RICKART ∗-RINGS WITH PLANAR ZERO-DIVISOR GRAPHS

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Abstract In this paper, we characterize all finite Rickart ∗-rings whose zero-divisor graphs are planar.

1 Introduction

An involution ∗ on an associative ring $A$ is a mapping such that $(a + b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$, for all $a, b \in A$. A ring with involution ∗ is called a ∗-ring. Clearly, identity mapping is an involution if and only if the ring is commutative. An element $e$ of a ∗-ring $A$ is a projection if $e = e^2$ and $e = e^*$. For a nonempty subset $B$ of $A$, we write $r(B) = \{x \in A : bx = 0, \forall b \in B\}$, and call a right annihilator of $B$ in $A$. A Rickart (resp. Baer) ∗-ring is a ∗-ring in which right annihilator of every element (resp. every subset) is generated, as a right ideal, by a projection in $A$. Every Baer ∗-ring is Rickart ∗-ring and every Rickart ∗-ring contains unity. For each element $a$ in a Rickart ∗-ring, there is unique projection $e$ such that $ae = a$ and $ax = 0$ if and only if $ex = 0$, called a right projection of $a$ denoted by $RP(a)$.

In fact, $r(\{a\}) = (1 - RP(a))A$. Similarly, the left annihilator $l(\{a\})$ and the left projection $LP(a)$ are defined for each element $a$ in a Rickart ∗-ring $A$. The set of projections $P(A)$ in a Rickart ∗-ring $A$ forms a lattice, denoted by $L(P(A))$, under the partial order ‘$e \leq f$ if and only if $e = ef = fe^*$’. In fact, $e \vee f = f + RP(e(1 - f))$ and $e \wedge f = e - LP(e(1 - f))$. More details about Rickart ∗-ring can be found in Berberian [4].

Beck [3] introduced the concept of zero-divisor graph $\Gamma(R)$ of a commutative ring $R$ with unity as follows. Let $G$ be a simple graph whose vertices are the elements of $R$ and two vertices $x$ and $y$ are adjacent if $xy = 0$. The graph $G$ is known as the zero-divisor graph of $R$. He was mainly interested in the coloring of this graph. An interesting question was proposed by Anderson, Frazier, Laue, and Livingston: For which finite commutative rings $R$, $\Gamma(R)$ is planar? Cf. [2, Question 5.3]. In [1], Akbari et al. answered this question affirmatively and gave classification of commutative rings with zero-divisor graph planar.

In [6], Patil and Waphare extended the concept of zero divisor graph to ∗-rings as follows: Let $A$ be a ∗-ring. A simple undirected graph $\Gamma^*(A)$ is associated to $A$ whose vertex set is $V(\Gamma^*(A)) = \{a(\neq 0) \in A | ab = 0, \text{for some nonzero } b \in A\}$ (i.e. nonzero left zero-divisors) and two distinct vertices $x$ and $y$ are adjacent if and only if $xy^* = 0$. Patil and Waphare [7] studied the zero-divisor graphs of Rickart ∗-rings in detail.

In this paper, we characterize all finite Rickart ∗-rings whose zero-divisor graphs are planar.

2 Main Results

In this section we consider $A$ as a finite Rickart ∗-ring such that $\Gamma^*(A)$ is non-empty. We need the following results proved by Thakare and Waphare ([8, Theorem 3]), which completely classifies ∗-rings with finitely many elements into Baer ∗-rings and non-Baer ∗-rings.

Theorem 2.1. A ∗-ring $A$ with finitely many elements is a Baer ∗-ring if and only if $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$, where $A_i$ is a field or $A_i$ is a $2 \times 2$-matrix ring over a finite field $F(p^\alpha)$ with $\alpha$ odd positive integer and $p$ is a prime number of the form $4k + 3$. 
The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use $K_n$ for the complete graph with $n$ vertices. A planar graph is a graph that can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930.

**Theorem 2.2** (Kuratowski, [9, Theorem 6.2.2, p. 246]). A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

**Proposition 2.3.** If $A$ is a finite Rickart *-ring such that either $\Gamma^*(A)$ or $\Gamma^*(A)^c$ is planar then $A$ is a commutative ring.

**Proof.** Let $A$ be a finite Rickart *-ring. By Berberian [4, §4, Proposition 1], $A$ is a Baer *-ring. Consequently, by Theorem 2.1, $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$ where $A_i$ is a field or $A_i$ is a 2 × 2-matrix ring over a finite field $F(p^\alpha)$ with $\alpha$ odd positive integer and $p$ is a prime number of the form $4k + 3$. If $A$ is non-commutative, then it has at least one component of the type $M_2(F(p^\alpha))$ in its direct sum representation. In that case, $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $I - e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ are elements of $A$ such that $e(I - e)^* = eb^* = e(2b)^* = 0$. Then $U = \{e, a, 2a\}$ and $\overline{V} = \{I - e, b, 2b\}$ are independent sets of $\Gamma^*(A)$ such that each element of $U$ is adjacent to every element of $V$, hence $\Gamma^*(A)$ contains $K_{3,3}$ as a subgraph, hence it can not be planar.

On the other hand, the set $S = \{\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\}$ is an independent set in $\Gamma^*(A)$, i.e., it forms $K_{5}$ in $\Gamma^*(A)^c$, hence $\Gamma^*(A)^c$ can not be planar, a contradiction. Therefore $A$ does not contain a component of the type $M_2(F(p^\alpha))$ in its direct sum representation, hence $A$ is a commutative ring.

**Corollary 2.4.** If $A$ is a finite Rickart *-ring such that $\Gamma^*(A)$ is disconnected then $\Gamma^*(A)$ can not be planar.

**Proof.** In [5] it proved that if $\Gamma^*(A)$ is connected, then $A = M_2(F)$ where $F$ is a finite field with $p^\alpha$ elements, $\alpha$ is odd positive integer and $p$ is prime of the form $4k + 3$. The result follows from Proposition 2.3.

Now we prove the main result of this section.

**Theorem 2.5.** Let $A$ be a finite Rickart *-ring. Then $\Gamma^*(A)$ is planar if and only if $A$ is one of the type: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$, $F_1 \oplus F_2$, where $F_1$ and $F_2$ are finite fields with $|F_1| \leq 3$ or $|F_2| \leq 3$.

**Proof.** If $A$ is one of the given rings then $\Gamma^*(A)$ is one of the graphs given in Figure 1. Hence $\Gamma^*(A)$ is planar. Now, let $A$ be a finite Rickart *-ring such that $\Gamma^*(A)$ is planar. Then by Proposition 2.3, $A$ is commutative Baer *-ring. Hence $A$ is direct sum of finite fields, say $A = F_1 \oplus F_2 \oplus \cdots \oplus F_n$. If $n \geq 4$ then $(1,0,0,0), (0,1,0,0), (1,1,0,0)$ are all adjacent to $(0,0,1,0), (0,0,0,1), (0,0,0,1)$, hence $\Gamma^*(A)$ contains a subdivision of $K_{3,3}$, a contradiction. Therefore $A = F_1 \oplus F_2 \oplus F_3$. If $|F_2| \geq 3$ and $|F_3| \geq 3$ then $\Gamma^*(A)$ contains a subdivision of $K_5$. Let $A = F_1 \oplus F_2$. If $|F_1|$ and $|F_2|$ both exceed 3, then $\Gamma^*(A)$ will contain $K_{3,3}$, a contradiction. Hence either $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ or $A = F_1 \oplus F_2$, where $F_1$ and $F_2$ are finite fields with $|F_1| \leq 3$ or $|F_2| \leq 3$.

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Figure 1.

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