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A numerical algorithm to inversing a Toeplitz heptadiagonal matrix

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Abstract In this paper, we compare two algorithms for computing the inverse of a heptadiagonal Toeplitz matrix. The numerical results are given to compare the effectiveness of this method.

1 Introduction:

Heptadiagonal matrices frequently arise from boundary value problems, the finite element method and the spectral method, and could be applied to the mathematical representation of high dimensional, nonlinear electromagnetic interference signals [1-7]. Heptadiagonal matrices are a certain class of special matrices, and other common types of special matrices are Jordan, Frobenius, generalized Vandermonde, Hermite, centrosymmetric, and arrowhead matrices [8-12].
The heptadiagonal systems emanate in many numerical models. These kinds of matrices appear in many areas of science and engineering. So a good technique for computing the inverse of such matrices is required. The paper is organized as follows: In Section 2,3 the first and the second algorithms for computing the inverse of hetadiagonal toeplitz matrix. The illustrative examples are presented in Section 4. Conclusions of the work are given in Section 5.

2 Inverse of a Toeplitz Heptadiagonal matrix algoritm 1:

Definition 2.1. We consider the heptadiagonal $Toeplitz$ matrix:

$$\mathbf{H} = \begin{bmatrix} a & b & c & d & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \alpha & a & b & c & d & \ddots & & & & & \vdots \\ \beta & \alpha & a & b & c & \ddots & \ddots & & & & \vdots \\ \gamma & \beta & \alpha & a & b & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \gamma & \beta & \alpha & a & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & a & b & c & d & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \alpha & a & b & c & d \\ \vdots & & & \ddots & \ddots & \ddots & \beta & \alpha & a & b & c \\ \vdots & & & \ddots & \ddots & \ddots & \gamma & \beta & \alpha & a & b \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \gamma & \beta & \alpha & a \end{bmatrix}$$

where: $\mathbf{H} \in \mathbf{M}_{n,n}(\mathbb{K})$. Also γ, β, α, a, b, c and d are an arbitrary numbers. Assume that $\mathbf{H}$ is non-singular and denote:

$$\mathbf{H}^{-1} = (D_1, D_2, ..., D_n)$$

where $(D_i)_{1\leq i\leq n}$ are the columns of the inverse $\mathbf{H}^{-1}$.

From the relation $\mathbf{H}^{-1}\mathbf{H} = I_n$ where I_n denotes the identity matrix of order n. We deduce the relations:

$$D_{n-3} = \frac{1}{d}(E_n - cD_{n-2} - bD_{n-1} - aD_n)$$

$$D_{n-4} = \frac{1}{d}(E_{n-1} - cD_{n-3} - bD_{n-2} - aD_{n-1} - \alpha D_n) \tag{2.1}$$

$$D_{n-5} = \frac{1}{d}(E_{n-2} - cD_{n-4} - bD_{n-3} - aD_{n-2} - \alpha D_{n-1} - \beta D_n)$$

And:

$$D_{j-3} = \frac{1}{d}(E_j - cD_{j-1} - bD_j - aD_{j+1} - \alpha D_{j+2} - \beta C_{j+3} - \gamma D_{j+4}) \quad for\ \ 4 \leq j \leq n-3$$

Where $E_j = [(\delta_{i,j})_{1\leq i\leq n}]^t \in \mathbb{K}^n$ is the vector of order j of the canonical basis of $\mathbb{K}^n$.
Consider the sequence of numbers $(A_i)_{(0\leq i\leq n)}$, $(B_i)_{(0\leq i\leq n)}$ and $(C_i)_{(0\leq i\leq n)}$ characterized by a term recurrence relation:

$$A_0 = 0$$

$$A_1 = 0$$

$$A_2 = 1$$

$$aA_0 + bA_1 + cA_2 + dA_3 = 0 = 0$$

$$\alpha A_0 + aA_1 + bA_3 + cA_4 + dA_5 = 0$$

$$\beta A_0 + \alpha A_1 + aA_2 + bA_3 + cA_4 + dA_5 = 0$$

$$\gamma A_0 + \beta A_1 + \alpha A_2 + aA_3 + bA_4 + cA_5 + dA_6 = 0$$

$$\gamma A_{j-4} + \beta A_{j-3} + \alpha A_{j-2} + aA_{j-1} + bA_j + cA_{j+1} + dA_{j+2} = 0 \quad for\ \ 5 \leq j \leq n-3$$

$$\gamma A_{n-6} + \beta A_{n-5} + \alpha A_{n-4} + aA_{n-3} + bA_{n-2} + cA_{n-1} + A_n = 0$$

$$\gamma A_{n-5} + \beta A_{n-4} + \alpha A_{n-3} + aA_{n-2} + bA_{n-1} + A_{n+1} = 0$$

$$\gamma A_{n-4} + \beta A_{n-3} + \alpha A_{n-2} + aA_{n-1} + A_{n+2} = 0$$

And:

$$B_0 = 0$$

$$B_1 = 1$$

$$B_2 = 0$$

$$aB_0 + bB_1 + cB_2 + dB_3 = 0 = 0$$

$$\alpha B_0 + aB_1 + bB_2 + cB_3 + dB_4 = 0$$

$$\beta B_0 + \alpha B_1 + aB_2 + bB_3 + cB_4 + dB_5 = 0$$

$$\gamma B_0 + \beta B_1 + \alpha B_2 + aB_3 + bB_4 + cB_5 + dB_6 = 0 \tag{2.2}$$

$$\gamma B_{j-4} + \beta B_{j-3} + \alpha B_{j-2} + aB_{j-1} + bB_j + cB_{j+1} + dB_{j+2} = 0 \quad for\ \ 5 \leq j \leq n-3$$

$$\gamma B_{n-6} + \beta B_{n-5} + \alpha B_{n-4} + aB_{n-3} + bB_{n-2} + cB_{n-1} + B_n = 0$$

$$\gamma B_{n-5} + \beta B_{n-4} + \alpha B_{n-3} + aB_{n-2} + bB_{n-1} + B_{n+1} = 0$$

$$\gamma B_{n-4}+\beta B_{n-3}+\alpha B_{n-2}+aB_{n-1}+B_{n+2}=0$$

We have also:

$$C_0=1$$

$$C_1=0$$

$$C_2=0$$

$$aC_0+bC_1+cC_2+dC_3=0=0$$

$$\alpha C_0+aC_1+bC_2+cC_3+dC_4=0$$

$$\beta C_0+\alpha C_1+aC_2+bC_3+cC_4+dC_5=0$$

$$\gamma C_0+\beta C_1+\alpha C_2+aC_3+bC_4+cC_5+dC_6=0 \tag{2.3}$$

$$\gamma C_{j-4}+\beta C_{j-3}+\alpha C_{j-2}+aC_{j-1}+bC_j+cC_{j+1}+dC_{j+2}=0 \quad for\;\; 5\le j\le n-3$$

$$\gamma C_{n-6}+\beta C_{n-5}+\alpha C_{n-4}+aC_{n-3}+bC_{n-2}+cC_{n-1}+C_n=0$$

$$\gamma C_{n-5}+\beta C_{n-4}+\alpha C_{n-3}+aC_{n-2}+bC_{n-1}+C_{n+1}=0$$

$$\gamma C_{n-4}+\beta C_{n-3}+\alpha C_{n-2}+aC_{n-1}+C_{n+2}=0$$

We can give a matrix form to this term recurrence:

$$\mathbf{HA}=-A_nE_{n-2}-A_{n+1}E_{n-1}-A_{n+2}E_n. \tag{2.4}$$

$$\mathbf{HB}=-B_nE_{n-2}-B_{n+1}E_{n-1}-B_{n+2}E_n. \tag{2.5}$$

And:

$$\mathbf{HC}=-C_nE_{n-2}-C_{n+1}E_{n-1}-C_{n+2}E_n. \tag{2.6}$$

where $\mathbf{A}=[A_0,A_1,...,A_{n-1}]^t$, $\mathbf{B}=[B_0,B_1,...,B_{n-1}]^t$ and $\mathbf{C}=[C_0,C_1,...,C_{n-1}]^t$
Let's define for each $0\le i\le n+2$

$$Q_i=det\begin{pmatrix}A_n & A_{n+1} & A_i\\ B_n & B_{n+1} & B_i\\ C_n & C_{n+1} & C_i\end{pmatrix}$$

$$P_i=det\begin{pmatrix}A_n & A_i & A_{n+2}\\ B_n & B_i & B_{n+2}\\ C_n & C_i & C_{n+2}\end{pmatrix}$$

$$R_i=det\begin{pmatrix}A_i & A_{n+1} & A_{n+2}\\ B_i & B_{n+1} & B_{n+2}\\ C_i & C_{n+1} & C_{n+2}\end{pmatrix}$$

Immediately we have:

$$\mathbf{HQ}=-Q_{n+2}E_{n-2}\quad and\quad \mathbf{HP}=-P_{n+1}E_{n-1}\quad and\quad \mathbf{HR}=-R_nE_n \tag{2.7}$$

Where: $\mathbf{Q}=[Q_0,Q_1,...,Q_{n-1}]^t$, $\mathbf{P}=[P_0,P_1,...,P_{n-1}]^t$ and $\mathbf{R}=[R_0,R_1,...,R_{n-1}]^t$

Lemma 2.2. *If $Q_{n+2}=0$, then the matrix **H** is singular.*

Proof. If $Q_{n+2}=0$, The $\mathbf{Q}$ is non-null and by (8) we have $\mathbf{HQ}=0$. We conclude that $\mathbf{H}$ is singular. □

Theorem 2.3. *We supposed that $Q_{n+2} \neq 0$, then **H** is non-singular.*
And:

$$D_n = \frac{-1}{Q_{n+2}}[Q_0, Q_1, ..., Q_{n-1}]^t$$

$$D_{n-1} = \frac{-1}{P_{n+1}}[P_0, P_1, ..., P_{n-1}]^t \tag{2.8}$$

And:

$$D_{n-2} = \frac{-1}{R_n}[R_0, R_1, ..., R_{n-1}]^t$$

Proof. We have $det(\mathbf{H}) = (-1)^{n+1}(d)^{n-3}Q_{n+2} \neq 0$. So **H** is invertible from the relations Equations 7 and 8 we have $\mathbf{H}D_n = E_n,\ \mathbf{H}D_{n-1} = E_{n-1}\ and\ \mathbf{H}D_{n-2} = E_{n-2}$.
The proof is completed. □

Algorithm: New efficient computational algorithm for computing the Inverse of the heptatadiagonal matrix
INPUT: $the\ dimension\ n$, β, α, γ, a, b, c and d.
OUTPUT: The inverse $H^{-1} = (D_1, D_2, ..., D_n)$.
First step:

$$D_n = \frac{-1}{Q_{n+2}}[Q_0, Q_1, ..., Q_{n-1}]^t$$

$$D_{n-1} = \frac{-1}{P_{n+1}}[P_0, P_1, ..., P_{n-1}]^t$$

And

$$D_{n-2} = \frac{-1}{R_n}[R_0, R_1, ..., R_{n-1}]^t$$

Second step:

$$D_{n-3} = \frac{1}{d}(E_n - cD_{n-2} - bD_{n-1} - aD_n)$$

$$D_{n-4} = \frac{1}{d}(E_{n-1} - cD_{n-3} - bD_{n-2} - aD_{n-1} - \alpha D_n)$$

$$D_{n-5} = \frac{1}{d}(E_{n-2} - cD_{n-4} - bD_{n-3} - aD_{n-2} - \alpha D_{n-1} - \beta D_n)$$

Step 3:

$$D_{j-3} = \frac{1}{d}(E_j - cD_{j-1} - bD_j - aD_{j+1} - \alpha D_{j+2} - \beta C_{j+3} - \gamma D_{j+4}) \qquad for\ \ 4 \leq j \leq n-3$$

3. Inverse of a Toeplitz Heptadiagonal matrix algorithm 2:

Definition 2.4. The Hessenberg (or lower Hessenberg) matrices are the matrices $H = [h_{ij}]$ satisfying the condition $h_{ij} = 0$ for $j - i > 1$. More general, the matrix is $K - Hessenberg$ if and only if $H_{ij} = 0$ for $j - i > K$.

Theorem [17] : Let H be a a strict $K - Hessenberg$ matrix with the block decomposition:

$$\mathbf{H} = \begin{bmatrix} B & A \\ D & C \end{bmatrix}\ ,\ \ A \in \mathbf{M}_{n-k}(F)\ ,\ B \in \mathbf{M}_{(n-k)\times k}(F)\ ,\ C \in \mathbf{M}_{k\times(n-k)}(F)\ ,\ D \in \mathbf{M}_k.$$

The H is invertible if and only if $CA^{-1}B - D$ is invertible and if **H** is invertible we have:

$$H^{-1} = \begin{bmatrix} 0 & 0 \\ A^{-1} & 0 \end{bmatrix} - \begin{bmatrix} I_k \\ -A^{-1}B \end{bmatrix}(CA^{-1}B - D)^{-1}\begin{bmatrix} -CA^{-1} & I_k \end{bmatrix}$$

Roksana Slowik [16] worked in the particular case of Toeplitz-Hessenberg matrices. Our work is applied on a Toeplitz heptadiagonal matrix **H**. Then we find:

$$\mathbf{H}^{-1} = \begin{bmatrix} 0 & 0 \\ A^{-1} & 0 \end{bmatrix} - \begin{bmatrix} I_3 \\ -A^{-1}B \end{bmatrix} (CA^{-1}B - D)^{-1} \begin{bmatrix} -CA^{-1} & I_3 \end{bmatrix} \tag{2.9}$$

The blocks B, C,D and A are given as:

$$B = \begin{bmatrix} a & b & c \\ \alpha & a & b \\ \beta & \alpha & a \\ \gamma & \beta & \alpha \\ 0 & \gamma & \alpha \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & \cdots & 0 & \gamma & \beta & \alpha & a & b & c \\ 0 & \cdots & 0 & 0 & \gamma & \beta & \alpha & a & b \\ 0 & \cdots & 0 & 0 & 0 & \gamma & \beta & \alpha & a \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And:

$$A = \begin{bmatrix} d & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ c & d & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ b & c & d & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a & b & c & d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \alpha & a & b & c & d & 0 & \cdots & \cdots & \cdots & \vdots \\ \beta & \alpha & a & b & c & d & 0 & \cdots & \cdots & \vdots \\ \gamma & \beta & \alpha & a & b & c & d & 0 & \ddots & \vdots \\ 0 & \gamma & \beta & \alpha & a & b & c & d & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma & \beta & \alpha & a & b & c & d \end{bmatrix}$$

Then the inverse of the matrix A will be:

$$A^{-1} = \frac{1}{d}\sum_{k=0}^{n-2}\sum_{r=1}^{n-1-k} l_k E_{r+k,r} \tag{2.10}$$

Where:

$$l_0 = 1; \quad l_1 = \frac{-c}{d}; \quad l_2 = (\frac{-c}{d})^2 - \frac{b}{d} = -\frac{1}{d}(cl_1 + bl_0); \quad l_3 = (\frac{-c}{d})^3 + \frac{2bc}{d^2} - \frac{a}{d}$$

$$l_4 = (\frac{-c}{d})^4 - \frac{3bc^2}{d^3} + \frac{2ac+b^2}{d^2} - \frac{\alpha}{d}; \quad l_5 = (\frac{-c}{d})^5 + \frac{4bc^3}{d^4} - \frac{3b^2c+3ac^2}{d^3} + \frac{2ab}{d^2} + \frac{\alpha c}{d^2} - \frac{\beta}{d}$$

$$And \quad l_k = -\frac{1}{d}(cl_{k-1} + bl_{k-2} + al_{k-3} + \alpha l_{k-4} + \beta l_{k-5} + \gamma l_{k-6})$$

Proof. Since A is a triangular Toeplitz matrix, so is A^{-1}. We prove (11) inductively on n. We have:

$$\begin{bmatrix} 1 &&&&& \\ 0 & 1 &&&& \\ 0 & -\frac{c}{d} & 1 &&& \\ 0 & (-\frac{c}{d})^2 & -\frac{2c}{d} & 1 && \\ 0 & (-\frac{c}{d})^3 & -\frac{b}{d}+\frac{3c^2}{d^2} & -\frac{2c}{d} & 1 & \\ 0 & (-\frac{c}{d})^4 & (-\frac{c}{d})^3+\frac{3cb}{d^2} & \frac{3c^2}{d^2}-\frac{2b}{d} & -\frac{2c}{d} & 1 \end{bmatrix} \begin{bmatrix} 1 &&&&& \\ -\frac{c}{d} & 1 &&&& \\ -\frac{b}{d} & 0 & 1 &&& \\ -\frac{a}{d} & -\frac{b}{d} & \frac{c}{d} & 1 && \\ -\frac{\alpha}{d} & -\frac{a}{d} & 0 & \frac{c}{d} & 1 & \\ -\frac{\beta}{d} & -(\frac{b}{d})^2-\frac{\alpha}{d} & \frac{cb}{d^2}-\frac{a}{d} & \frac{c}{d} & 1 & \end{bmatrix} = \begin{bmatrix} 1 &&&&& \\ l_1 & 1 &&&& \\ l_2 & l_1 & 1 &&& \\ l_3 & l_2 & l_1 & 1 && \\ l_4 & l_3 & l_2 & l_1 & 1 & \\ l_5 & l_4 & l_3 & l_2 & l_1 & 1 \end{bmatrix} \tag{2.11}$$

With:

$$l_1 = \frac{-c}{d},\ l_2 = (\frac{-c}{d})^2 - \frac{b}{d} = -\frac{1}{d}(cl_1 + bl_0),\ l_3 = (\frac{-c}{d})^3 + \frac{2bc}{d^2} - \frac{a}{d},\ l_4 = (\frac{-c}{d})^4 - \frac{3bc^2}{d^3} + \frac{2ac+b^2}{d^2}$$

$$And\ l_5 = (\frac{-c}{d})^5 + \frac{4bc^3}{d^4} - \frac{3b^2c+3ac^2}{d^3} + \frac{2ab}{d^2} + \frac{\alpha c}{d^2} - \frac{\beta}{d}$$

The first step of induction holds.
Consider now $n > 4$. We have:

$$\begin{bmatrix} 1 &&&&&&&&& \\ 0 & 1 &&&&&&&& \\ 0 & l_1 & 1 &&&&&&& \\ 0 & l_2 & l_1 & 1 &&&&&& \\ 0 & l_3 & l_2 & l_1 & 1 &&&&& \\ 0 & l_4 & l_3 & l_2 & l_1 & 1 &&&& \\ 0 & l_5 & l_4 & l_3 & l_2 & l_1 & 1 &&& \\ 0 & l_6 & l_5 & l_4 & l_3 & l_2 & l_1 & 1 && \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & l_{n-3} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & l_1 & 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ -\frac{c}{d} & 1 && \\ -\frac{b}{d} &&& \\ -\frac{a}{d} &&& \\ && \ddots & \\ -\frac{\alpha}{d} &&& \\ -\frac{\beta}{d} &&& \\ -\frac{\gamma}{d} &&& \\ 0 &&& \\ \vdots &&& \ddots \\ 0 &&& 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 &&&&&&&& \\ l_1 & 1 &&&&&&& \\ l_2 & l_1 & 1 &&&&&& \\ l_3 & l_2 & l_1 & 1 &&&&& \\ l_4 & l_3 & l_2 & l_1 & 1 &&&& \\ l_5 & l_4 & l_3 & l_2 & l_1 & 1 &&& \\ l_6 & l_5 & l_4 & l_3 & l_2 & l_1 & 1 && \\ l_7 & l_6 & l_5 & l_4 & l_3 & l_2 & l_1 & 1 & \\ \vdots & \vdots & \vdots &&&&& \ddots & \\ l_{n-2} & l_{n-3} & \cdots && \cdots && \cdots & l_1 & 1 \end{bmatrix}$$

Then:

$$l_{n-2} = -\tfrac{1}{d}(cl_{n-3} + bl_{n-4} + al_{n-5} + \alpha l_{n-6} + \beta l_{n-7} + \gamma l_{n-8})$$

□

4. Examples

In this section, we give a numerical example to illustrate the effectiveness of our algorithm. Our algorithm is tested by MATLAB R2014a.
Consider the following 9-by-9 pentadiagonal matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 2 & 3 & 7 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & 3 & 7 & 0 & 0 & 0 & 0 \\ 0.5 & -2 & 1 & 2 & 3 & 7 & 0 & 0 & 0 \\ 4 & 0.5 & -2 & 1 & 2 & 3 & 7 & 0 & 0 \\ 0 & 4 & 0.5 & -2 & 1 & 2 & 3 & 7 & 0 \\ 0 & 0 & 4 & 0.5 & -2 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 4 & 0.5 & -2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 & 0.5 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0.5 & -2 & 1 \end{bmatrix}$$

The columns of the inverse $\mathbf{H}^{-1}$ are:

$$D_1=\begin{bmatrix}0.1504\\0.1162\\0.0810\\0.0535\\-0.0197\\0.0040\\-0.0749\\-0.0231\\-0.0250\end{bmatrix},\ D_2=\begin{bmatrix}-0.1562\\-0.0050\\0.0774\\-0.0094\\0.0809\\-0.0333\\0.1042\\-0.0521\\-0.0231\end{bmatrix},\ D_3=\begin{bmatrix}-0.0075\\-0.1942\\0.0329\\0.0424\\-0.0020\\0.0719\\-0.0088\\0.1042\\-0.0749\end{bmatrix},\ D_4=\begin{bmatrix}0.1352\\-0.0073\\0.0143\\-0.0234\\0.0456\\-0.0266\\0.0719\\-0.0333\\0.0040\end{bmatrix},\ D_5=\begin{bmatrix}-0.0568\\0.0970\\-0.0451\\-0.0002\\-0.0171\\0.0456\\-0.0020\\0.0809\\-0.0197\end{bmatrix}$$

$$D_6=\begin{bmatrix}0.0419\\-0.0518\\0.1552\\-0.0577\\-0.0002\\-0.0234\\0.0424\\-0.0094\\0.0535\end{bmatrix},\ D_7=\begin{bmatrix}-0.2098\\-0.0457\\-0.2617\\0.1552\\-0.0451\\0.0143\\0.0329\\0.0774\\0.0810\end{bmatrix},\ D_8=\begin{bmatrix}0.2727\\0.1136\\-0.0457\\-0.0518\\0.0970\\-0.0073\\-0.1942\\-0.0050\\0.1162\end{bmatrix},\ D_9=\begin{bmatrix}-0.2094\\0.2727\\-0.2098\\0.0419\\-0.0568\\0.1352\\-0.0075\\-0.1562\\0.1504\end{bmatrix}$$

Let's employ the same example and give the blocks A, B, C, and D of the matrix $\mathbf{H}$

$$A=\begin{bmatrix}7&0&0&0&0&0\\3&7&0&0&0&0\\2&3&7&0&0&0\\1&2&3&7&0&0\\-2&1&2&3&7&0\\0.5&-2&1&2&3&7\end{bmatrix}$$

$$B=\begin{bmatrix}1&2&3\\-2&1&2\\0.5&-2&1\\4&0.5&-2\\0&4&0.5\\0&0&4\end{bmatrix}$$

$$C=\begin{bmatrix}4&0.5&-2&1&2&3\\0&4&0.5&-2&1&2\\0&0&4&0.5&-2&1\end{bmatrix}$$

$$D=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Therefore the inverse $\mathbf{H}^{-1}$ is :

$$\mathbf{H}^{-1} = \begin{bmatrix} 0.1504 & -0.1562 & -0.0075 & 0.1352 & -0.0568 & 0.0419 & -0.2098 & 0.2727 & -0.2094 \\ 0.1162 & -0.0050 & -0.1942 & -0.0073 & 0.0970 & -0.0518 & -0.0457 & 0.1136 & 0.2727 \\ 0.0810 & 0.0774 & 0.0329 & 0.0143 & -0.0451 & 0.1552 & -0.2617 & -0.0457 & -0.2098 \\ 0.0535 & -0.0094 & 0.0424 & -0.0234 & -0.0002 & -0.0577 & 0.1552 & -0.0518 & 0.0419 \\ -0.0197 & 0.0809 & -0.0020 & 0.0456 & -0.0171 & -0.0002 & -0.0451 & 0.0970 & -0.0568 \\ 0.0040 & -0.0333 & 0.0719 & -0.0266 & 0.0456 & -0.0234 & 0.0143 & -0.0073 & 0.1352 \\ -0.0749 & 0.1042 & -0.0088 & 0.0719 & -0.0020 & 0.0424 & 0.0329 & -0.1942 & -0.0075 \\ -0.0231 & -0.0521 & 0.1042 & -0.0333 & 0.0809 & -0.0094 & 0.0774 & -0.0050 & -0.1562 \\ -0.0250 & -0.0231 & -0.0749 & 0.0040 & -0.0197 & 0.0535 & 0.0810 & 0.1162 & 0.1504 \end{bmatrix}$$

In the table we give a comparison of the running time between 'toepltiz-hessenberg' algorithm and our algorithm in MATLAB R2014a.
The running time (in seconds) of two algorithms in MATLAB R2014a.

Table 1. The running time

Size of the matrix (n)	Algorithm 1	Algorithm 2	LU method
100	0.036954	0.137472	0.918161
200	0.061992	0.520568	2.547606
300	0.090051	1.201593	6.816085
500	0.149696	3.235135	24.165349
1000	0.314484	13.066815	149.750575

3 Conclusion

In this work new numeric and symbolic algorithms have been developed for finding the inverse of any nonsingular heptadiagonal matrix. And we show the efficient of our fast algorithm (algorithm 1) by comparing it with the toeplitz-hessenberg algorithm (algorithm 2) and with the LU method.

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