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CERTAIN RESULTS ON $M-$ PROJECTIVE CURVATURE TENSOR ON $(k, \mu)-$ CONTACT SPACE FORMS

SHYAM KISHOR, PUSHPENDRA VERMA

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Abstract The object of the present paper is to study $(k, \mu)-$contact space forms satisfying certain curvature tensor. We also study $\xi - M-$projectively flat, $M-$projectively flat and $(k, \mu)-$contact space forms satisfying $\widetilde{F}.S = 0$ and $Q.\widetilde{F} = 0$. Also we study $\phi-M-$projectively semi-symmetric $(k, \mu)-$contact space form.

1 Introduction

The notion of $(k, \mu)-$ contact metric manifold was introduced by Blair, Koufogiorgos and Papantoniou [4]. A class of contact metric manifolds with contact metric structure (ϕ, ξ, η, g) in which the curvature tensor R satisfies the condition

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y),$$

for all $X, Y \in TM$ is called $(k, \mu)-$ contact metric manifolds.

The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a $\phi-$sectional curvature. If the $(k, \mu)-$contact metric manifold M has constant $\phi-$sectional curvature c, then it is called a $(k, \mu)-$ contact space form and is denoted by $M(c)$. $(k, \mu)-$ contact space forms have been studied by K. Arslan, R. Ezentas, I. Mihai, C. Murthan and Özgür, C. [2] and A. Akbar and A. Sarkar [1] and many others.

The $M-$projective curvature tensor is important tensor from the differential geometric point of view. Let M be a $(2n + 1)-$dimensional Riemannian manifold. M is said to be locally $M-$projectively flat for $n \geq 1$, if and only if the $M-$projective curvature tensor $\widetilde{F}$ vanishes, which is defined by

$$\begin{aligned} \widetilde{F}(X, Y)Z &= R(X, Y)Z - \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY], \end{aligned} \tag{1.1}$$

for all $X, Y, Z \in TM$, where R is the curvature tensor and S is the Ricci tensor.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). Since at each point $p \in M$ the tangent space $T_P M$ can be decomposed into direct sum $T_P M = \phi(T_P M\ ) \oplus \{\xi_P\}$, where $\{\xi_P\}$ is the $1-$ dimensional linear subspace of $T_P M$ generated by $\{\xi_P\}$, the conformal curvature tensor C is a map

$$C : T_P M \times T_P M \times T_P M \longrightarrow \phi(T_P M) \oplus \{\xi_P\}, p \in M.$$

It may be natural to consider the following particular cases:
(1) the projection of the image of C in $\phi(T_P M)$ is zero;
(2) the projection of the image of C in $\{\xi_P\}$ is zero;
(3) the projection of the image of $C|_{\phi(T_P M)\times\phi(T_P M)\times\phi(T_P M)}$ in $\phi(T_P M)$ is zero.

An almost contact metric manifold satisfying the case (1), (2), and (3) is said to be conformally symmetric [18], $\xi-$conformally flat [19], and $\phi-$conformally flat[7] respectively. In an anlogous way, we define $\xi - M-$projectively flat $(k, \mu)-$contact space forms.

Definition A contact metric manifold is called $M - projectively\ flat$ if the manifold satisfies $\widetilde{F}(X, Y)\xi = 0$ for all vector fields X, Y.

As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor satisfies

$$R(X, Y).R = 0,$$

where $R(X, Y)Z$ is considered as a field of linear operators acting on R.

A natural extension of such curvature conditions from curvature conditions of pseudosymmetry type. The condition $Q \cdot R = 0$ have been studied by $Verstraelen$ et. al. in [15].

In this paper, we characterize $(k, \mu)-$ contact space forms $Q \cdot P = 0$.

Motivated by the above studied, in this paper we characterize a $(k, \mu)-$contact space form satisfying certain curvature conditions on the $M-$projectively curvature tensor. The paper is organized as follows:

In section 2, we give necessary details about $(k, \mu)-$contact space forms. In section 3, we study $M-$Projectively flat $(k, \mu)-$contact space forms. Section 4 deals with the study of $(k, \mu)-$contact space forms satisfying $\widetilde{F}.S = 0$. In section 5, $\xi-M-$projectively flat $(k, \mu)-$contact space forms have been studied. Section 6, we study $(k, \mu)-$contact space forms satisfying $Q.\widetilde{F} = 0$. Finally, we study $\phi - M-$projectively semisymmetric $(k, \mu)-$contact space form.

2 Preliminaries

A $(2n + 1)-$ dimensional differential manifold M is called an almost contact manifold [3] if there is an almost contact structure (ϕ, ξ, η) consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a $1-$ form η satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta o\phi = 0 \tag{2.1}$$

An almost contact stucture is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{2.2}$$

is integrable where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$.

The condition for being normal is equivalent to vanishing of the torsion tensor $[\phi, \phi]+2d\eta\otimes\xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ.

Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is,

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.3}$$

or equivalently,

$$g(X, \xi) = \eta(X), g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

for all $X, Y \in TM$.

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.5}$$

for all $X, Y \in TM$.

Given a contact metric manifold $M(\phi, \xi, \eta, g)$, we define a $(1, 1)$ tensor field h by $h = \frac{1}{2}L_\xi\phi$ where L denotes the Lie differentiation. Then h is symmetric and satisfies

$$h\xi = 0, h\phi + \phi h = 0, \tag{2.6}$$

$$\bigtriangledown\xi = -\phi - \phi h, trace(h) = trace(\phi h) = 0, \tag{2.7}$$

where $\bigtriangledown$ is the Levi-Civita connection.

A contact metric manifold is said to be an $\eta-$Einstein manifold if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.8}$$

where a, b are smooth functions and $X, Y \in TM$, S is the Ricci tensor.

A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\bigtriangledown_X \phi) = g(X,Y)\xi - \eta(Y)X. \tag{2.9}$$

On a Sasakian manifold the following relation holds

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

for all $X, Y \in TM$.

Blair, Koufogiorgos and Papantoniou [4] considered the $(k,\mu)-$ nullity condition and gave several reasons for studying it. The $(k,\mu)-$ nullity distribution $N(k,\mu)$ [4] of a contact metric manifold M is defined by

$$N(k,\mu) : p \to N_P(k,\mu) = [U \in T_pM \mid R(X,Y)U = (kI + \mu h)(g(Y,U)X - g(X,U)Y)],$$

for all $X, Y \in TM$, where $(k,\mu) \in \mathbb{R}^2(Y)$.

A Contact metric manifold M with $\xi \in N(k,\mu)$ is called a $(k,\mu)-$contact metric manifold. Then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.11}$$

for all $X, Y \in TM$. For $(k,\mu)-$contact metric manifolds, it follows that $h^2 = (k-1)\phi^2$. This class contains Sasakian manifolds for $k = 1$ and $h = 0$. In fact, for a $(k,\mu)-$contact metric manifold, the condition of being Sasakian manifold, $K-$contact manifold, $k = 1$ and $h = 0$ are equivalent. If $\mu = 0$, then the $(k,\mu)-$nullity distribution $N(k,\mu)$ is reduced to $k-$nullity distribution $N(k)$ [12]. If $\xi \in N(k)$, then we call a contact metric manifold M an $N(k)-$contact metric manifold.

The sectional curvature $K(X,\phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a $\phi-$sectional curvature. If the $(k,\mu)-$contact metric manifold M has constant $\phi-$sectional curvature c, then it is called a $(k,\mu)-$contact space form and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [14]

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] \\ &+\frac{c-1}{4}[2g(X,\phi Y)\phi Z + g(X,\phi Z)\phi Y \\ &-g(Y,\phi Z)\phi X] + \frac{c+3-4k}{4}[\eta(X)\eta(Z)Y \\ &-\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \\ &+\frac{1}{2}[g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY \\ &-g(\phi hY,Z)\phi hX + g(\phi Y,\phi Z)hX - g(\phi X,\phi Z)hY \\ &+g(hX,Z)\phi^2 Y - g(hY,Z)\phi^2 X] + \mu[\eta(Y)\eta(Z)hX \\ &-\eta(X)\eta(Z)hY + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi], \end{aligned} \tag{2.12}$$

for all $X, Y, Z \in T(M)$, where $c + 2k = -1 = k - \mu$ if $k < 1$.

From (2.12), we obtain for $(k,\mu)-$contact space forms:

$$\begin{aligned} R(X,Y)\phi Z &= \frac{c+3}{4}[g(Y,\phi Z)X - g(X,\phi Z)Y] \\ &+\frac{c-1}{4}[-2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi - g(X,Z)\phi Y \\ &+\eta(Z)\eta(X)\phi Y + g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X] \\ &+\frac{c+3-4k}{4}[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi] \\ &+\frac{1}{2}[g(hY,\phi Z)hX - g(hX,\phi Z)hY + g(hX,Z)\phi hY \\ &-g(hY,Z)\phi hX - g(\phi Y,Z)hX + g(\phi X,Z)hY \\ &-g(hX,\phi Z)Y + g(hX,\phi Z)\eta(Y)\xi - g(hY,\phi Z)X \\ &-g(hY,\phi Z)\eta(X)\xi] + \mu[g(hY,\phi Z)\eta(X)\xi \\ &-g(hX,\phi Z)\eta(Y)\xi], \end{aligned} \tag{2.13}$$

$$\begin{aligned} \phi R(X,Y)Z &= \frac{c+3}{4}[g(Y,Z)\phi X - g(X,Z)\phi Y] \\ &+\frac{c-1}{4}[-2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi \\ &-g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)X \\ &-g(Y,\phi Z)\eta(X)\xi] \\ &+\frac{c+3-4k}{4}[\eta(Z)\eta(X)\phi Y - \eta(Y)\eta(Z)\phi X] \\ &+\frac{1}{2}[g(hY,Z)\phi hX - g(hX,Z)\phi hY - g(\phi hX,Z)hY \\ &+g(\phi hY,Z)hX + g(\phi Y,\phi Z)\phi hX - g(\phi X,\phi Z)\phi hY \\ &-g(hX,Z)\phi Y + g(hY,Z)\phi X] \\ &+\mu[\eta(Y)\eta(Z)\phi hX - \eta(Z)\eta(X)\phi hY], \end{aligned} \tag{2.14}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.15}$$

$$R(X,\xi)\xi = k[X - \eta(X)\xi] + \mu hX, \tag{2.16}$$

$$R(\xi,Y)Z = k[g(Y,Z)\xi - \eta(Z)Y] + \mu[g(hY,Z)\xi - \eta(Z)hY], \tag{2.17}$$

$$\begin{aligned} S(Y,Z) &= \frac{1}{2}[c(n+1) + 3(n-1) + 2k]g(Y,Z) \\ &+\frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)]\eta(Y)\eta(Z) \\ &+[2n-2+\mu]g(hY,Z), \end{aligned} \tag{2.18}$$

$$\begin{aligned} S(Y,hZ) &= \frac{1}{2}[c(n+1) + 3(n-1) + 2k]g(Y,hZ) \\ &+(k-1)[2n-2+\mu]g(Y,Z) \\ &-(k-1)[2n-2+\mu]\eta(Y)\eta(Z), \end{aligned} \tag{2.19}$$

$$S(Y,\xi) = 2nk\eta(Y), \tag{2.20}$$

$$S(\xi,\xi) = 2nk, \tag{2.21}$$

$$\begin{aligned} QY &= \frac{1}{2}[c(n+1)+3(n-1)+2k]Y \\ &+\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)]\eta(Y)\xi \\ &+[2n-2+\mu]hY, \end{aligned} \tag{2.22}$$

$$Q\xi = 2nk\xi. \tag{2.23}$$

Definition 2.1. The $M-$projectively curvature tensor $\widetilde{F}$ of type $(1,3)$ on $(k,\mu)-$contact metric form M of dimension $(2n+1)$ is defined as

$$\begin{aligned} \widetilde{F}(X,Y)Z &= R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y \\ &\quad +g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{2.24}$$

for any vector field X,Y,Z on M. The manifold is called $M-$projectively flat if $\widetilde{F}$ vanishes identically on M.

From (2.24) using (2.15), (2.17), (2.18), (2.20), (2.21), (2.22) and (2.23), we have

$$\widetilde{F}(X,Y)\xi = a[\eta(Y)X - \eta(X)Y] + b[\eta(Y)hX - \eta(X)hY], \tag{2.25}$$

$$\widetilde{F}(\xi,Y)\xi = a[\eta(Y)\xi - Y] - bhY, \tag{2.26}$$

$$\widetilde{F}(\xi,Y)Z = a[g(Y,Z)\xi - \eta(Z)Y] + b[g(hY,Z)\xi - \eta(Z)hY], \tag{2.27}$$

$$\widetilde{F}(\xi,Y)hZ = ag(Y,hZ)\xi + bg(hY,hZ)\xi, \tag{2.28}$$

where

$$a = \frac{k}{2} - \frac{1}{2(4n)}[c(n+1)+3(n-1)+2k],$$

and

$$b = \mu - \frac{1}{4n}[2n-2+\mu].$$

3 $M-$Projectively flat $(k,\mu)-$Contact Space Forms

Theorem 3.1. *A (2n+1)-dimensional $M-$Projectively flat $(k,\mu)-$contact space form is an $\eta-$Einstein manifold.*

Proof. From the definition of $M-$Projectively flat $(k,\mu)-$contact space forms we have

$$\widetilde{F}(X,Y)Z = 0.$$

Applying this in (2.24), we obtain

$$R(X,Y)Z = \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{3.1}$$

Taking the inner product with W of (3.1), we obtain

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{1}{4n}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &\quad + g(Y,Z)g(QX,W) - g(X,Z)g(QY,W). \end{aligned} \tag{3.2}$$

Putting $X = W = \xi$ in (3.2) and using (2.17), (2.18), (2.19), (2.20) and (2.21), we have

$$g(hY,Z) = \frac{1}{4n\mu}S(Y,Z) - \frac{k}{2}g(Y,Z). \tag{3.3}$$

By using (3.3) in (2.18), we get

$$S(Y,Z) = a_1 g(Y,Z) + b_1\eta(Y)\eta(Z), \tag{3.4}$$

where

$$a_1 = \frac{2n\mu[c(n+1) + 3(n-1) + 2k - \frac{k}{2}(2n-2+\mu)]}{4n\mu - (2n-2+\mu)},$$

and

$$b_1 = \frac{2n\mu[-c(n+1) - 3(n-1) + 2k(2n-1)]}{4n\mu - (2n-2+\mu)}.$$

□

4 $(k,\mu)-$Contact Space Forms Satisfying $\widetilde{F}.S = 0$

Theorem 4.1. *A $(2n+1)$-dimensional $(k,\mu)-$contact space forms satisfying $\widetilde{F}.S = 0$ is an $\eta-$Einstein manifold.*

Proof. Let $M(c)$ be a $(2n+1)$-dimensional $(k,\mu)-$contact space forms satisfying $\widetilde{F}.S = 0$ which implies that

$$S(\widetilde{F}(X,Y)U,V) + S(U,\widetilde{F}(X,Y)V) = 0, \tag{4.1}$$

By putting $U = X = \xi$, we get

$$S(\widetilde{F}(\xi,Y)\xi,V) + S(\xi,\widetilde{F}(\xi,Y)V) = 0. \tag{4.2}$$

By using (2.18), (2.19), (2.20) and (2.24), we obtain

$$g(hY,Z) = c_1 g(Y,V) + d_1\eta(Y)\eta(V), \tag{4.3}$$

where,

$$c_1 = \frac{\{\frac{a}{2}[c(n+1) + 3(n-1) + 2k] + (k-1)b(2n-2+\mu) + 2nka\}}{\{2nkb - a(2n-2+\mu) - \frac{b}{2}[c(n+1) + 3(n-1) + 2k]\}},$$

and

$$d_1 = \frac{\{\frac{a}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)] - b(k-1)(2n-2+\mu)\}}{\{2nkb - a(2n-2+\mu) - \frac{b}{2}[c(n+1) + 3(n-1) + 2k]\}}.$$

By using (4.3) in (2.18), we get

$$S(Y,V) = c_2 g(Y,V) + d_2\eta(Y)\eta(V),$$

where

$$c_2 = \frac{1}{2}[c(n+1) + 3(n-1) + 2k] + (2n-2+\mu)c_1,$$

and

$$d_2 = \frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)] + (2n-2+\mu)d_1.$$

□

5 $\xi - M-$Projectively Flat $(k, \mu)-$Contact Space Forms

Theorem 5.1. *Let $M(c)$ be a $\xi - M-$projectively flat $(k, \mu)-$contact space forms. Then $M(c)$ is either a Sasakian space form or a $N(k)$- contact space form for particular $n = 1$.*

Proof. Assume that $M(c)$ is a $\xi - M-$Projectively flat $(k, \mu)-$contact space form. Then

$$\widetilde{F}(X, Y)\xi = 0. \tag{5.1}$$

putting $Z = \xi$ in (1.1), we obtain

$$\begin{aligned} \widetilde{F}(X, Y)\xi &= R(X, Y)\xi - \frac{1}{4n}[S(Y, \xi)X - S(X, \xi)Y \\ &\quad +g(Y, \xi)QX - g(X, \xi)QY], \end{aligned} \tag{5.2}$$

Using (2.11) and (2.20) in (5.2), we get

$$a[\eta(Y)X - \eta(X)Y] + b[\eta(Y)hX - \eta(X)hY] = 0. \tag{5.3}$$

From (5.3), we may conclude that if $a = 0$ then either $b = 0$ or

$$\eta(Y)hX - \eta(X)hY = 0 \tag{5.4}$$

Putting $Y = \xi$ in above equation , we have

$$hX = 0$$

If $\mu = 0$, then $M(c)$ is a $N(k)$- contact space form for particular $n = 1$.
If $h = 0$, then $M(c)$ is a Sasakian space form. □

6 $(k, \mu)-$Contact Space Forms Satisfying $Q.\widetilde{F} = 0$

Theorem 6.1. *A $(k, \mu)-$Contact Space Forms Satisfying $Q.\widetilde{F} = 0$ is either $(0, 1)-$contact space form of constant $\phi-$sectional curvature -1 or $N(k)$- contact space form for particular $n = 1$ or, a Sasakian space form.*

Proof. A $(k, \mu)-$contact space forms satisfying $Q.\widetilde{F} = 0$, where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. Suppoese $M(c)$ be a $(k, \mu)-$ contact space form satisfying $Q.\widetilde{F} = 0$. Then

$$Q(\widetilde{F}(X, Y)Z) - \widetilde{F}(QX, Y)Z - \widetilde{F}(X, QY)Z - \widetilde{F}(X, Y)QZ = 0. \tag{6.1}$$

Putting $Z = \xi$ in (6.1) and using (2.25), we have

$$\begin{aligned} &a[\eta(QX)Y - \eta(QY)X] - 2nka[\eta(Y)X - \eta(X)Y] + \\ &b\eta(Y)[Q(hX) - hQX] - b\eta(X)[Q(hY) - hQY] + \\ &b[\eta(QX)hY - \eta(QY)hX] - 2nkb[\eta(Y)hX - \eta(X)hY] = 0. \end{aligned} \tag{6.2}$$

Using (2.22), we obtain

$$\begin{aligned} Q(hY) - hQY &= \frac{1}{2}[c(n+1) + 3(n-1) + 2k]hY \\ &+\frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)]\eta(hY)\xi \\ &+[2n - 2 + \mu]h^2Y - \frac{1}{2}[c(n+1) + 3(n-1) + 2k]hY \\ &-\frac{1}{2}[-c(n+1) - 3(n-1) + 2k(2n-1)] \\ &\eta(Y)h\xi - [2n - 2 + \mu]h^2Y = 0, \end{aligned} \tag{6.3}$$

and

$$\eta(QX)Y-\eta(QY)X=2nk[\eta(X)Y-\eta(Y)X], \tag{6.4}$$

and

$$\eta(QX)hY-\eta(QY)hX=2nk[\eta(X)hY-\eta(Y)hX]. \tag{6.5}$$

Using (6.3), (6.4) and (6.5) in (6.2), we have

$$4nk\{a[\eta(X)Y-\eta(Y)X]+b[\eta(X)hY-\eta(Y)hX\}=0. \tag{6.6}$$

From (6.6), we may conclude that if $a=0$ then either $k=0$ or $b=0$ or

$$[\eta(X)hY-\eta(Y)hX]=0. \tag{6.7}$$

Putting $Y=\xi$ in the above equation yields

$$hX=0.$$

If $k=0$, then from (2.12), we have $\mu=1$ and constant $\phi-$sectional curvature $c=-1$.
If $\mu=0$ for particular $n=1$, then $M(c)$ is a $N(k)$- contact space form.
If $h=0$, then $M(c)$ is a Sasakian space form. □

7 $\phi-M-$Projectively Semisymmetric $(k,\mu)-$Contact Space Forms

Definition 7.1. A $(k,\mu)-$contact space form is said to be $\phi-M-$projectively semi-symmetric if $\widetilde{F}(X,Y)\cdot\phi=0$ for all $X,Y\in TM$.

Proposition 7.2. *Let $M(c)$ be a $\phi-M-$projectively semi-symmetric $(k,\mu)-$contact space form, then $\mu=\frac{2}{2n+1}$.*

Proof. Suppose $M(c)$ be a $\phi-M-$projectively semi-symmetric $(k,\mu)-$contact space form. Then

$$\widetilde{F}(X,Y)\phi Z-\phi(\widetilde{F}(X,Y)Z)=0. \tag{7.1}$$

From (1.1), it follows that

$$\begin{aligned}\widetilde{F}(X,Y)\phi Z &= R(X,Y)\phi Z-\frac{1}{4n}[S(Y,\phi Z)X-S(X,\phi Z)Y\\ &\quad +g(Y,\phi Z)QX-g(X,\phi Z)QY].\end{aligned} \tag{7.2}$$

Using (2.18) in (7.2), we get

$$\begin{aligned}\widetilde{F}(X,Y)\phi Z &= R(X,Y)\phi Z-\\ &\quad \frac{1}{4n}\{[c(n+1)+3(n-1)+2k][g(Y,\phi Z)X\\ &\quad -g(X,\phi Z)Y]+\frac{1}{2}[-c(n+1)-3(n-1)+\\ &\quad 2k(2n-1)][g(Y,\phi Z)\eta(X)\xi-g(X,\phi Z)\eta(Y)\xi]+\\ &\quad [2n-2+\mu][g(hY,\phi Z)X-g(hX,\phi Z)Y\\ &\quad +g(Y,\phi Z)hX-g(X,\phi Z)hY]\}\end{aligned} \tag{7.3}$$

Again,

$$\begin{aligned}\phi(\widetilde{F}(X,Y)Z) &= \phi R(X,Y)Z\\ &\quad -\frac{1}{4n}\{[c(n+1)+3(n-1)+2k][g(Y,Z)\phi X\\ &\quad -g(X,Z)\phi Y+\frac{1}{2}[-c(n+1)-3(n-1)+\\ &\quad 2k(2n-1)][\eta(Y)\eta(Z)\phi X-\eta(Z)\eta(X)\phi Y+\\ &\quad g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi]+\\ &\quad [2n-2+\mu][g(hY,Z)\phi X-g(hX,Z)\phi Y\\ &\quad +g(Y,Z)h\phi X-g(X,Z)h\phi Y]\}.\end{aligned} \tag{7.4}$$

Using (7.3) and (7.4) in (7.2), we have

$$\begin{aligned}(\widetilde{F}(X,Y)\cdot\phi)Z &= R(X,Y)\phi Z-\phi R(X,Y)Z\\ &-\frac{1}{4n}\{[c(n+1)+3(n-1)+2k][g(Y,\phi Z)X\\ &-g(X,\phi Z)Y]+\frac{1}{2}[-c(n+1)-3(n-1)+\\ &2k(2n-1)][g(Y,\phi Z)\eta(X)\xi-g(X,\phi Z)\eta(Y)\xi]+\\ &[2n-2+\mu][g(hY,\phi Z)X-g(hX,\phi Z)Y+g(Y,\phi Z)hX\\ &-g(X,\phi Z)hY]\}+\frac{1}{4n}\{[c(n+1)+\\ &3(n-1)+2k][g(Y,Z)\phi X-g(X,Z)\phi Y\\ &+\frac{1}{2}[-c(n+1)-3(n-1)+2k(2n-1)][\eta(Y)\eta(Z)\phi X\\ &-\eta(Z)\eta(X)\phi Y+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi]\\ &+[2n-2+\mu][g(hY,Z)\phi X-g(hX,Z)\phi Y\\ &+g(Y,Z)h\phi X-g(X,Z)h\phi Y]\}=0. \qquad (7.5)\end{aligned}$$

Putting the value of $R(X,Y)\phi Z$ and $\phi R(X,Y)Z$ in (7.5) and taking inner product with W of (7.5) and contracting Y and W, we obtain

$$\begin{aligned}&\{\frac{c+3}{4}(1-2n)+\frac{c-1}{4}(2n-1)+\frac{1}{4n}(2n-1)\\ &[c(n+1)+3(n-1)+2k]\}g(\phi Z,X)+\\ &\{\frac{1}{2}(1-2n)+\frac{[2n-2+\mu]}{4n}(2n+1)\}g(\phi Z,hX)=0. \qquad (7.6)\end{aligned}$$

Putting $X=hX$ in the above equation yields

$$\begin{aligned}&\{\frac{c+3}{4}(1-2n)+\frac{c-1}{4}(2n-1)+\frac{1}{4n}(2n-1)\\ &[c(n+1)+3(n-1)+2k]\}g(\phi Z,hX)+\\ &\{\frac{1}{2}(1-2n)+\frac{[2n-2+\mu]}{4n}(2n+1)\}g(\phi Z,h^2X)=0. \qquad (7.7)\end{aligned}$$

Taking trace in both sides of (7.7) and using $trace(h)=0$, we get

$$\mu=\frac{2}{2n+1}.$$

□

From the above proposition we can state the following:

Theorem 7.3. *A three dimensional $\phi-M-$projectively semi-symmetric $(k,\mu)-$contact space form reduces to an $N(k)-$contact space form.*

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Author information

SHYAM KISHOR, PUSHPENDRA VERMA, Department of Mathematics and Astronomy, University of Lucknow, INDIA.
E-mail: `skishormath@gmail.com`,
`pushpendra140690@gmail.com`

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