Palestine Journal of Mathematics

Vol. 10(1)(2021) , 50–52

RICKART $*$-RINGS WITH PLANAR ZERO-DIVISOR GRAPHS

Avinash Patil

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 05C25, Secondary 05C75.

Keywords and phrases: Zero-divisor graph, planar graph, Rickart $*$-ring.

Abstract In this paper, we characterize all finite Rickart $*$-rings whose zero-divisor graphs are planar.

1 Introduction

An *involution* $*$ on an associative ring A is a mapping such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$, for all $a, b \in A$. A ring with involution $*$ is called a $*$-*ring*. Clearly, identity mapping is an involution if and only if the ring is commutative. An element e of a $*$-ring A is a *projection* if $e = e^2$ and $e = e^*$. For a nonempty subset B of A, we write $r(B) = \{x \in A \colon bx = 0, \forall b \in B\}$, and call a *right annihilator* of B in A. A *Rickart* (resp. *Baer*) $*$-*ring* is a $*$-ring in which right annihilator of every element (resp. every subset) is generated, as a right ideal, by a projection in A. Every Baer $*$-ring is Rckart $*$-ring and every Rickart $*$-ring contains unity. For each element a in a Rickart $*$-ring, there is unique projection e such that $ae = a$ and $ax = 0$ if and only if $ex = 0$, called a *right projection* of a denoted by $RP(a)$. In fact, $r(\{a\}) = (1 - RP(a))A$. Similarly, the left annihilator $l(\{a\})$ and the left projection $LP(a)$ are defined for each element a in a Rickart $*$-ring A. The set of projections $P(A)$ in a Rickart $*$-ring A forms a lattice, denoted by $L(P(A))$, under the partial order '$e \leq f$ if and only if $e = fe = ef$'. In fact, $e \vee f = f + RP(e(1-f))$ and $e \wedge f = e - LP(e(1-f))$. More details about Rickart $*$-ring can be found in Berberian [4].

Beck [3] introduced the concept of zero-divisor graph $\Gamma(R)$ of a commutative ring R with unity as follows. Let G be a simple graph whose vertices are the elements of R and two vertices x and y are adjacent if $xy = 0$. The graph G is known as the *zero-divisor graph* of R. He was mainly interested in the coloring of this graph. An interesting question was proposed by Anderson, Frazier, Lauve, and Livingston: For which finite commutative rings R, $\Gamma(R)$ is planar? Cf. [2, Question 5.3]. In [1], Akbari et al. answered this question affirmatively and gave classification of commutative rings with zero-divisor graph planar.

In [6], Patil and Waphare extended the concept of zero divisor graph to $*$-rings as follows: Let A be a $*$-ring. A simple undirected graph $\Gamma^*(A)$ is associated to A whose vertex set is $V(\Gamma^*(A)) = \{a(\neq 0) \in A \mid ab = 0$, for some nonzero $b \in A\}$ (i.e. nonzero left zero-divisors) and two distinct vertices x and y are adjacent if and only if $xy^* = 0$. Patil and Waphare [7] studied the zero-divisor graphs of Rickart $*$-rings in detail.

In this paper, we characterize all finite Rickart $*$-rings whose zero-divisor graphs are planar.

2 Main Results

In this section we consider A as a finite Rickart $*$-ring such that $\Gamma^*(A)$ is non-empty. We need the following results proved by Thakare and Waphare ([8, Theorem 3]), which completely classifies $*$-rings with finitely many elements into Baer $*$-rings and non-Baer $*$-rings.

Theorem 2.1. *A $*$-ring A with finitely many elements is a Baer $*$-ring if and only if $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$ where A_i is a field or A_i is a 2×2-matrix ring over a finite field $F(p^\alpha)$ with α odd positive integer and p is a prime number of the form $4k + 3$.*

The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. We use K_n for the complete graph with n vertices. A *planar graph* is a graph that can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930.

Theorem 2.2 (Kuratowski, [9, Theorem 6.2.2, p. 246]). *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Proposition 2.3. *If A is a finite Rickart $*$-ring such that either $\Gamma^*(A)$ or $\Gamma^*(A)^c$ is planar then A is a commutative ring.*

Proof. Let A be a finite Rickart $*$-ring. By Berberian [4, §4, Proposition 1], A is a Baer $*$-ring. Consequently, by Theorem 2.1, $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$ where A_i is a field or A_i is a 2×2-matrix ring over a finite field $F(p^\alpha)$ with α odd positive integer and p is a prime number of the form $4k+3$. If A is non-commutative, then it has at least one component of the type $M_2(F_{p^\alpha})$ in its direct sum representation. In that case, $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $I - e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ are elements of A such that $e(I-e)^* = eb^* = e(2b)^* = 0$. Then $U = \{e, a, 2a\}$ and $V = \{I-e, b, 2b\}$ are independent sets of $\Gamma^*(A)$ such that each element of U is adjacent to every element of V, hence $\Gamma^*(A)$ contains $K_{3,3}$ as a subgraph, hence it can not be planar. On the other hand, the set $S = \left\{ \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}$ is an independent set in $\Gamma^*(A)$, *i.e.,* it forms K_5 in $\Gamma^*(A)^c$, hence $\Gamma^*(A)^c$ can not be planar, a contradiction. Therefore A does not contain a component of the type $M_2(F_{p^\alpha})$ in its direct sum representation, hence A is a commutative ring. □

Corollary 2.4. *If A is a finite Rickart $*$-ring such that $\Gamma^*(A)$ is disconnected then $\Gamma^*(A)$ can not be planar.*

Proof. In [5] it proved that if $\Gamma^*(A)$ is connected, then $A = M_2(F)$ where F is a finite field with p^α elements, α is odd positive integer and p is prime of the form $4k+3$. The result follows from Proposition 2.3. □

Now we prove the main result of this section.

Theorem 2.5. *Let A be a finite Rickart $*$-ring. Then $\Gamma^*(A)$ is planar if and only if A is one of the type: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$, $F_1 \oplus F_2$, where F_1 and F_2 are finite fields with $|F_1| \leq 3$ or $|F_2| \leq 3$.*

Proof. If A is one of the given rings then $\Gamma^*(A)$ is one of the graphs given in Figure 1 Hence $\Gamma^*(A)$ is planar graph. Now, let A be a finite Rickart $*$-ring such that $\Gamma^*(A)$ is planar. Then by Proposition 2.3, A is commutative Baer $*$-ring. Hence A is direct sum of finite fields, say $A = F_1 \oplus F_2 \oplus \cdots \oplus F_n$. If $n \geq 4$ then $(1,0,0,0)$, $(0,1,0,0)$, $(1,1,0,0)$ are all adjacent to $(0,0,1,0)$, $(0,0,0,1)$, $(0,0,1,1)$, hence $\Gamma^*(A)$ contains a subdivision of $K_{3,3}$, a contradiction. Therefore $A = F_1 \oplus F_2 \oplus F_3$. If $|F_2| \geq 3$ and $|F_3| \geq 3$ then $\Gamma^*(A)$ contains a subdivision of K_5. Let $A = F_1 \oplus F_2$. If $|F_1|$ and $|F_2|$ both exceed 3, then $\Gamma^*(A)$ will contain $K_{3,3}$, , a contradiction. Hence either $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ or $A = F_1 \oplus F_2$, where F_1 and F_2 are finite fields with $|F_1| \leq 3$ or $|F_2| \leq 3$. □

Acknowlegement: The author is grateful to the anounimus referee for careful reading of the paper, valuable comments and fruitful suggestions.

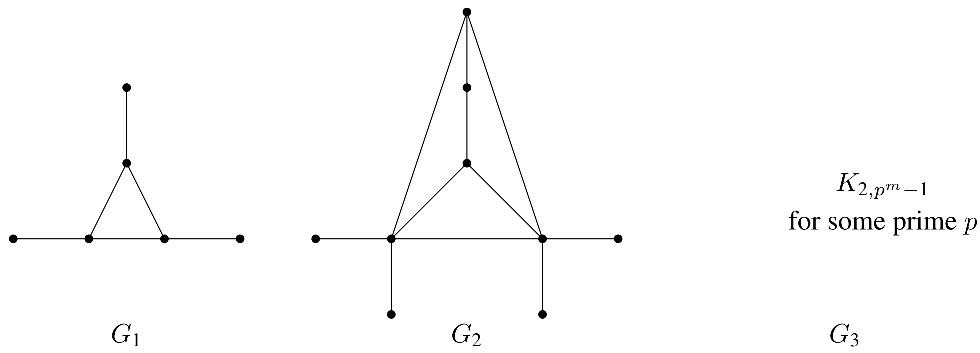

Figure 1.

References

[1] S. Akbari, H.R. Maimani, S. Yassemi, *When a zero-divisor graph is planar or a complete r-partite graph*, J. Algebra, **270** (2003) 169-180.

[2] D. F. Anderson, A. Frazier, A. Lauve, P. S. Livingston, *The zero-divisor graph of a commutative ring- II*, in: Lecture Notes in Pure and Appl. Math; vol. **220** Dekker, New York, (2001) pp. 61-72.

[3] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988) 208-226.

[4] S. K. Berberian, *Baer $*$-Rings*, Springer-Verlag, Berlin and New York(1972).

[5] Avinash Patil, Zero-Divisor Graphs of Rickart $*$-Rings, Dismantlable Lattices and Related Algebraic Structures, Ph. D. Thesis submitted to SPP University, Pune, India (2016).

[6] Avinash Patil, B. N. Waphare, *The zero-divisor graph of a ring with involution*, J. Algebra Appl. **17**(3) (2018) 1850050 (17 pages). doi.org/10.1142/S0219498818500500.

[7] Avinash Patil, B. N. Waphare, *On the zero-divisor graph of Rickart *-rings*, Asian-Eur. J. Math.**10** (01) (2017) 1750015 (17 pages). doi.org/10.1142/S1793557117500152.

[8] N. K. Thakare and B. N. Waphare, *Baer $*$-rings with Finitely Many Elements*, J. Combin. Math. Combin. Comput. **26** (1998) 161-164.

[9] D. B. West, *Introduction to Graph Theory*, Second Edition, Prentice-Hall of India, New Delhi(2002).

Author information

Avinash Patil, JET's Z. B. Patil College, Dhule (MS)- 424 002, India.
E-mail: avipmj@gmail.com

Received: February 2, 2019.
Accepted: May 20, 2019