Palestine Journal of Mathematics

Vol. 10(1)(2021) , 37–49

On Weakly Firm Commutative Rings and their Connection to the Total and Zero-divisor Graphs

Elham Mehdi-Nezhad and Amir M. Rahimi

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13A15, 13B30; Secondary 05C25.

Keywords and phrases: Commutative ring; (weakly) essential prime ideal; (weakly) firm ring; zero-component of a prime ideal; (countably) McCoy ring; zero-divisor and total graphs.
The research of the second author was in part supported by grant no. 98130018 from IPM.

Abstract We generalize the notion of the firm commutative rings to weakly firm ones. A ring is said to be (weakly) firm if it contains a (weakly) essential prime ideal and the zero-component of each (weakly) essential prime ideal is (weakly) essential. An (weakly) essential ideal is one with nonzero intersection with every nonzero (prime) ideal. For a prime ideal P of a commutative ring A with identity, we denote (as usual) by O_P its zero-component; that is, the set of members of P that are annihilated by non-members of P. We study rings in which O_P is a weakly essential ideal whenever P is a weakly essential prime ideal. We prove that the classical ring of quotients of any ring of this kind is itself of this kind. We show that direct products of rings of this kind are themselves of this kind. We show that a ring is not weakly firm (consequently, not firm) if its zero divisor set is an ideal and by an example show that the class of firm rings is properly contained in the class of weakly firm rings. We also observe some connections between these type of rings and their total and zero-divisor graphs via the set of their zero divisors.

1 Introduction

The main goal of this paper is to extend the work of Dube [13], on *firm commutative rings*, to *weakly firm Commutative rings* (Definitions 1.1 and 3.3, respectively). We introduce the notion of a weakly firm ring, which is a generalization of a firm ring, and easily show (Theorem 3.6) that any ring R is not weakly firm (consequently, not firm) provided that $Z(R)$, its set of zero divisors, is an ideal of R. Thus, it is a natural approach to relate the *zero-divisor type graphs* of commutative rings to (weakly) firm rings via Theorem 3.6 and we will apply, in the last two sections (Sections 4 and 5), many results (related to the *zero-divisor and total graphs*) from [5, 20, 3] in this context.

In the next section, we will provide some facts about commutative rings and (undirected) graphs that are relevant to our discussion. We will define the zero-divisor and total graphs of a commutative ring with some of their properties in Sections 4 and 5, respectively, for the sake of completeness. We will also be consistent with the notations and literature of the above four mentioned papers as much as possible.

Throughout the paper all rings are commutative with identity $1 \neq 0$, unless the contrary is explicitly stated, and $\mathrm{Nil}(R)$ is the ideal of nilpotent elements of the ring R. Recall that an essential ideal is one with nonzero intersection with every nonzero ideal. One of the interesting things about these ideals is that the socle of a ring, which is "built from below" by taking the union of all minimal prime ideals and then generating an ideal, can also be "built from above" by intersecting all essential ideals. In [16], the authors study the ideal obtained by intersecting all essential maximal ideals of a semi-primitive ring. They then characterize those Tychonoff spaces X for which the socle of $C(X)$ is the intersection of the essential maximal ideals. The work of [13] is in part motivated by reading [16].

• We now recall the definitions of firm and strongly firm rings, which are taken from Sections 3 and 4 of [13], respectively, for the sake of completeness and comparison. For the definitions of the zero-component and pure part of an ideal, see Subsection 2.1 below.

Definition 1.1. A ring A is *firm* [resp. *strongly firm*] if it has an essential prime ideal [resp. essential ideal] and O_P, the zero-component of P, [resp. the pure part of every essential prime ideal] is essential whenever P is an essential prime ideal in A. On the other hand, we say A is *anti-firm* if it has an essential prime ideal P for which O_P is not essential. A ring can of course fail to have an essential prime ideal (for instance any field), so whenever we assert that a particular ring is firm we will need to demonstrate that it actually does have an essential prime ideal. We should emphasize that strong firmness is *formally* stronger than firmness because it implies firmness since mP, the pure part of P, is contained in O_P for every prime ideal P.

• The work of Dube in [13] was in part motivated by reading [16] which part of it is a characterization of those Tychonoff spaces X for which the socle of $C(X)$ is the intersection of the essential maximal ideals. In his work [13], besides many interesting examples, he defines (strongly) firm rings and characterizes them in terms of the lattices of their radical ideals provided that the rings have no nonzero nilpotent elements. It is shown that any proper ideal of a firm reduced ring, when viewed as a ring in its own right, is firm [resp. the classical ring of quotients of any ring (not necessarily reduced) of this kind is itself of this kind, direct products of (finitely many) rings of this kind are themselves of this kind, the ring of real-valued continuous functions on a Tychonoff space is of this kind precisely when the underlying set of the space is infinite]. It is also shown that for some (different) classes of rings, firm and strongly firm coincide.

• The organization of this paper is as follows: In Section 2, we collect some facts about commutative rings and (undirected) graphs that are relevant to our discussion in this paper. In Section 3, we introduce the notion of the weakly firm rings and study some of their algebraic properties. The key result in this section (paper) is Theorem 3.6 that excludes a class of rings R of being (weakly) firm when $Z(R)$ is an ideal of R. Finally, the last two sections are devoted on (non)(weakly) firmness of a ring R that are related (mainly via Theorem 3.6) to some graph-theoretic properties of the zero-divisor and total graphs of R, respectively.

2 Preliminaries: Rings and Graphs

This section consists of two parts that will be relevant for our discussion, where the first and second part, respectively, provide some facts about commutative rings and (undirected) graphs.

2.1 Rings

Recall that a ring is called *reduced* if it has no nilpotent elements apart from 0. We adhere to the convention that prime ideals are assumed to be proper ideals. In general, by “ideal" we do not necessarily mean a proper ideal. We shall thus always say “proper ideal" when we mean a proper ideal. The symbols Min(A) and Max(A) have their usual meanings; namely, the sets of minimal prime and maximal ideals of A, respectively. We shall frequently write the zero ideal simply as 0, unless it becomes necessary to write it as $\{0\}$. The annihilator of a set I will be written as Ann(I), and Ann(a) abbreviates Ann$(\{a\})$.

• Let P be a prime ideal of a ring A. The *zero-component* of P, denoted O_P, is defined by

$$O_P = \{a \in P \mid ab = 0 \text{for some} b \in A \setminus P\}.$$

Observe that O_P is an ideal consisting entirely of zero-divisors. If A is a reduced ring, then

$$O_P = \bigcap\{Q \in \text{Min}(A) \mid Q \subseteq P\}.$$

• The *pure part* of an ideal I of A, denoted mI, is the ideal

$$mI = \{a \in I \mid a = ab \text{ for some} b \in I\} = \bigcup\{\mathrm{Ann(1-x) \mid x \in I}\}.$$

Observe that the containment $mP \subseteq O_P$ holds for every prime ideal P, and for any maximal ideal M we have $mM = O_M$. Indeed, let $a \in O_M$, and take $b \notin M$ such that $ab = 0$. Since M is a maximal ideal, there exist $c \in M$ and $d \in A$ such that $1 = c + db$. Then $a = a(c + db) = ac$, which shows that $a \in mM$.

Whenever convenient, we shall use the language and notation of *contraction and extension* of ideals. To recall, let $\phi\colon A \to B$ be a ring homomorphism, I be an ideal of A, and J an ideal of B. The ideal $J^c = \phi^{-1}[J]$ of A is called the *contraction* of J, and the (possibly improper) ideal I^e of B generated by $\phi[I]$ is called the extension of I. Ideals of B of the form I^e are called *extended ideals*. Recall that if A is a ring and S a multiplicatively closed subset, then the ideals of the ring of fractions $A[S^{-1}]$ are exactly the ideals $I^e = \{\frac{u}{s} \mid u \in I, s \in S\}$. Prime ideals of $A[S^{-1}]$ are precisely the extensions of the prime ideals of A that miss S.

As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, and F_q will denote the integers, rational numbers, integers modulo n, and the finite field with q elements, respectively. The group of units of a commutative ring R will be denoted by $U(R)$, the nonzero elements of $A \subseteq R$ will be denoted by A^*, and $\subset$ will denote proper inclusion. We say that R is reduced if $\mathrm{Nil}(R) = \{0\}$. General references for ring theory are [17] and [18].

• Recall that for an R-module M, the *idealization* of M over R is the commutative ring formed from $R \times M$ by defining addition and multiplication as $(r, m) + (s, n) = (r + s, m + n)$ and $(r, m)(s, n) = (rs, rn + sm)$, respectively. A standard notation for this "idealized ring" is $R(+)M$; see [17] for basic properties of rings resulting from the idealization construction. The zero-divisor graph $\Gamma(R(+)M)$ has been studied in [6] and [8].

2.2 Graphs

Let G be a graph. We say that G is *connected* if there is a *path* between any two distinct vertices of G. At the other extreme, we say that G is *totally disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(x, y) \mid x \textit{ and } y \text{ are vertices of } G\}$. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is *complete*; i.e., each pair of distinct vertices forms an *edge*. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest *cycle* in G ($\mathrm{gr}(G) = 0$ if G contains no cycles). We denote the complete graph on n vertices by K_n and the *complete bipartite graph* on m and n vertices by $K_{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K_{1,n}$ a *star graph*. We say that two *(induced) subgraphs* G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 [respectively, G_2] is *adjacent* (in G) to any vertex not in G_1 [respectively, G_2].

A general reference for graph theory is [12]. Also, the reader can refer to [5, 20, 3] for all necessary definitions that are related to graphs in this paper.

3 Weakly Firm Rings and some of their Properties

We need a name for the rings which will be the subject of this discussion. Recall that an ideal of a ring A is said to be *essential* if it has nonzero intersection with every nonzero ideal of A. If I is an ideal of A and $\mathrm{Ann}(I) = 0$, then I is essential. For reduced rings, an ideal is essential if and only if its annihilator is 0.

Definition 3.1. An ideal I of a commutative ring R is *weakly essential* if it has nonzero intersection with every nonzero prime ideal of R.

Example 3.2. Clearly, if I and $\mathrm{Nil}(R)$ having nonzero intersection, then I is weakly essential since $\mathrm{Nil}(R)$ is the intersection of all prime ideals of R. Further, if $\mathrm{Nil}(R)$ is not zero (i.e. R is not reduced), then every prime ideal of R is a weakly essential ideal. Also, Nil(R) is a weakly essential ideal of R for any nonreduced ring R.

Definition 3.3. A ring R is weakly firm if it has a weakly essential prime ideal and the zero-component of every weakly essential prime is weakly essential. On the other hand, we say R is *weakly anti-firm* if it has a weakly essential prime ideal P for which O_P is not weakly essential.

Clearly, each firm ring is weakly firm by definition. In the following two examples we show that the class of firm rings is properly contained in the class of weakly firm rings.

Example 3.4. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then R is a weakly firm ring which is not firm. Let $M_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\}$, $M_2 = \mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_2$, $M_3 = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $I_1 = \{0\} \times \{0\} \times \mathbb{Z}_2$, $I_2 = \{0\} \times \mathbb{Z}_2 \times \{0\}$, and $I_3 = \mathbb{Z}_2 \times \{0\} \times \{0\}$. Clearly, M_i's are the only maximal (weakly essential prime) ideals of R and neither of I_i's is prime or essential. Note that $Z(R)$ is not an ideal of R.

Example 3.5. Recall that when R is a finite reduced ring, then it is a direct product of finitely many finite fields. Suppose $R = R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a field ($1 \leq i \leq n$). Then

(1) R is not firm for any finite $n \geq 1$.

(2) R is not (weakly) firm if $n = 1$ or 2.

(3) R is a nonfirm weakly firm ring if $n \geq 3$ (see also Example 3.14(c)).

We next provide an example of a class of nonweakly firm rings, which obviously is a class of nonfirm rings. Namely, those rings whose each set of zero divisors is an ideal. Recall that $Z(R)$ is an ideal of R when it is closed under addition). Note that since $Z(R)$ is a union of prime ideals of R [by [18, p. 3], we always have $xy \in Z(R)$ for $x, y \in R$ implies $x \in Z(R)$ or $y \in Z(R)$. So if $Z(R)$ is an ideal of R, then $Z(R)$ is actually a prime ideal of R, and hence $R/Z(R)$ is an integral domain. Moreover, if R is a finite commutative ring and $Z(R)$ is an ideal of R, then R is local with $Z(R) = \text{Nil}(R)$ its unique maximal ideal and hence not weakly firm (see Example 3.14(d)). By a local ring, we mean a ring with a unique maximal ideal.

Theorem 3.6. *Let R be a commutative ring such that $P = Z(R)$ is an ideal of R. Then R is not weakly firm and consequently, not firm. Further, if $Z(R) \neq 0$, then R is weakly anti-firm.*

Proof. Clearly, if R is an integral domain, then the result is immediate since $O_P = 0$ for any prime ideal of R. Now, the proof follows directly since $Z(R)$ is prime by hypothesis and has a nonzero intersection with any nonzero prime ideal and hence is weakly essential prime by definition and $O_P = 0$, which is obviously not (weakly) essential. For the further part, see Definition 3.3. □

Remark 3.7. Note that the converse of the above theorem need not be true in general. That is, there are some examples of (nonweakly firm) nonfirm rings whose set of zero divisors is not an ideal (see for example, Examples 3.4 and 3.5). Also, in order to check that $Z(R)$ is not an ideal of R, it suffices to show That $x + y \notin Z(R)$ for some distinct elements $x, y \in Z(R)$ (i.e., $x + y$ is a regular element of R) since $Z(R)$ is always closed under multiplication by elements of R. For example, the set of the zero divisors of the direct product of unital rings with more than one factor is not an ideal.

The following corollary could be easily obtained from the above theorem since Nil(R) is an ideal in a commutative ring.

Corollary 3.8. *Any commutative ring R with* $\text{Nil}(R) = Z(R)$ *is excluded of being weakly firm (or firm). In particular, an integral domain is never weakly firm and consequently not firm.*

Proposition 3.9. *Let R be a commutative ring such that $P = Z(R)$ is an ideal of R. Then the classical ring of quotients of R (or in this case, localization of R at P) is not weakly firm and hence not a firm ring.*

Proof. The result follows directly since R_P is a local ring (see Example 3.14(d)). □

Remark 3.10. Corollary 3.1 of Dube's paper [13] states that The classical ring of quotients of a firm ring is firm. Now from the above result and Theorem 3.6, we see an example of a nonfirm ring whose ring of quotients is not firm.

It is possible for a ring not to have an (weakly) essential prime ideal (see Examples 3.4 and 3.5). If we assume the Axiom of Choice (as we shall do whenever we need it), then a ring has an (weakly) essential prime ideal if and only if it has an (weakly) essential maximal ideal. A ring with no (weakly) essential prime ideal is neither (weakly) firm nor (weakly) anti-firm.

• A ring R is said to be *McCoy [resp, countably McCoy]* if each finitely [resp, countably] generated ideal $I \subseteq Z(R)$ has a nonzero annihilator.

Proposition 3.11. *Let R be a reduced McCoy [resp, countably McCoy] ring. If R contains an essential prime ideal P such that O_P is finitely [resp, countably] generated ideal, then R is not firm (it is actually anti-firm).*

Proof. The result follows since $O_P \subseteq Z(R)$ has a nonzero annihilator by the assumption and hence is not essential since R is reduced. □

In contrast to [13, Proposition 3.1] that states every ideal in a reduced firm ring is firm (when viewed as a ring in its own right), we show in the following example that this is not true for reduced weakly firm rings in general.

Example 3.12. (cf. [13, Proposition 3.1]) Let $R = R_1 \times R_2 \times R_3$ be a finite reduced ring as defined in Example 3.5, where each R_i is a field $(1 \leq i \leq 3)$ and let $I = R_1 \times R_2 \times \{0\}$. Clearly, by Example 3.5, R is weakly firm but I is not.

The following easy criterion shows that in order to check whether a ring is (weakly) firm we need only limit to maximal ideals. It will be particularly useful when we deal with direct products.

Proposition 3.13. (cf. [13, Proposition 3.2]) *The following two conditions are equivalent for a ring A which has an (weakly) essential ideal.*

(a) A is (weakly) firm.

(b) O_M is (weakly) essential for every (weakly) essential maximal ideal M of A.

Proof. The proof is similar to the proof of [13, Proposition 3.2]. $(a) \Leftrightarrow (b)$: The left-to-right implication is trivial because our blanket assumption is that all rings have the identity, so that maximal ideals are prime. Conversely, suppose O_M is (weakly) essential for every (weakly) essential maximal ideal M. Let P be an (weakly) essential prime ideal of A (it exists by the assumption and the fact that any ideal is contained in a maximal (prime) ideal) and clearly, if $I \subseteq J$ is (weakly) essential, then J is (weakly) essential by definition. Pick a maximal ideal M with $M \supseteq P$. Then M is (weakly) essential, and hence O_M is (weakly) essential by the present hypothesis. But $O_M \subseteq O_P$, so O_P is (weakly) essential. Therefore A is (weakly) firm. □

Let us now give some examples of (weakly) firm and anti-firm rings. More examples will present themselves as we proceed.

Example 3.14. (cf. [13, Examples 3.1])

(a) Every von Neumann regular ring with at least one (weakly) essential ideal is (weakly) firm because $O_P = P$ for any prime ideal P in a Von Neumann regular ring.

(b) An integral domain is never (weakly) firm; and it is anti-firm if and only if it is not a field. Recall that an integral domain is reduced.

(c) A reduced Noetherian ring is never firm. The reason is that, in any ring, O_P consists entirely of zero-divisors, and in a reduced Noetherian ring any ideal consisting entirely of zero-divisors is non-essential (see [18, Theorem 82]). But a finite reduced ring is a direct product of finitely many finite fields which is weakly firm when number of factors of the direct product of the fields are more than 2. Thus, this is an example of a weakly firm ring which is not firm since each finite ring is Noetherian (see also Examples 3.4 and 3.5) above.

(d) A local ring is never (weakly) firm ($O_M = 0$ for the unique maximal ideal M); and it is anti-firm if and only if it has at least one nonzero nonunit. Thus, a subring of an anti-firm ring need not be anti-firm. For instance, the field of real numbers $\mathbb{R}$ is not anti-firm, but the ring $\mathbb{R}[[X]]$ of power series is anti-firm.

Example 3.15. Note that the ring of integers modulo p^n (power of a prime p) is an example of a finite local ring, which of course, is not (weakly) firm by Part (d) of the above example.

We conclude this section with a few results related to the extensions and product of firm rings (that are taken from [13, Section 3]) to show that certain extensions and product of (weakly) firm rings are (weakly) firm. To start, we record the following lemma. The proof is routine, so we omit it.

Lemma 3.16. (cf. [13, Lemma 3.1]) *Let A be a ring and suppose B is a subring of A such that $A = B[S^{-1}]$ for some $S \subseteq B$ consisting entirely of units of A.*

(a) If I is an (weakly) essential proper ideal in B with $I \cap S = \emptyset$, then I^e is an (weakly) essential proper ideal in A.

(b) The contraction of any (weakly) essential proper ideal of A is an (weakly) essential proper ideal in B.

Proposition 3.17. (cf. [13, Proposition 3.3]) *Let A be a ring and suppose B is a subring of A such that $A = B[S^{-1}]$ for some $S \subseteq B$ consisting entirely of units of A. If B is (weakly) firm, then A is (weakly) firm.*

Proof. The proof is similar to the proof of Proposition 3.3 in [13] by using the above lemma. □

We write $q(A)$ for the classical ring of quotients of A. This of course is the ring obtained from A by inverting all non-divisors of zero.

Corollary 3.18. (cf. [13, Corollary 3.1]) *The classical ring of quotients of a (weakly) firm ring is (weakly) firm.*

Proof. The proof is similar to the proof of [13, Corollary 3.1]. □

We now turn to products. Instead of presenting a proof for the upcoming result, we shall indicate how it can be put together from various ingredients. Recall that for any collection $\{A_\lambda \mid \lambda \in \Lambda\}$ of rings (with identity), an ideal of the direct product $\prod A_\lambda$ is maximal if and only if it is of the form $\pi_\ell^{-1}[M]$, for some index ℓ and $M \in \text{Max}(A_\ell)$; where π_ℓ denotes the projection map $\pi_\ell \colon \prod A_\lambda \to A_\ell$. It is not difficult to show that for any $\lambda \in \Lambda$ and $M \in \text{Max}(A_\lambda)$,

$$O_{\pi_\ell^{-1}[M]} = \pi_\ell^{-1}[O_M].$$

With this observation in hand, one can prove the following proposition. Recall that the inverse function of any map between two sets preserves the intersection operation and inclusion relation.

Proposition 3.19. (cf. [13, Proposition 3.4]) *A direct product of reduced (weakly) firm rings is (weakly) firm.*

Proof. The result follows by Proposition 3.13 and the fact that for any epimorphism $f \colon R \to S$ of rings with kernel K, the image of every prime ideal of R that contains the kernel is a prime ideal in S; the inverse image of any prime ideal of S is a prime ideal of R that contains k; and There is a one-to-one correspondence between the set of all prime ideals in R that contain K and the set of all prime ideals in S, given by $P \mapsto f(P)$. □

4 Zero-divisor Graphs and (Weakly) Firmness

In this section, we study the firmness and weakly firmness of a (finite) commutative ring by applying some known results related to the zero-divisor graphs of commutative rings that are taken from [5] and [20], respectively. In [5], Anderson and Livingston introduced the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [11], who let all the elements of R be vertices and was mainly interested in colorings. Among other things, they proved that $\Gamma(R)$ is always connected and its diameter, diam$(\Gamma(R))$, is always less than or equal to 3 [5, Theorem 2.3]. They also proved that $\Gamma(R)$ is a complete graph if and only if either R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$ [5, Theorem 2.8]. For some other works on graphs associated to algebraic structures, see [1, 2, 4, 6, 7, 8, 14, 9, 10, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. In the first part of this section, we apply a few results from [5] to show that a ring R is not weakly firm when its zero-divisor graph $\Gamma(R)$ has a vertex adjacent to every other vertex (i.e., when $\Gamma(R)$ has a *spanning tree* which is a star graph). Special cases of this are when either $\Gamma(R)$ is a complete graph or a star graph. Then we continue the section by applying some results from [20] to characterize (mainly) the diameter of the zero-divisor graphs of R, $R[x]$, and $R[[x]]$ in connection to the (weakly) firmness of R (see $(*)$ below).

Theorem 4.1. (cf. [5, Theorem 2.5]) *Let R be a reduced commutative ring such that $\Gamma(R)$ has a vertex adjacent to every other vertex. Then R is either anti-firm or not weakly firm.*

Proof. If $\Gamma(R)$ has a vertex adjacent to every other vertex, then, by [5, Theorem 2.5], R must have the form $\mathbb{Z}_2 \times A$ for A an integral domain or $Z(R)$ is an annihilator ideal. Clearly, $Z(R)$ is not an annihilator ideal Since R is reduced (i.e., $Z(R) \neq \text{Ann}(a)$, a adjacent to every other vertex) and hence R is of the form $\mathbb{Z}_2 \times A$. Thus R is either anti-firm (by Remark 3.4 [13] and the fact that a nonfield integral domain is anti-firm [13, Examples 3.1(b)] [or see Example 3.14(b) above]) or not weakly firm (by Example 3.5(b)) which depends on whether A is a nonfield integral domain or a field, respectively. □

• Note that Remark 3.4 of [13] states that if A is a reduced anti-firm ring, then $A \times B$ is anti-firm for any ring B.

Since $Z(R)$ is always a union of prime ideals [18, p. 3], $Z(R)$ is a prime ideal if (and only if) it is an ideal. If R is also Noetherian, then $Z(R)$ is an annihilator ideal if and only if it is an (prime) ideal [18, Theorems 6 and 82].

The following provides some examples of nonweakly firm rings for some special cases (see also Corollary 3.8).

Example 4.2. Recall that $\{0\}$ is a primary ideal of R if and only if $Z(R) = \text{Nil}(R)$. If dim $R = 0$, then $Z(R) = \text{Nil}(R)$ if and only if $Z(R)$ is a (prime) ideal of R; if R is finite, this is equivalent to R being local. Thus R is not weakly firm in each case.

We next specialize to the case when R is finite.

Proposition 4.3. (cf. [5, Corollary 2.7]) *Let R be a finite commutative ring such that $\Gamma(R)$ has a vertex adjacent to every other vertex. Then R is not weakly firm.*

Proof. If $\Gamma(R)$ has a vertex adjacent to every other vertex, then, by [5, Corollary 2.7], either $R = \mathbb{Z}_2 \times F$, where F is a finite field, or R is local. Now the result follows from Example 3.5 or Example 3.14(d). □

We next discuss when $\Gamma(R)$ is a complete graph (i.e., any two vertices are adjacent). By definition, $\Gamma(R)$ is complete if and only if $xy = 0$ for all distinct $x, y \in Z(R)$. Except for the case when $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, our next theorem shows that we must also have $x^2 = 0$ for all $x \in Z(R)$ when $\Gamma(R)$ is complete. So, except for that one case, nilpotent elements are detected by complete graphs.

Theorem 4.4. (cf. [5, Theorem 2.8]) *Let R be a commutative ring such that $\Gamma(R)$ is a complete graph. Then R is not weakly firm.*

Proof. If $\Gamma(R)$ is complete, then by [5, Theorem 2.8], either $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$. Now the result follows from Example 3.5(b) or Theorem 3.6, respectively, since $Z(R)$, in this case, is an ideal of R. That is, $x(x + y) = x^2 + xy = 0$ implies $x + y \in Z(R)$ for any two distinct x and $y \in Z(R)$. Actually, $x \in Z(R)$ and $x^2 = 0$ implies $Z(R) \subseteq \text{Nil}(R)$ and hence $Z(R) = \text{Nil}(R)$ which is an ideal in a commutative ring (see also Corollary 3.8). □

Theorem 4.5. (cf. [5, Theorem 2.10]) *Let R be a finite commutative ring. If $\Gamma(R)$ is complete, then R is not weakly firm.*

Proof. If $\Gamma(R)$ is complete, then by [5, Theorem 2.10], either $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local. Now the result follows from Example 3.5(b) or Example 3.14(d). □

We next consider when $\Gamma(R)$ has exactly one vertex which is adjacent to every other vertex.

Proposition 4.6. (cf. [5, Lemma 2.12]) *Let R be a finite commutative ring. If $\Gamma(R)$ has exactly one vertex adjacent to every other vertex and no other adjacent vertices, then R is not (weakly) firm.*

Proof. By [5, Lemma 2.12], either $R = \mathbb{Z}_2 \times F$, where F is a finite field with $|F| \geq 3$, or R is local. Now the result follows from Example 3.5 or Example 3.14(d). □

Theorem 4.7. (cf. [5, Theorem 2.13]) *Let R be a finite commutative ring with $|\Gamma(R)| \geq 4$. If $\Gamma(R)$ is a star graph, then R is not weakly firm.*

Proof. By [5, Theorem 2.13], R is a star graph if and only $R = \mathbb{Z}_2 \times F$, where F is a finite field and hence the result follows by Example 3.5. □

Remark 4.8. Example 2.14 of [5], which is an example of a finite local ring, shows that the converse of the above result need not be true in general. That is, the zero-divisor graph of a nonweakly firm ring need not be a star graph. On the other hand, [5, Theorem 2.13] can be regarded as an example of a nonweakly firm ring whose zero-divisor graph is a star graph.

$(*)$ The rest of this section is devoted on a characterization of (mainly) the diameter of the zero-divisor graphs of R, $R[x]$, and $R[[x]]$ (by applying some results from [20]) in connection to the (weakly) firmness of R.

In our first result we provide a sufficient condition for $\Gamma(R)$ to have diameter 3 when R is a reduced ring. A similar equivalence holds for nonreduced rings, but in this case the number of minimal primes is irrelevant.

Theorem 4.9. (cf. [20, Theorem 2.1]) *Let R be a reduced (weakly) firm ring. If R has more than two minimal primes, then diam$(\Gamma(R)) = 3$.*

Proof. The result follows from [20, Theorem 2.1] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

Theorem 4.10. (cf. [20, Theorem 2.2]) *Let R be a reduced (weakly) firm ring. Then the diameter of $\Gamma(R)$ is less than or equal to 2 if and only if R has exactly two minimal primes.*

Proof. The result follows from [20, Theorem 2.2] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

Theorem 4.11. (cf. [20, Theorem 2.4 and Corollary 2.5]) *If R is a nonreduced (weakly) firm ring, then diam$(\Gamma(R)) = 3$.*

Proof. By Theorem 3.6, $Z(R)$ is not an ideal since R is (weakly) firm. Thus, there exist $a, b \in Z(R)$ such that $(0 : (a, b)) = (0)$ and hence the result follows by [20, Theorem 2.4] (see also [20, Corollary 2.5]). □

Finally, we conclude this section with a few results related to the (weakly) firmness of the polynomial rings and power series rings. Section 5 of [20] provides many (interesting) examples of commutative rings whose each set of zero divisors forms an ideal, and hence examples of nonweakly firm rings.

Theorem 4.12. (cf. [20, Theorem 3.3]) *Let R be a McCoy ring such that $Z(R)$ is an ideal. Then $R[x]$ is not weakly firm.*

Proof. The proof follows from sufficient part of [20, Theorem 3.3] which states that $Z(R[x])$ is an ideal of $R[x]$, and hence Theorem 3.6 implies the result. □

Remark 4.13. Since $Z(R)$ in the hypothesis of the above theorem is an ideal of R, then R is not weakly firm by Theorem 3.6. Thus, this can be regarded as an example of a nonweakly firm ring whose polynomial ring is likewise not weakly firm.

Theorem 4.14. (cf. [20, Theorem 4.4]) *If R is a reduced (weakly) firm ring with more than two minimal primes, then $diam(\Gamma(R[[x]])) = 3$.*

Proof. The proof follows directly from [20, Theorem 4.4] since $Z(R)$ is not an ideal of R by the (weakly) firmness of R (Theorem 3.6). □

The ring R in [20, Example 5.4] is a reduced ring such that $Z(R)$ is an (nonzero prime) ideal, which consequently (by Theorem 3.6) is not weakly firm, and $\text{diam}(\Gamma(R[[x]])) = 2$.

Our last result provides a condition which is sufficient to give $\text{diam}(\Gamma(R[x])) = 2$ when R is nonreduced with nonnilpotent zero divisors.

Theorem 4.15. (cf. [20, Theorem 5.10]) *Let R be a nonreduced ring such that $Z(R)$ is not the nilradical of R. If $Z(R)$ has a nonzero annihilator, then R is not weakly firm, R is a McCoy ring, $diam(\Gamma(R)) = diam(\Gamma(R[x])) = 2$ and $Z(R)[[x]] \subseteq Z(R[[x]])$. Moreover, if $Z(R)\text{Nil}(R) = (0)$, then $diam(\Gamma(R[[x]])) = 2$.*

Proof. The proof is immediate since $Z(R)$ is an ideal of R by [20, Theorem 5.10], and hence R is not weakly firm by Theorem 3.6. □

5 Total Graphs and (Weakly) Firmness

In this section, we study some graph-theoretic properties of a (finite, weakly) firm ring by applying some known results related to the total graphs of commutative rings that are taken mainly from Section 3 of [3]. That is, we relate (apply) the results of Section 3 of [3] to (weakly) firm rings when $Z(R)$ is not an ideal of R which is a consequence of the (weakly) firmness of R by Theorem 3.6 above. The work of Anderson and Badawi in [3] for the study of the total graph of a commutative ring R, denoted by $T(\Gamma(R))$, is (mainly) divided into two cases depending on whether or not $Z(R)$ is an ideal of R ([3, Sections 2 and 3]), respectively.

In this section, we adopt all notations and definitions (exactly) from [3]. Let R be a commutative ring with $T(R)$ its total quotient ring, $\text{Nil}(R)$ its ideal of nilpotent elements, $Z(R)$ its set of zero-divisors, and $Reg(R)$ its set of regular elements. In their paper, Anderson and Badawi introduce and investigate the total graph of R, denoted by $T(\Gamma(R))$. It is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. They also study the three (induced) subgraphs $Nil(\Gamma(R))$, $Z(\Gamma(R))$, and $Reg(\Gamma(R))$ of $T(\Gamma(R))$, with vertices $\text{Nil}(R)$, $Z(R)$,and $Reg(R)$, respectively.

Note that if A is a subring of a commutative ring B, then $T(\Gamma(A))$ need not be an induced subgraph of $T(\Gamma(B))$. Although $x, y \in A$ are adjacent in $T(\Gamma(B))$ if they are adjacent in $T(\Gamma(A))$ since $Z(A) \subseteq Z(B)$, they may be adjacent in $T(\Gamma(B))$, but not adjacent in $T(\Gamma(A))$. In fact, $T(\Gamma(A))$ is an induced subgraph of $T(\Gamma(B))$ if and only if $Z(B) \cap A = Z(A)$.

We now begin with some examples of nonweakly firm rings.

Example 5.1. Parts (a), (b), and (c) of [3, Example 2.7] provide three examples of commutative rings R when $Z(R)$ is an ideal with some graph-theoretic properties of $Reg(\Gamma(R))$. Thus, (a), (b), and (c) are three examples of nonweakly firm rings by Theorem 3.6 since $Z(R)$ is an ideal of R.

Example 5.2. Let R be a commutative ring such that $Z(\Gamma(R))$ is complete. Then R is not weakly firm.

Proof. The result follows from Theorem 3.6 since $Z(R)$, as an implication of the hypothesis, is an ideal of R. □

• The rest of this section is devoted on the case when $Z(R)$ is not an ideal of R which is a consequence of weakly firmness of R. Since $Z(R)$ is always closed under multiplication by elements of R, this just means that there are distinct $x, y \in Z(R)^*$ such that $x + y \in Reg(R)$. In this case, $Z(\Gamma(R))$ is always connected (but never complete), $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ are never disjoint subgraphs of $T(\Gamma(R))$, and $|Z(R)| \geq 3$.

The following result shows that, for a (weakly) firm ring, $T(\Gamma(R))$ is connected when $Reg(\Gamma(R))$ is connected. However, we give an example to show that the converse fails.

Theorem 5.3. (cf. [3, Theorem 3.1]) *Let R be a (weakly) firm commutative ring.*

(1) *$Z(\Gamma(R))$ is connected with diam$(Z(\Gamma(R))) = 2$.*

(2) *Some vertex of $Z(\Gamma(R))$ is adjacent to a vertex of $Reg(\Gamma(R))$. In particular, the subgraphs $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ of $T(\Gamma(R))$ are not disjoint.*

(3) *If $Reg(\Gamma(R))$ is connected, then $T(\Gamma(R))$ is connected.*

Proof. The result follows from [3, Theorem 3.1] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

We next, for a (weakly) firm ring, determine when $T(\Gamma(R))$ is connected and compute diam$(T(\Gamma(R)))$. In particular, $T(\Gamma(R))$ is connected if and only if diam$(T(\Gamma(R))) < \infty$.

Theorem 5.4. (cf. [3, Theorem 3.3]) *Let R be a (weakly) firm commutative ring. Then $T(\Gamma(R))$ is connected if and only if $(Z(R)) = R$ (i.e., $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in$Z(R)). In particular, if R is a finite (weakly) firm commutative ring, then $T(\Gamma(R))$ is connected.*

Proof. The result follows from [3, Theorem 3.3] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

Theorem 5.5. (cf. [3, Theorem 3.4]) *Let R be a (weakly) firm commutative ring such that $(Z(R)) = R$ (i.e., $T(\Gamma(R))$ is connected). Let $n \geq 2$ be the least integer such that $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in Z(R)$. Then diam$(T(\Gamma(R))) = n$. In particular, if R is a finite (weakly) firm commutative ring, then diam$(T(\Gamma(R))) = 2$.*

Proof. The result follows from [3, Theorem 3.4] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

Corollary 5.6. (cf. [3, Corollary 3.5]) *Let R be a (weakly) firm commutative ring and suppose that $T(\Gamma(R))$ is connected.*

(1) *diam$(T(\Gamma(R))) = d(0, 1)$.*

(2) *If diam$(T(\Gamma(R))) = n$, then diam$(Reg(\Gamma(R))) \geq n - 2$.*

Proof. The result follows from [3, Corollary 3.5] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

Remark 5.7. Let R be a firm or a weakly firm commutative ring. Then diam$(Z(\Gamma(R))) = 2$ since by Theorem 3.6 $Z(R)$ is not an ideal of R and hence $(x, 0, y)$ is a path in $Z(\Gamma(R))$ for some distinct nonadjacent vertices x and y. Moreover, we have $2 \leq$ diam$(T(\Gamma(R))) < \infty$ when $T(\Gamma(R))$ is connected.

The next example shows that we may also have either diam$(T(\Gamma(R)))$ = diam$(Reg(\Gamma(R)))$ or diam$(T(\Gamma(R)))$ > diam$(Reg(\Gamma(R)))$ when R is not a (weakly) firm ring (by Example 3.5(b)) and also $Z(R)$ is not an ideal of R.

Example 5.8. (cf. [3, Example 3.9])

(a) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then diam$(T(\Gamma(R))) = 2$ by [3, Theorem 3.4] (or [3, Corollary 3.7]), and it is easy to check that diam$(Reg(\Gamma(R))) = 2$. Thus diam$(T(\Gamma(R)))$ = diam$(Reg(\Gamma(R)))$.

(b) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then diam$(T(\Gamma(R))) = 2$ by [3, Theorem 3.4] (or [3, Corollary 3.7]), and it is easy to check that diam$(Reg(\Gamma(R))) = 1$. Thus diam$(T(\Gamma(R)))$ > diam$(Reg(\Gamma(R)))$.

We next briefly discuss the diameter of $Reg(\Gamma(R \times S))$ for commutative rings R and S. Note that $Reg(R \times S) = Reg(R) \times Reg(S)$. So for distinct $(a,b),(c,d) \in Reg(R \times S)$, (a,b)—$(-a,-d)$—(c,d) is a path of length at most two in $Reg(\Gamma(R \times S))$. Thus $Reg(\Gamma(R \times S))$ is connected with diam$(Reg(\Gamma(R \times S))) \leq 2$. In particular, if $Z(\mathbb{Z}_m)$ is not an ideal of $\mathbb{Z}_m$, then $Reg(\Gamma(\mathbb{Z}_m))$ is always connected (cf. [3, Example 2.7(a)]). For example, $Reg(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))$, $Reg(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3))$, and $Reg(\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5))$ have diameters $0, 1,$ and2, respectively.

Theorem 5.9. (cf. [3, Theorem 3.10]) *Let R be a (weakly) firm commutative ring. Then $T(\Gamma(T(R)))$ is connected with diam$(T(\Gamma(T(R)))) = 2$. In particular, if R is a finite (weakly) firm commutative ring, then $T(\Gamma(R))$ is connected with diam$(T(\Gamma(R))) = 2$.*

Proof. The result follows from [3, Theorem 3.10] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

We next investigate the girth of $Z(\Gamma(R))$, $Reg(\Gamma(R))$, and $T(\Gamma(R))$ when R is (weakly) firm and hence $Z(R)$ is not an ideal of R. Recall that $|Z(R)| \geq 3$ if $Z(R)$ is not an ideal of R. We start with a lemma.

Lemma 5.10. (cf. [3, Lemma 3.13]) *Let R be a (weakly) firm commutative ring. Then char $R = 2$ if and only if $2Z(R) = \{0\}$.*

Proof. If char $R = 2$, then clearly $2Z(R) = \{0\}$. Conversely, suppose that $2z = 0$ for all $z \in Z(R)$. Since (by (weakly) firmness of R), $Z(R)$ is not an ideal of R (Theorem 3.6), there are distinct $x, y \in Z(R)$ such that $z = x + y \in Reg(R)$. Then $2z = 2x + 2y = 0$; so $2 = 0$ since $z \in Reg(R)$, i.e., char $R = 2$. □

Theorem 5.11. *Theorem 3.14. Let R be a (weakly) firm commutative ring.*

(1) *Either gr$(Z(\Gamma(R))) = 3$ or gr$(Z(\Gamma(R))) = \infty$. Moreover, if gr$(Z(\Gamma(R))) = \infty$, then $R = \mathbb{Z}_2 \times \mathbb{Z}_2$; so $Z(\Gamma(R))$ is a $K_{1,2}$ star graph with center 0.*

(2) *gr$(T(\Gamma(R))) = 3$ if and only if gr$(Z(\Gamma(R))) = 3$ (if and only if $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$).*

(3) *gr$(T(\Gamma(R))) = 4$ if and only if gr$(Z(\Gamma(R))) = 0$ (if and only if $R = \mathbb{Z}_2 \times \mathbb{Z}_2$).*

(4) *If char $R = 2$, then gr$(Reg(\Gamma(R))) = 3$ or ∞. In particular, gr$(Reg(\Gamma(R))) = 3$ if char $R = 2$ and $Reg(\Gamma(R))$ contains a cycle.*

(5) *gr$(Reg(\Gamma(R))) = 3, 4,$ or ∞. In particular, gr$(Reg(\Gamma(R))) \leq 4$ if $Reg(\Gamma(R))$ contains a cycle.*

Proof. The result follows from [3, Theorem 3.14] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

Example 3.15 of [3] provides 3 possibilities for the girth of $Reg(\Gamma(R))$ when $Z(R)$ is not an ideal of R and R is not weakly firm. Note that $R = \mathbb{Z}_3 \times \mathbb{Z}_4$ in Part (b) of this example is not weakly firm since $I = \mathbb{Z}_3 \times \{0,2\}$ is a weakly essential prime ideal of R, but $O_I = \mathbb{Z}_3 \times \{0\}$ is not a weakly essential ideal of R.

Let M be an R-module. We conclude this paper with some results about the graphs of the idealization $R(+)M$ of a module over a (weakly) firm ring R. In this theorem, we assume that $Z(R)(+)M = Z(R(+)M)$. Note that $Z(R)(+)M \subseteq Z(R(+)M)$ always holds, but the inclusion may be proper since $Z(\mathbb{Z}(+)\mathbb{Z}_2) = 2\mathbb{Z}(+)\mathbb{Z}_2$. However, equality holds if either M is an ideal of R or R is an integral domain and M is *torsionfree*.

Theorem 5.12. (cf. [3, Theorem 3.16]) *Let R be a (weakly) firm commutative ring, and let M be an R-module such that $Z(R(+)M) = Z(R)(+)M$.*

(1) *$T(\Gamma(R(+)M))$ is connected if and only if $T(\Gamma(R))$ is connected.*

(2) *diam*$(T(\Gamma(R(+)M))) = \text{diam}(T(\Gamma(R)))$.

Proof. The result follows from [3, Theorem 3.16] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

In view of the above theorem, we have the following corollary.

Corollary 5.13. (cf. [3, Corollary 3.17]) *Let R be a (weakly) firm commutative ring, and let M be an R-module. If $T($*
$)isconnected, then$T$(\Gamma(R(+)M))$ is connected with diam$(T(\Gamma(R(+)M))) \leq$ diam$(T(\Gamma(R)))$.

Proof. The result follows from [3, Corollary 3.17] since $Z(R)$ is not an ideal of R by Theorem 3.6. □

References

[1] F. Aliniaeifard, M. Behboodi, E. Mehdi-Nezhad, A. M. Rahimi, *The annihilating-Ideal graph of a commutative ring with respect to an ideal*, Comm. Algebra, 42 (2014), 2269-2284.

[2] D. F. Anderson, A. Badawi, *On the zero-divisor graph of a ring*, Comm. Algebra 36 (2008) 3073-3092.

[3] D. F. Anderson, A. Badawi, *The total graph of a commutative ring*, J. Algebra 320(7), (2008) 2706-2719.

[4] D. F. Anderson, R. Levy, J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras*, J. Pure Appl. Algebra 180 (2003) 221-241.

[5] D. F. Anderson, P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra 217 (1999) 434-447.

[6] D. F. Anderson, S. B. Mulay, *On the diameter and girth of a zero-divisor graph*, J. Pure Appl. Algebra 210 (2007) 543-550.

[7] M. Axtel, J. Coykendall, J. Stickles, *Zero-divisor graphs of polynomials and power series over commutative rings*, Comm. Algebra 33 (2005) 2043-2050.

[8] M. Axtel, J. Stickles, *Zero-divisor graphs of idealizations*, J. Pure Appl. Algebra 204 (2006) 235-243.

[9] A. Badawi, *On the Annihilator Graph of a Commutative Ring*, Communications in Algebra, 42(1), (2014), 108-121.

[10] A. Badawi, *On the Dot Product Graph of a Commutative Ring*, Communications in Algebra 43(1), (2015), 43-50.

[11] I. Beck, *Coloring of commutative rings*, J. Algebra 116 (1988) 208-226.

[12] B. Bollaboas, *Graph Theory, An Introductory Course*, Springer-Verlag, New York, 1979.

[13] T. Dube, *Commutative rings in which zero-components of essential primes are essential*, Journal of Algebra and Its Applications, Vol. 16, No. 11 (2017),15 pages.

[14] Ch. Eslahchi, A. M. Rahimi, *The k-zero-divisor hypergraph of a commutative ring*, Int. J. Math. Math. Sci. ID 50875, (2007), 1-15.

[15] N. Ganesan, *Properties of rings with a finite number of zero-divisors*, Math. Ann. 157 (1964) 215-218.

[16] M. Ghirati and A. Taherifar, *Intersections of essential (resp. free) maximal ideals in $C(X)$*, Topology Appl. 167, (2014) 62-68.

[17] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York/Basel, 1988.

[18] I. Kaplansky, *Commutative Rings*, rev. ed., University of Chicago Press, Chicago, 1974.

[19] J.D. LaGrange, *Complemented zero-divisor graphs and Boolean rings*, J. Algebra 315 (2007) 600-611.

[20] T. G. Lucas, *The diameter of a zero-divisor graph*, J. Algebra 301 (2006) 174-193.

[21] E. Mehdi-Nezhad, A. M Rahimi, *On some graphs associated to commutative semirings*, Results in Mathematics, volume 68, Issue 3, (2015), 293-312.

[22] E. Mehdi-Nezhad, A. M Rahimi, *The annihilation graphs of commutator posets and lattices with respect to an element*, Journal of Algebra and its applications, Volume 16, Issue 06, (2017), [20 pages].

[23] E. Mehdi-Nezhad, A. M Rahimi, *The annihilation graphs of commutator posets and lattices with respect to an ideal*, Journal of Algebra and its applications, Volume 17, Issue 06, (2018).

[24] E. Mehdi-Nezhad, A. M Rahimi, *Dominating sets of the comaximal and ideal-based zero-divisor graphs of commutative rings*, Quaestiones Mathematicae, 38(5), (2015) 1-17.

[25] D.A. Mojdeh and A.M. Rahimi, *Dominating sets of some graphs associated to commutative rings*, Communications in Algebra 40 (2012), 3389-3396.

[26] A. M. Rahimi, *Smarandache vertices of the graphs associated to the commutative rings*, Comm. Algebra 41 (2013), 1989-2004.

[27] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003) no. 9, 4425-4443.

[28] N. O. Smith, *Planar zero-divisor graphs*, Comm. Algebra 35 (2007) 171-180.

Author information

Elham Mehdi-Nezhad, Department of Mathematics and Applied Mathematics, North West University, South Africa.,.
E-mail: Elham.Mehdinezhad@NWU.ac.za

Amir M. Rahimi, Department of Mathematical Sciences, University of South Africa, P.O. Box 392, 0003 Pretoria, South Africa, And
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.,.
E-mail: amrahimi@ipm.ir

Received: February 7, 2019.
Accepted: August 14, 2019