, \phi Z)\eta(X)\xi - g(hX, \phi Z)\eta(Y)\xi], \tag{2.14}
\]

\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.15}
\]

\[
R(X, \xi)\xi = k[X - \eta(X)\xi] + \mu hX, \tag{2.16}
\]

\[
R(\xi, Y)Z = k[g(Y, Z)\xi - \eta(Z)Y] + \mu[g(hY, Z)\xi - \eta(Z)hY], \tag{2.17}
\]

\[
S(Y, Z) = \frac{1}{2} [c(n + 1) + 3(n - 1) + 2k]g(Y, Z) + \frac{1}{2} [-c(n + 1) - 3(n - 1) + 2k(2n - 1)]\eta(Y)\eta(Z) + [2n - 2 + \mu]g(hY, Z), \tag{2.18}
\]

\[
S(Y, hZ) = \frac{1}{2} [c(n + 1) + 3(n - 1) + 2k]g(Y, hZ) + (k - 1)[2n - 2 + \mu]g(Y, Z) - (k - 1)[2n - 2 + \mu]\eta(Y)\eta(Z), \tag{2.19}
\]
\[ S(Y, \xi) = 2nk\eta(Y), \]  
\[ S(\xi, \xi) = 2nk, \]  
\[ QY = \frac{1}{2} [c(n+1) + 3(n-1) + 2k]Y \]  
\[ + \frac{1}{2} [-c(n+1) - 3(n-1) + 2k(2n-1)]\eta(Y)\xi \]  
\[ + [2n - 2 + \mu]hY, \]

\[ Q\xi = 2nk\xi. \]  

**Definition 2.1.** The \( M \)–projectively curvature tensor \( \tilde{F} \) of type \((1,3)\) on \((k,\mu)\)–contact metric form \( M \) of dimension \((2n+1)\) is defined as

\[
\tilde{F}(X,Y)Z = R(X,Y)Z - \frac{1}{4n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY],
\]

for any vector field \( X, Y, Z \) on \( M \). The manifold is called \( M \)–projectively flat if \( \tilde{F} \) vanishes identically on \( M \).

From (2.24) using (2.15), (2.17), (2.18), (2.20), (2.21), (2.22) and (2.23), we have

\[
\tilde{F}(X,Y)\xi = a[\eta(Y)X - \eta(X)Y] + b[\eta(Y)hX - \eta(X)hY],
\]

\[
\tilde{F}(\xi,Y)\xi = a[\eta(Y)\xi - Y] - bhY,
\]

\[
\tilde{F}(\xi,Y)Z = a[g(Y,Z)\xi - \eta(Z)Y] + b[g(hY,Z)\xi - \eta(Z)hY],
\]

\[
\tilde{F}(\xi,Y)hZ = ag(Y,hZ)\xi + bh(hY,hZ)\xi,
\]

where

\[
a = \frac{k}{2} - \frac{1}{2(4n)} [c(n+1) + 3(n-1) + 2k],
\]

and

\[
b = \mu - \frac{1}{4n} [2n - 2 + \mu].
\]

### 3 \( M \)–Projectively flat \((k,\mu)\)–Contact Space Forms

**Theorem 3.1.** A \((2n+1)\)-dimensional \( M \)–Projectively flat \((k,\mu)\)–contact space form is an \( \eta \)–Einstein manifold.

**Proof.** From the definition of \( M \)–Projectively flat \((k,\mu)\)–contact space forms we have

\[
\tilde{F}(X,Y)Z = 0.
\]

Applying this in (2.24), we obtain

\[
R(X,Y)Z = \frac{1}{4n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].
\]
Taking the inner product with $W$ of (3.1), we obtain
\[ g(R(X,Y)Z,W) = \frac{1}{4n} [S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)]. \quad (3.2) \]

Putting $X = W = \xi$ in (3.2) and using (2.17), (2.18), (2.19), (2.20) and (2.21), we have
\[ g(hY,Z) = \frac{1}{4n\mu} S(Y,Z) - \frac{k}{2} g(Y,Z). \quad (3.3) \]

By using (3.3) in (2.18), we get
\[ S(Y,Z) = a_1 g(Y,Z) + b_1 \eta(Y) \eta(Z), \quad (3.4) \]
where
\[ a_1 = \frac{2n\mu[c(n+1) + 3(n-1) + 2k - \frac{k}{2}(2n - 2 + \mu)]}{4n\mu - (2n - 2 + \mu)}, \]
and
\[ b_1 = \frac{2n\mu[-c(n+1) - 3(n-1) + 2k(2n - 1)]}{4n\mu - (2n - 2 + \mu)}. \]

\[ \Box \]

4 \hspace{1em} (k, \mu) - Contact Space Forms Satisfying $\tilde{F}.S = 0$

**Theorem 4.1.** A $(2n + 1)$-dimensional $(k, \mu)$-contact space forms satisfying $\tilde{F}.S = 0$ is an $\eta$-Einstein manifold.

**Proof.** Let $M(c)$ be a $(2n + 1)$-dimensional $(k, \mu)$-contact space forms satisfying $\tilde{F}.S = 0$ which implies that
\[ S(\tilde{F}(X,Y)U,V) + S(U,\tilde{F}(X,Y)V) = 0, \quad (4.1) \]

By putting $U = X = \xi$, we get
\[ S(\tilde{F}(\xi,Y)\xi,V) + S(\xi,\tilde{F}(\xi,Y)V) = 0. \quad (4.2) \]

By using (2.18), (2.19), (2.20) and (2.24), we obtain
\[ g(hY,Z) = c_1 g(Y,V) + d_1 \eta(Y) \eta(V), \quad (4.3) \]
where,
\[ c_1 = \frac{\frac{3}{2}[c(n+1) + 3(n-1) + 2k] + (k-1)b(2n - 2 + \mu) + 2nka}{2nkb - a(2n - 2 + \mu) - \frac{k}{2}[c(n+1) + 3(n-1) + 2k]}, \]
and
\[ d_1 = \frac{\frac{3}{2}[-c(n+1) - 3(n-1) + 2k(2n - 1)] - b(k-1)(2n - 2 + \mu)}{2nkb - a(2n - 2 + \mu) - \frac{k}{2}[c(n+1) + 3(n-1) + 2k]}. \]

By using (4.3) in (2.18), we get
\[ S(Y,V) = c_2 g(Y,V) + d_2 \eta(Y) \eta(V), \]
where
\[ c_2 = \frac{1}{2}[c(n+1) + 3(n-1) + 2k] + (2n - 2 + \mu)c_1, \]
and
\[ d_2 = \frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n - 1)] + (2n - 2 + \mu)d_1. \]

\[ \Box \]
5 \( \xi - M - \text{Projectively Flat} (k, \mu) - \text{Contact Space Forms} \)

**Theorem 5.1.** Let \( M(c) \) be a \( \xi - M - \text{projectively flat} (k, \mu) - \text{contact space forms} \). Then \( M(c) \) is either a Sasakian space form or a \( N(k) \)-contact space form for particular \( n = 1 \).

**Proof.** Assume that \( M(c) \) is a \( \xi - M - \text{projectively flat} (k, \mu) - \text{contact space form} \). Then

\[
\tilde{F}(X, Y)\xi = 0.
\] (5.1)

Putting \( Z = \xi \) in (1.1), we obtain

\[
\tilde{F}(X, Y)\xi = R(X, Y)\xi - \frac{1}{4n} [S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY].
\] (5.2)

Using (2.11) and (2.20) in (5.2), we get

\[
a[\eta(Y)X - \eta(X)Y] + b[\eta(Y)hX - \eta(X)hY] = 0.
\] (5.3)

From (5.3), we may conclude that if \( a = 0 \) then either \( b = 0 \) or

\[
\eta(Y)hX - \eta(X)hY = 0
\] (5.4)

Putting \( Y = \xi \) in above equation, we have

\[
hX = 0
\]

If \( \mu = 0 \), then \( M(c) \) is a \( N(k) \)-contact space form for particular \( n = 1 \).

If \( h = 0 \), then \( M(c) \) is a Sasakian space form. \( \square \)

6 \( (k, \mu) - \text{Contact Space Forms Satisfying} \ Q.\tilde{F} = 0 \)

**Theorem 6.1.** A \( (k, \mu) - \text{Contact Space Forms Satisfying} \ Q.\tilde{F} = 0 \) is either \( (0, 1) - \text{contact space form of constant } \phi - \text{sectional curvature} \) or \( N(k) \)-contact space form for particular \( n = 1 \) or, a Sasakian space form.

**Proof.** A \( (k, \mu) - \text{contact space forms satisfying} \ Q.\tilde{F} = 0 \), where \( Q \) is the Ricci operator defined by \( S(X, Y) = g(QX, Y) \). Suppose \( M(c) \) be a \( (k, \mu) - \text{contact space form satisfying} \ Q.\tilde{F} = 0 \). Then

\[
Q(\tilde{F}(X, Y)Z) - \tilde{F}(QX, Y)Z - \tilde{F}(X, QY)Z = 0.
\] (6.1)

Putting \( Z = \xi \) in (6.1) and using (2.25), we have

\[
a[\eta(QX)Y - \eta(QY)X] - 2nka[\eta(Y)X - \eta(X)Y] + \\
b[\eta(Y)[Q(hX) - hQX] - b\eta(X)[Q(hY) - hQY] + \\
b[\eta(QX)hY - \eta(QY)hX] - 2nkb[\eta(Y)hX - \eta(X)hY] = 0.
\] (6.2)

Using (2.22), we obtain

\[
Q(hY) - hQY = \frac{1}{2} [c(n + 1) + 3(n - 1) + 2k]hY \\
+ \frac{1}{2} [-c(n + 1) - 3(n - 1) + 2k(2n - 1)]hY
\]

\[
+ [2n - 2 + \mu]h^2Y - \frac{1}{2} [c(n + 1) + 3(n - 1) + 2k]hY \\
- \frac{1}{2} [-c(n + 1) - 3(n - 1) + 2k(2n - 1)]
\]

\[
\eta(Y)hX - [2n - 2 + \mu]h^2Y = 0.
\] (6.3)
\[ \eta(QX)Y - \eta(QY)X = 2nk[\eta(X)Y - \eta(Y)X], \]  \hspace{1cm} (6.4)  

and

\[ \eta(QX)hY - \eta(QY)hX = 2nk[\eta(X)hY - \eta(Y)hX]. \]  \hspace{1cm} (6.5)  

Using (6.3), (6.4) and (6.5) in (6.2), we have

\[ 4nk\{a[\eta(X)Y - \eta(Y)X] + b[\eta(X)hY - \eta(Y)hX]\} = 0. \]  \hspace{1cm} (6.6)  

From (6.6), we may conclude that if \( a = 0 \) then either \( k = 0 \) or \( b = 0 \) or

\[ [\eta(X)hY - \eta(Y)hX] = 0. \]  \hspace{1cm} (6.7)  

Putting \( Y = \xi \) in the above equation yields

\[ hX = 0. \]

If \( k = 0 \), then from (2.12), we have \( \mu = 1 \) and constant \( \phi \)-sectional curvature \( c = -1 \).

If \( \mu = 0 \) for particular \( n = 1 \), then \( M(c) \) is a \( N(k) \)-contact space form.

If \( h = 0 \), then \( M(c) \) is a Sasakian space form. \( \square \)

### 7 \( \phi - M \)-Projectively Semisymmetric \((k, \mu)\)-Contact Space Forms

**Definition 7.1.** A \((k, \mu)\)-contact space form is said to be \( \phi - M \)-projectively semi-symmetric if \( \tilde{F}(X, Y) \cdot \phi = 0 \) for all \( X, Y \in TM \).

**Proposition 7.2.** Let \( M(c) \) be a \( \phi - M \)-projectively semi-symmetric \((k, \mu)\)-contact space form, then \( \mu = \frac{2}{2n+1} \).

**Proof.** Suppose \( M(c) \) be a \( \phi - M \)-projectively semi-symmetric \((k, \mu)\)-contact space form. Then

\[ \tilde{F}(X, Y)\phi Z - \phi(\tilde{F}(X, Y)Z) = 0. \]  \hspace{1cm} (7.1)  

From (1.1), it follows that

\[ \tilde{F}(X, Y)\phi Z = R(X, Y)\phi Z - \frac{1}{4n}[S(Y, \phi Z)X - S(X, \phi Z)Y + g(Y, \phi Z)QX - g(X, \phi Z)QY]. \]  \hspace{1cm} (7.2)  

Using (2.18) in (7.2), we get

\[ \tilde{F}(X, Y)\phi Z = R(X, Y)\phi Z - \frac{1}{4n}\{[c(n + 1) + 3(n - 1) + 2k][g(Y, \phi Z)X - g(X, \phi Z)Y] + \}

\[ 2k(2n - 1)][g(Y, \phi Z)\eta(X)\xi - g(X, \phi Z)\eta(Y)\xi] + [2n - 2 + \mu][g(hY, \phi Z)X - g(hX, \phi Z)Y + g(Y, \phi Z)hX - g(X, \phi Z)hY]\} \]  \hspace{1cm} (7.3)  

Again,

\[ \phi(\tilde{F}(X, Y)Z) = \phi R(X, Y)Z - \frac{1}{4n}\{[c(n + 1) + 3(n - 1) + 2k][g(Y, Z)\phi X - g(X, Z)\phi Y + \}

\[ 2k(2n - 1)][\eta(Y)\eta(Z)\phi X - \eta(Z)\eta(X)\phi Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] + [2n - 2 + \mu][g(hY, Z)\phi X - g(hX, Z)\phi Y + g(Y, Z)h\phi X - g(X, Z)h\phi Y]\}. \]  \hspace{1cm} (7.4)
Using (7.3) and (7.4) in (7.2), we have
\[
(\vec{F}(X,Y) \cdot \phi)Z = R(X,Y)\phi Z - \phi R(X,Y)Z \\
\quad - \frac{1}{4n}\{[c(n+1)+3(n-1)+2k][g(Y,\phi Z)X \\
\quad - g(X,\phi Z)Y] + \frac{1}{2}(-c(n+1)-3(n-1)+ \\
2k(2n-1)][g(Y,\phi Z)\eta(X)\xi - g(X,\phi Z)\eta(Y)\xi] + \\
[2n-2+\mu][g(hY,\phi Z)X - g(hX,\phi Z)Y + g(Y,\phi Z)hX \\
\quad - g(X,\phi Z)hY]) + \frac{1}{4n}\{[c(n+1) + \\
3(n-1)+2k][g(Y,Z)\phi X - g(X,Z)\phi Y \\
\quad + \frac{1}{2}(-c(n+1)-3(n-1)+2k(2n-1)][\eta(Y)\eta(Z)\phi X \\
- \eta(Z)\eta(X)\phi Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] + \\
[2n-2+\mu][g(hY,\phi Z)\phi X - g(hX,\phi Z)\phi Y \\
\quad + g(Y,Z)h\phi X - g(X,Z)h\phi Y]\} = 0. \quad \text{(7.5)}
\]

Putting the value of \(R(X,Y)\phi Z\) and \(\phi R(X,Y)Z\) in (7.5) and contracting \(Y\) and \(W\), we obtain
\[
\{\frac{c+3}{4}(1-2n) + \frac{c-1}{4}(2n-1) + \frac{1}{4n}(2n-1) \\
\quad [c(n+1)+3(n-1)+2k]\}g(\phi Z, X) + \\
\{\frac{1}{2}(1-2n) + \frac{2n-2+\mu}{4n}(2n+1)\}g(\phi Z, hX) = 0. \quad \text{(7.6)}
\]

Putting \(X = hX\) in the above equation yields
\[
\{\frac{c+3}{4}(1-2n) + \frac{c-1}{4}(2n-1) + \frac{1}{4n}(2n-1) \\
\quad [c(n+1)+3(n-1)+2k]\}g(\phi Z, hX) + \\
\{\frac{1}{2}(1-2n) + \frac{2n-2+\mu}{4n}(2n+1)\}g(\phi Z, h^2X) = 0. \quad \text{(7.7)}
\]

Taking trace in both sides of (7.7) and using \(\text{trace}(h) = 0\), we get
\[
\mu = \frac{2}{2n+1}.
\]

\[\square\]

From the above proposition we can state the following:

**Theorem 7.3.** A three dimensional \(\phi - M - \text{projectively semi-symmetric} (k, \mu) - \text{contact space form reduces to an } N(k) - \text{contact space form.}**

**References**


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