Palestine Journal of Mathematics

Vol. 10(1)(2021) , 251–265

Hermite-Hadamard type fractional integral inequalities for products of two generalized beta (r, g)-preinvex functions

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary: 26A51. Secondary: 26A33, 26D07, 26D10, 26D15.

Keywords and phrases: Hermite-Hadamard type inequality, Hölder's inequality, Minkowski's inequality, Cauchy's inequality, power mean inequality, Riemann-Liouville fractional integral.

Abstract In the present paper, a new class of generalized beta (r, g)-preinvex functions is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving products of two generalized beta (r, g)-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for products of two generalized beta (r, g)-preinvex functions via Riemann-Liouville fractional integrals are established. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities and also extend some results appeared in the literature, see [1]. At the end, some conclusions and future research are given.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, see [1],[3]-[8],[10]-[22],[24],[25],[28],[29],[32],[33] and the references cited therein.

Definition 1.2. Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Now, let us recall some basic definitions of various convex functions.

Definition 1.3. [7] A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0,1].$$

Definition 1.4. [9] A function $f : \mathbb{R}_\circ \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \geq 0, \ \lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated in [9].

Definition 1.5. [2] A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true, see [2],[31] and the references therein.

Definition 1.6. [27] The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star |f|$, see [30].
Recently, Liu in [23] obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.3 of P-function. Also in [26], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized beta (r,g)-preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving products of two generalized beta (r,g)-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type integral inequalities for products of two generalized beta (r,g)-preinvex functions via Riemann-Liouville fractional integrals are given. In Section 4, some conclusions and future research are given. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula for products of two generalized beta (r,g)-preinvex functions and Hermite-Hadamard type inequalities via Riemann-Liouville fractional integral.

2 New integral inequalities for products

Definition 2.1. [8] A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

Definition 2.3. [11] A positive function f on the invex set K is said to be logarithmically preinvex, if

$$f(u + t\eta(v,u)) \leq f^{1-t}(u) f^t(v)$$

for all $u, v \in K$ and $t \in [0,1]$.

Definition 2.4. [11] The function f on the invex set K is said to be r-preinvex with respect to η, if

$$f(u+t\eta(v,u)) \leq M_r(f(u),f(v);t)$$

holds for all $u,v \in K$ and $t\in[0,1]$, where

$$M_r(x,y;t)=\begin{cases}\left[(1-t)x^r+ty^r\right]^{\frac{1}{r}}, & \text{if } r\neq 0;\\ x^{1-t}y^t, & \text{if } r=0,\end{cases}$$

is the weighted power mean of order r for positive numbers x and y.

We next give new definition, to be referred as generalized beta (r,g)-preinvex function.

Definition 2.5. Let $K\subseteq\mathbb{R}$ be an open m-invex set with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$, $g:[0,1]\longrightarrow[0,1]$ be a differentiable function and $\varphi: I\longrightarrow K$ is a continuous function. The function $f:K\longrightarrow(0,+\infty)$ is said to be generalized beta (r,g)-preinvex with respect to η, if

$$f\left(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m)\right)\leq M_r(mf(\varphi(x)),f(\varphi(y)),p,q;g(t)) \tag{2.1}$$

holds for some fixed $m\in(0,1]$, for any fixed $p,q>-1$, and for all $x,y\in I, t\in[0,1]$, where

$$M_r(mf(\varphi(x)),f(\varphi(y)),p,q;g(t))$$

$$=\begin{cases}\left[mg^p(t)(1-g(t))^q f^r(\varphi(x))+g^q(t)(1-g(t))^p f^r(\varphi(y))\right]^{\frac{1}{r}}, & \text{if } r\neq 0;\\ \left[mf(\varphi(x))\right]^{g^p(t)(1-g(t))^q}\left[f(\varphi(y))\right]^{g^q(t)(1-g(t))^p}, & \text{if } r=0,\end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2.6. In Definition 2.5, it is worthwhile to note that the class of generalized beta (r,g)-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 1.4. Also, for $r=1, p=0, q=s$, $g(t)=t$, $\forall\, t\in[0,1]$ and $\varphi(x)=x$, $\forall\, x\in I$, we get the notion of generalized (s,m)-preinvex function, see [8].

Let see the following relevant example of generalized beta (r,g)-preinvex function wich is not convex.

Example 2.7. Let take $m=r=\dfrac{1}{2}$ two fixed numbers, $p,\ q>1$, $g(t)=e^{\alpha t}$, where $\alpha>1$ and $\varphi(t)=t^2$. Consider the function $f:[0,+\infty)\longrightarrow[0,+\infty)$ by

$$f(x)=\begin{cases}x, & 0\leq x\leq 2;\\ 2, & x>2.\end{cases}$$

Also, we define the function $\eta:[0,+\infty)\times[0,+\infty)\times(0,1]\longrightarrow\mathbb{R}$ by

$$\eta(y,x,m)=\begin{cases}-my, & 0\leq y\leq 2;\\ m(x+y), & y>2,\end{cases}$$

Then f is generalized beta $\left(\dfrac{1}{2},e^{\alpha t}\right)$-preinvex with respect to η. But f is not convex (consider $x=0, y=3$ and $t\in(0,1]$).

In this section, in order to prove our main results regarding some new integral inequalities involving products of two generalized beta (r,g)-preinvex functions, we need the following lemma.

Lemma 2.8. *Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ is a differentiable function. Assume that $f,h: K=[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ are continuous* *functions on* K° *with respect to the same* $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ *for* $\eta(\varphi(b), \varphi(a), m) > 0$. *Then for some fixed* $m \in (0,1]$ *and* $p, q > 0$, *we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m)$$
$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))$$
$$\times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)].$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx$$
$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$
$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q$$
$$\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)]$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m)$$
$$\times \int_0^1 g^p(t)(1 - g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))$$
$$\times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m))d[g(t)].$$

This completes the proof of the lemma. □

The following definition will be used in the sequel.

Definition 2.9. The Euler beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

Theorem 2.10. *Let* $k > 1, r > 1$ *and* $r^{-1} + l^{-1} = 1$. *Let* $\varphi : I \longrightarrow K$ *be a continuous function and* $g : [0,1] \longrightarrow [0,1]$ *is a differentiable function. Assume that* $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ *are continuous functions on* K° *with* $\eta(\varphi(b), \varphi(a), m) > 0$. *If* $f^{\frac{k}{k-1}}, h^{\frac{k}{k-1}}$ *are respectively generalized beta* (r, g)*-preinvex function and generalized beta* (l, g)*-preinvex function on an open* m*-invex set* K° *with respect to the same* $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0,1]$ *where* $s, \gamma > -1$, *then for any fixed* $p, q > 0$, *we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx$$
$$\leq \left(\frac{1}{2}\right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{1}{k}}(g(t); k, p, q)$$
$$\times \left[\left\{ m f^{\frac{rk}{k-1}}(\varphi(a)) B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2\gamma, 2s\right) + f^{\frac{rk}{k-1}}(\varphi(b)) B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2s, 2\gamma\right) \right\}^{\frac{2}{r}} \right.$$
$$\left. + \left\{ m h^{\frac{lk}{k-1}}(\varphi(a)) B^{\frac{l}{2}}\left(g(t); \frac{1}{l}, 2\gamma, 2s\right) + h^{\frac{lk}{k-1}}(\varphi(b)) B^{\frac{l}{2}}\left(g(t); \frac{1}{l}, 2s, 2\gamma\right) \right\}^{\frac{2}{l}} \right]^{\frac{k-1}{k}}, \quad (2.2)$$

where $B(g(t); k, p, q) := \int_0^1 g^{kp}(t)(1 - g(t))^{kq} d[g(t)]$.

Proof. Since $f^{\frac{k}{k-1}}$ and $h^{\frac{k}{k-1}}$ are respectively generalized beta (r,g)-preinvex function and generalized (l,g)-preinvex function on an open m-invex set K°, combining with Lemma 2.8, Hölder inequality, Cauchy and Minkowski inequality, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^{p}(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{q} f(x)h(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\left[\int_0^1 g^{kp}(t)(1-g(t))^{kq}d[g(t)]\right]^{\frac{1}{k}}$$

$$\times\left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))\right.$$

$$\left.\times h^{\frac{k}{k-1}}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\int_0^1 \left(mg^{\gamma}(t)(1-g(t))^{s}f^{\frac{rk}{k-1}}(\varphi(a))+g^{s}(t)(1-g(t))^{\gamma}f^{\frac{rk}{k-1}}(\varphi(b))\right)^{\frac{1}{r}}\right.$$

$$\left.\times\left(mg^{\gamma}(t)(1-g(t))^{s}h^{\frac{lk}{k-1}}(\varphi(a))+g^{s}(t)(1-g(t))^{\gamma}h^{\frac{lk}{k-1}}(\varphi(b))\right)^{\frac{1}{l}}d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\leq \left(\frac{1}{2}\right)^{\frac{k-1}{k}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\int_0^1 \left(mg^{\gamma}(t)(1-g(t))^{s}f^{\frac{rk}{k-1}}(\varphi(a))+g^{s}(t)(1-g(t))^{\gamma}f^{\frac{rk}{k-1}}(\varphi(b))\right)^{\frac{2}{r}}d[g(t)]\right.$$

$$\left.+\int_0^1 \left(mg^{\gamma}(t)(1-g(t))^{s}h^{\frac{lk}{k-1}}(\varphi(a))+g^{s}(t)(1-g(t))^{\gamma}h^{\frac{lk}{k-1}}(\varphi(b))\right)^{\frac{2}{l}}d[g(t)]\right]^{\frac{k-1}{k}}$$

$$\leq \left(\frac{1}{2}\right)^{\frac{k-1}{k}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\left\{\left(\int_0^1 m^{\frac{2}{r}}g^{\frac{2\gamma}{r}}(t)(1-g(t))^{\frac{2s}{r}}f^{\frac{2k}{k-1}}(\varphi(a))d[g(t)]\right)^{\frac{r}{2}}\right.\right.$$

$$\left.+\left(\int_0^1 g^{\frac{2s}{r}}(t)(1-g(t))^{\frac{2\gamma}{r}}f^{\frac{2k}{k-1}}(\varphi(b))d[g(t)]\right)^{\frac{r}{2}}\right\}^{\frac{2}{r}}$$

$$+\left\{\left(\int_0^1 m^{\frac{2}{l}}g^{\frac{2\gamma}{l}}(t)(1-g(t))^{\frac{2s}{l}}h^{\frac{2k}{k-1}}(\varphi(a))d[g(t)]\right)^{\frac{l}{2}}\right.$$

$$\left.\left.+\left(\int_0^1 g^{\frac{2s}{l}}(t)(1-g(t))^{\frac{2\gamma}{l}}h^{\frac{2k}{k-1}}(\varphi(b))d[g(t)]\right)^{\frac{l}{2}}\right\}^{\frac{2}{l}}\right]^{\frac{k-1}{k}}$$

$$=\left(\frac{1}{2}\right)^{\frac{k-1}{k}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{1}{k}}(g(t);k,p,q)$$

$$\times\left[\left\{mf^{\frac{rk}{k-1}}(\varphi(a))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2\gamma,2s\right)+f^{\frac{rk}{k-1}}(\varphi(b))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2s,2\gamma\right)\right\}^{\frac{2}{r}}\right.$$

$$+\left\{mh^{\frac{lk}{k-1}}(\varphi(a))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2\gamma,2s\right)+h^{\frac{lk}{k-1}}(\varphi(b))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2s,2\gamma\right)\right\}^{\frac{2}{l}}\Bigg]^{\frac{k-1}{k}}.$$

So, the proof of this theorem is completed. □

Corollary 2.11. *Under the same conditions as in Theorem 2.10 for $r=l=2$ and $g(t)=t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\leq\left(\frac{\beta(s+1,\gamma+1)}{2}\right)^{\frac{k-1}{k}}\beta^{\frac{1}{k}}(kp+1,kq+1)\eta^{p+q+1}(\varphi(b),\varphi(a),m)$$

$$\times\left[m\left(f^{\frac{2k}{k-1}}(\varphi(a))+h^{\frac{2k}{k-1}}(\varphi(a))\right)+\left(f^{\frac{2k}{k-1}}(\varphi(b))+h^{\frac{2k}{k-1}}(\varphi(b))\right)\right]^{\frac{k-1}{k}}.$$

Theorem 2.12. *Let $l\geq 1, r>1$ and $r^{-1}+r_1^{-1}=1$. Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ is a differentiable function. Assume that $f,h: K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ are continuous functions on K° with $\eta(\varphi(b),\varphi(a),m)>0$. If f^l,h^l are respectively generalized beta (r,g)-preinvex function and generalized beta (r_1,g)-preinvex function on an open m-invex set K° with respect to the same $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ where $s,\gamma>-1$, then for any fixed $p,q>0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{l}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg[\Bigg\{mf^{rl}(\varphi(a))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(p+\gamma),2(q+s)\right)$$

$$+f^{rl}(\varphi(b))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(p+s),2(q+\gamma)\right)\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{mh^{r_1l}(\varphi(a))B^{\frac{r_1}{2}}\left(g(t);\frac{1}{r_1},2(p+\gamma),2(q+s)\right)$$

$$+h^{r_1l}(\varphi(b))B^{\frac{r_1}{2}}\left(g(t);\frac{1}{r_1},2(p+s),2(q+\gamma)\right)\Bigg\}^{\frac{2}{r_1}}\Bigg]^{\frac{1}{l}}. \tag{2.3}$$

Proof. Since f^l and h^l are respectively generalized beta (r,g)-preinvex function and generalized (r_1,g)-preinvex function on an open m-invex set K°, combining with Lemma 2.8, the well-known power mean inequality, Cauchy and Minkowski inequality, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\leq\eta^{p+q+1}(\varphi(b),\varphi(a),m)\left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)]\right]^{\frac{l-1}{l}}$$

$$\times\Bigg[\int_0^1 g^p(t)(1-g(t))^q f^l(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))$$

$$\times h^{l}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m))d[g(t)]\Bigg]^{\frac{1}{l}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg[\int_0^1 g^{p}(t)(1-g(t))^{q}\left(mg^{\gamma}(t)(1-g(t))^{s}f^{rl}(\varphi(a))+g^{s}(t)(1-g(t))^{\gamma}f^{rl}(\varphi(b))\right)^{\frac{1}{r}}$$

$$\times\left(mg^{\gamma}(t)(1-g(t))^{s}h^{r_1 l}(\varphi(a))+g^{s}(t)(1-g(t))^{\gamma}h^{r_1 l}(\varphi(b))\right)^{\frac{1}{r_1}}d[g(t)]\Bigg]^{\frac{1}{l}}$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{l}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{1}{k}}(g(t);1,p,q)$$

$$\times\Bigg[\int_0^1\left(mg^{p+\gamma}(t)(1-g(t))^{q+s}f^{rl}(\varphi(a))+g^{p+s}(t)(1-g(t))^{q+\gamma}f^{rl}(\varphi(b))\right)^{\frac{2}{r}}d[g(t)]$$

$$+\int_0^1\left(mg^{p+\gamma}(t)(1-g(t))^{q+s}h^{r_1 l}(\varphi(a))+g^{p+s}(t)(1-g(t))^{q+\gamma}h^{r_1 l}(\varphi(b))\right)^{\frac{2}{r_1}}d[g(t)]\Bigg]^{\frac{1}{l}}$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{l}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg[\Bigg\{\left(\int_0^1 m^{\frac{2}{r}}g^{\frac{2(p+\gamma)}{r}}(t)(1-g(t))^{\frac{2(q+s)}{r}}f^{2l}(\varphi(a))d[g(t)]\right)^{\frac{r}{2}}$$

$$+\left(\int_0^1 g^{\frac{2(p+s)}{r}}(t)(1-g(t))^{\frac{2(q+\gamma)}{r}}f^{2l}(\varphi(b))d[g(t)]\right)^{\frac{r}{2}}\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{\left(\int_0^1 m^{\frac{2}{r_1}}g^{\frac{2(p+\gamma)}{r_1}}(t)(1-g(t))^{\frac{2(q+s)}{r_1}}h^{2l}(\varphi(a))d[g(t)]\right)^{\frac{r_1}{2}}$$

$$+\left(\int_0^1 g^{\frac{2(p+s)}{r_1}}(t)(1-g(t))^{\frac{2(q+\gamma)}{r_1}}h^{2l}(\varphi(b))d[g(t)]\right)^{\frac{r_1}{2}}\Bigg\}^{\frac{2}{r_1}}\Bigg]^{\frac{1}{l}}$$

$$=\left(\frac{1}{2}\right)^{\frac{1}{l}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)B^{\frac{l-1}{l}}(g(t);1,p,q)$$

$$\times\Bigg[\Bigg\{mf^{rl}(\varphi(a))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(p+\gamma),2(q+s)\right)$$

$$+f^{rl}(\varphi(b))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(p+s),2(q+\gamma)\right)\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{mh^{r_1 l}(\varphi(a))B^{\frac{r_1}{2}}\left(g(t);\frac{1}{r_1},2(p+\gamma),2(q+s)\right)$$

$$+h^{r_1 l}(\varphi(b))B^{\frac{r_1}{2}}\left(g(t);\frac{1}{r_1},2(p+s),2(q+\gamma)\right)\Bigg\}^{\frac{2}{r_1}}\Bigg]^{\frac{1}{l}}.$$

So, the proof of this theorem is completed. □

Corollary 2.13. *Under the same conditions as in Theorem 2.12 for $r = r_1 = 2$ and $g(t) = t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)h(x)dx$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1)$$

$$\times\Big[m\beta(p + \gamma + 1, q + s + 1)\left(f^{2l}(\varphi(a)) + h^{2l}(\varphi(a))\right)$$

$$+\beta(p + s + 1, q + \gamma + 1)\left(f^{2l}(\varphi(b)) + h^{2l}(\varphi(b))\right)\Big]^{\frac{1}{l}}.$$

3 Hermite-Hadamard type fractional integral inequalities for products

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for products of two generalized (r, g)-preinvex functions via fractional integrals.

Theorem 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ with $\eta(\varphi(y), \varphi(x), m) > 0$, $\forall\, x, y \in I$. Assume that $f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \longrightarrow (0, +\infty)$ are respectively generalized beta (r, g)-preinvex function and generalized beta (l, g)-preinvex function on an open m-invex set K°. Then for $\alpha > 0$, $p, q > -1$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have*

$$\frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt$$

$$\leq \frac{1}{2}\Bigg[\Bigg\{mf^r(\varphi(x))B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2(\alpha + p - 1), 2q\right)$$

$$+f^r(\varphi(y))B^{\frac{r}{2}}\left(g(t); \frac{1}{r}, 2(\alpha + q - 1), 2p\right)\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{mh^l(\varphi(x))B^{\frac{l}{2}}\left(g(t); \frac{1}{l}, 2(\alpha + p - 1), 2q\right)$$

$$+h^l(\varphi(y))B^{\frac{l}{2}}\left(g(t); \frac{1}{l}, 2(\alpha + q - 1), 2p\right)\Bigg\}^{\frac{2}{l}}\Bigg]. \tag{3.1}$$

Proof. Since f and h are respectively generalized beta (r, g)-preinvex function and generalized beta (l, g)-preinvex function on an open m-invex set K°, combining with Cauchy and Minkowski inequalities, we get

$$\frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt$$

$$= \int_0^1 g^{\alpha-1}(t)f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))$$

$$\times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m))d[g(t)]$$

$$\leq \int_0^1 g^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)}(t)\Big[mg^p(t)(1 - g(t))^q f^r(\varphi(x)) + g^q(t)(1 - g(t))^p f^r(\varphi(y))\Big]^{\frac{1}{r}}$$

$$\times\Big[mg^p(t)(1 - g(t))^q h^l(\varphi(x)) + g^q(t)(1 - g(t))^p h^l(\varphi(y))\Big]^{\frac{1}{l}} d[g(t)]$$

$$\leq \frac{1}{2}\Bigg\{\int_0^1 \Big[mg^{\alpha+p-1}(t)(1-g(t))^q f^r(\varphi(x)) + g^{\alpha+q-1}(t)(1-g(t))^p f^r(\varphi(y))\Big]^{\frac{2}{r}} d[g(t)]$$

$$+\int_0^1 \Big[mg^{\alpha+p-1}(t)(1-g(t))^q h^l(\varphi(x)) + g^{\alpha+q-1}(t)(1-g(t))^p h^l(\varphi(y))\Big]^{\frac{2}{l}} d[g(t)]\Bigg\}$$

$$\leq \frac{1}{2}\Bigg[\Bigg\{\left(\int_0^1 m^{\frac{2}{r}} g^{\frac{2(\alpha+p-1)}{r}}(t)(1-g(t))^{\frac{2q}{r}} f^2(\varphi(x))d[g(t)]\right)^{\frac{r}{2}}$$

$$+\left(\int_0^1 g^{\frac{2(\alpha+q-1)}{r}}(t)(1-g(t))^{\frac{2p}{r}} f^2(\varphi(y))d[g(t)]\right)^{\frac{r}{2}}\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{\left(\int_0^1 m^{\frac{2}{l}} g^{\frac{2(\alpha+p-1)}{l}}(t)(1-g(t))^{\frac{2q}{l}} h^2(\varphi(x))d[g(t)]\right)^{\frac{l}{2}}$$

$$+\left(\int_0^1 g^{\frac{2(\alpha+q-1)}{l}}(t)(1-g(t))^{\frac{2p}{l}} h^2(\varphi(y))d[g(t)]\right)^{\frac{l}{2}}\Bigg\}^{\frac{2}{l}}\Bigg]$$

$$= \frac{1}{2}\Bigg[\Bigg\{mf^r(\varphi(x))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha+p-1),2q\right)$$

$$+f^r(\varphi(y))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha+q-1),2p\right)\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{mh^l(\varphi(x))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+p-1),2q\right)$$

$$+h^l(\varphi(y))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+q-1),2p\right)\Bigg\}^{\frac{2}{l}}\Bigg].$$

So, the proof of this theorem is completed. □

Corollary 3.2. *Under the same conditions as in Theorem 3.1 for* $p = 0, m = q = 1, \varphi(x) = x$, $g(t) = t$ *and* $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$, *we get (see [1], Theorem 3.3).*

Corollary 3.3. *Under the same conditions as in Theorem 3.1 for* $r = l = 2$ *and* $g(t) = t$, *we get*

$$\frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(y),\varphi(x),m)} J^\alpha_{(m\varphi(x)+\eta(\varphi(y),\varphi(x),m))^-} f(m\varphi(x))h(m\varphi(x))$$

$$\leq \frac{1}{2}\Big[m\beta(\alpha+p,q+1)\left(f^2(\varphi(x))+h^2(\varphi(x))\right)+\beta(\alpha+q,p+1)\left(f^2(\varphi(y))+h^2(\varphi(y))\right)\Big].$$

Theorem 3.4. *Let* $0 < r, l \leq 1, q > 1$ *and* $p^{-1} + q^{-1} = 1$. *Let* $\varphi : I \longrightarrow K$ *be a continuous function and* $g : [0,1] \longrightarrow [0,1]$ *is a differentiable function. Assume that* $f, h : K = [m\varphi(x), m\varphi(x)+\eta(\varphi(y),\varphi(x),m)] \longrightarrow (0,+\infty)$ *are continuous functions on* K° *with* $\eta(\varphi(y),\varphi(x),m) > 0$, $\forall x, y \in I$. *If* f^p, h^q *are respectively generalized beta* (r,g)*-preinvex function and generalized beta* (l,g)*-preinvex function on an open* m*-invex set* K° *with respect to the same* $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0,1]$ *where* $s, \gamma > -1$, *then for* $\alpha > 0$, *we have*

$$\frac{1}{\eta^\alpha(\varphi(y),\varphi(x),m)}\int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)} (t-m\varphi(x))^{\alpha-1} f(t)h(t)dt$$

$$\leq \Bigg[mf^{rp}(\varphi(x))B^r\left(g(t);\frac{1}{r},\gamma+rp(\alpha-1),s\right)$$

$$+f^{rp}(\varphi(y))B^r\left(g(t);\frac{1}{r},s+rp(\alpha-1),\gamma\right)\Bigg]^{\frac{1}{rp}}$$

$$\times\left[mh^{lq}(\varphi(x))B^l\left(g(t);\frac{1}{l},\gamma,s\right)+h^{lq}(\varphi(y))B^l\left(g(t);\frac{1}{l},s,\gamma\right)\right]^{\frac{1}{lq}}. \tag{3.2}$$

Proof. Since f^p, h^q are respectively generalized beta (r,g)-preinvex function and generalized beta (l,g)-preinvex function on an open m-invex set K°, combining with Hölder and Minkowski inequalities, we get

$$\frac{1}{\eta^\alpha(\varphi(y),\varphi(x),m)}\int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)}(t-m\varphi(x))^{\alpha-1}f(t)h(t)dt$$

$$=\int_0^1 g^{\alpha-1}(t)f(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))$$
$$\times h(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]$$

$$\leq\left(\int_0^1 g^{p(\alpha-1)}(t)f^p(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 h^q(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq\left(\int_0^1 g^{p(\alpha-1)}(t)\Big[mg^\gamma(t)(1-g(t))^s f^{rp}(\varphi(x))+g^s(t)(1-g(t))^\gamma f^{rp}(\varphi(y))\Big]^{\frac{1}{r}}d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\Big[mg^\gamma(t)(1-g(t))^s h^{lq}(\varphi(x))+g^s(t)(1-g(t))^\gamma h^{lq}(\varphi(y))\Big]^{\frac{1}{l}}d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq\Bigg\{\left(\int_0^1 m^{\frac{1}{r}}g^{p(\alpha-1)+\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}}f^p(\varphi(x))d[g(t)]\right)^r$$

$$+\left(\int_0^1 g^{p(\alpha-1)+\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}}f^p(\varphi(y))d[g(t)]\right)^r\Bigg\}^{\frac{1}{rp}}$$

$$\times\Bigg\{\left(\int_0^1 m^{\frac{1}{l}}g^{\frac{\gamma}{l}}(t)(1-g(t))^{\frac{s}{l}}h^q(\varphi(x))d[g(t)]\right)^l$$

$$+\left(\int_0^1 g^{\frac{s}{l}}(t)(1-g(t))^{\frac{\gamma}{l}}h^q(\varphi(y))d[g(t)]\right)^l\Bigg\}^{\frac{1}{lq}}$$

$$=\Bigg[mf^{rp}(\varphi(x))B^r\left(g(t);\frac{1}{r},\gamma+rp(\alpha-1),s\right)$$

$$+f^{rp}(\varphi(y))B^r\left(g(t);\frac{1}{r},s+rp(\alpha-1),\gamma\right)\Bigg]^{\frac{1}{rp}}$$

$$\times\left[mh^{lq}(\varphi(x))B^l\left(g(t);\frac{1}{l},\gamma,,s\right)+h^{lq}(\varphi(y))B^l\left(g(t);\frac{1}{l},s,\gamma\right)\right]^{\frac{1}{lq}}.$$

So, the proof of this theorem is completed. □

Corollary 3.5. *Under the same conditions as in Theorem 3.4 for $p = q = 2$ and $g(t) = t$, we get*

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}J^{\alpha}_{(m\varphi(x)+\eta(\varphi(y),\varphi(x),m))^-}f(m\varphi(x))h(m\varphi(x))$$
$$\leq \beta^{\frac{1}{2}}\left(\frac{\gamma}{l}+1,\frac{s}{l}+1\right)\left[mh^{2l}(\varphi(x))+h^{2l}(\varphi(y))\right]^{\frac{1}{2l}}$$
$$\times\left[mf^{2r}(\varphi(x))\beta^{r}\left(\frac{\gamma}{r}+2(\alpha-1)+1,\frac{s}{r}+1\right)\right.$$
$$\left.+f^{2r}(\varphi(y))\beta^{r}\left(\frac{s}{r}+2(\alpha-1)+1,\frac{\gamma}{r}+1\right)\right]^{\frac{1}{2r}}.$$

Theorem 3.6. *Let $q \geq 1, r > 1$ and $r^{-1} + l^{-1} = 1$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ is a differentiable function. Assume that $f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y),\varphi(x),m)] \longrightarrow (0,+\infty)$ are continuous functions on K° with $\eta(\varphi(y),\varphi(x),m) > 0$, $\forall\, x, y \in I$. If f, h^q are respectively generalized beta (r,g)-preinvex function and generalized beta (l,g)-preinvex function on an open m-invex set K° with respect to the same $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ where $s, \gamma > -1$, then for $\alpha > 0$, we have*

$$\frac{1}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}\int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)}(t-m\varphi(x))^{\alpha-1}f(t)h(t)dt$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{q}}\left[mf^{r}(\varphi(x))B^{r}\left(g(t);\frac{1}{r},\gamma+r(\alpha-1),s\right)\right.$$
$$\left.+f^{r}(\varphi(y))B^{r}\left(g(t);\frac{1}{r},s+r(\alpha-1),\gamma\right)\right]^{\frac{q-1}{rq}}$$
$$\times\left[\left\{mf^{r}(\varphi(x))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha+\gamma-1),2s\right)\right.\right.$$
$$\left.+f^{r}(\varphi(y))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha+s-1),2\gamma\right)\right\}^{\frac{2}{r}}$$
$$+\left\{mh^{lq}(\varphi(x))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+\gamma-1),2s\right)\right.$$
$$\left.\left.+h^{lq}(\varphi(y))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+s-1),2\gamma\right)\right\}^{\frac{2}{l}}\right]^{\frac{1}{q}}. \tag{3.3}$$

Proof. Since f and h^q are respectively generalized beta (r,g)-preinvex function and generalized beta (l,g)-preinvex function on an open m-invex set K°, combining with the well-known power mean inequality, Cauchy and Minkowski inequalities, we get

$$\frac{1}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}\int_{m\varphi(x)+g(0)\eta(\varphi(y),\varphi(x),m)}^{m\varphi(x)+g(1)\eta(\varphi(y),\varphi(x),m)}(t-m\varphi(x))^{\alpha-1}f(t)h(t)dt$$
$$=\int_0^1 g^{\alpha-1}(t)f(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))$$
$$\times h(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]$$
$$\leq\left(\int_0^1 g^{\alpha-1}(t)f(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 g^{\alpha-1}(t)f(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))\right.$$

$$\left.\times h^q(m\varphi(x)+g(t)\eta(\varphi(y),\varphi(x),m))d[g(t)]\right]^{\frac{1}{q}}$$

$$\leq\left(\int_0^1 g^{\alpha-1}(t)\Big[mg^{\gamma}(t)(1-g(t))^s f^r(\varphi(x))+g^s(t)(1-g(t))^{\gamma}f^r(\varphi(y))\Big]^{\frac{1}{r}}d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\left\{\int_0^1 g^{\alpha-1}(t)\Big[mg^{\gamma}(t)(1-g(t))^s f^r(\varphi(x))+g^s(t)(1-g(t))^{\gamma}f^r(\varphi(y))\Big]^{\frac{1}{r}}\right.$$

$$\left.\times\Big[mg^{\gamma}(t)(1-g(t))^s h^{lq}(\varphi(x))+g^s(t)(1-g(t))^{\gamma}h^{lq}(\varphi(y))\Big]^{\frac{1}{l}}d[g(t)]\right\}^{\frac{1}{q}}$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{\left(\int_0^1 m^{\frac{1}{r}}g^{\alpha-1+\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}}f(\varphi(x))d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1 g^{\alpha-1+\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}}f(\varphi(y))d[g(t)]\right)^r\right\}^{\frac{q-1}{rq}}$$

$$\times\left\{\int_0^1\Big[mg^{\alpha+\gamma-1}(t)(1-g(t))^s f^r(\varphi(x))+g^{\alpha+s-1}(t)(1-g(t))^{\gamma}f^r(\varphi(y))\Big]^{\frac{2}{r}}d[g(t)]\right.$$

$$\left.+\int_0^1\Big[mg^{\alpha+\gamma-1}(t)(1-g(t))^s h^{lq}(\varphi(x))+g^{\alpha+s-1}(t)(1-g(t))^{\gamma}h^{lq}(\varphi(y))\Big]^{\frac{2}{l}}d[g(t)]\right\}^{\frac{1}{q}}$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{\left(\int_0^1 m^{\frac{1}{r}}g^{\alpha-1+\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}}f(\varphi(x))d[g(t)]\right)^r\right.$$

$$\left.+\left(\int_0^1 g^{\alpha-1+\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}}f(\varphi(y))d[g(t)]\right)^r\right\}^{\frac{q-1}{rq}}$$

$$\times\left[\left\{\left(\int_0^1 m^{\frac{2}{r}}g^{\frac{2(\alpha+\gamma-1)}{r}}(t)(1-g(t))^{\frac{2s}{r}}f^2(\varphi(x))d[g(t)]\right)^{\frac{r}{2}}\right.\right.$$

$$\left.+\left(\int_0^1 g^{\frac{2(\alpha+s-1)}{r}}(t)(1-g(t))^{\frac{2\gamma}{r}}f^2(\varphi(y))d[g(t)]\right)^{\frac{r}{2}}\right\}^{\frac{2}{r}}$$

$$+\left\{\left(\int_0^1 m^{\frac{2}{l}}g^{\frac{2(\alpha+\gamma-1)}{l}}(t)(1-g(t))^{\frac{2s}{l}}h^{2q}(\varphi(x))d[g(t)]\right)^{\frac{l}{2}}\right.$$

$$\left.\left.+\left(\int_0^1 g^{\frac{2(\alpha+s-1)}{l}}(t)(1-g(t))^{\frac{2\gamma}{l}}h^{2q}(\varphi(y))d[g(t)]\right)^{\frac{l}{2}}\right\}^{\frac{2}{l}}\right]^{\frac{1}{q}}$$

$$=\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[mf^r(\varphi(x))B^r\left(g(t);\frac{1}{r},\gamma+r(\alpha-1),s\right)\right.$$

$$\left.+f^r(\varphi(y))B^r\left(g(t);\frac{1}{r},s+r(\alpha-1),\gamma\right)\right]^{\frac{q-1}{rq}}$$

$$\times\Bigg[\Bigg\{mf^{r}(\varphi(x))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha+\gamma-1),2s\right)$$

$$+f^{r}(\varphi(y))B^{\frac{r}{2}}\left(g(t);\frac{1}{r},2(\alpha+s-1),2\gamma\right)\Bigg\}^{\frac{2}{r}}$$

$$+\Bigg\{mh^{lq}(\varphi(x))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+\gamma-1),2s\right)$$

$$+h^{lq}(\varphi(y))B^{\frac{l}{2}}\left(g(t);\frac{1}{l},2(\alpha+s-1),2\gamma\right)\Bigg\}^{\frac{2}{l}}\Bigg]^{\frac{1}{q}}.$$

So, the proof of this theorem is completed. □

Corollary 3.7. *Under the same conditions as in Theorem 3.6 for $\gamma=0, m=q=s=1, \varphi(x)=x$, $g(t)=t$ and $\eta(\varphi(b),\varphi(a),m)=\eta(b,a)$, we get (see [1], Theorem 3.9). Also for $q=1$, we get Theorem 3.1.*

Corollary 3.8. *Under the same conditions as in Theorem 3.6 for $r=l=2$ and $g(t)=t$, we get*

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(y),\varphi(x),m)}J^{\alpha}_{(m\varphi(x)+\eta(\varphi(y),\varphi(x),m))^{-}}f(m\varphi(x))h(m\varphi(x))$$

$$\leq\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[mf^{2}(\varphi(x))\beta^{2}\left(\alpha+\frac{\gamma}{2},\frac{s}{2}+1\right)+f^{2}(\varphi(y))\beta^{2}\left(\alpha+\frac{s}{2},\frac{\gamma}{2}+1\right)\right]^{\frac{q-1}{2q}}$$

$$\times\Big[m\beta\left(\alpha+\gamma,s+1\right)\left(f^{2}(\varphi(x))+h^{2q}(\varphi(x))\right)$$

$$+\beta\left(\alpha+s,\gamma+1\right)\left(f^{2}(\varphi(y))+h^{2q}(\varphi(y))\right)\Big]^{\frac{1}{q}}.$$

Remark 3.9. For $\alpha>0$, for different choices of positive values $r,l=\frac{1}{2},\frac{1}{3}$, etc., for some fixed $m\in(0,1]$, $s,\gamma>-1$, for a particular choices of a differentiable function $g(t)=e^{t}$, $\ln(t+1)$, etc. and a particular choices of a continuous function $\varphi(x)=e^{x}$ for all $x\in\mathbb{R}$, x^{n} for all $x>0$ and $n\in\mathbb{N}$, etc., by Theorem 3.1, Theorem 3.4 and Theorem 3.6 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities for products of two generalized beta (r,g)-preinvex functions. The details are left to the interested reader.

4 Conclusion

Motivated by this new interesting class of generalized beta (r,g)-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for products of various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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Received: February 12, 2019.
Accepted: April 19, 2019.