3-divisor cordiality of some product related graphs

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Abstract Let $G$ be a $(p, q)$ graph and $2 \leq k \leq p$. Let $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a map. For each edge $uv$, assign the label 1 if either $f(u)$ or $f(v)$ divides the other and 0 otherwise. $f$ is called a $k$-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ if $i, j \in \{1, 2, \ldots, k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with $x$, where $x \in \{1, 2, \ldots, k\}$, $e_f(i)$ denote the number of edges labeled with $i$, $i \in \{0, 1\}$. A graph with a $k$-divisor cordial labeling is called a $k$-divisor cordial graph. In this paper, we discuss 3-divisor cordial labeling behavior of ladder, prism and book graphs.

1 Introduction

Graphs considered here are finite, undirected and simple. In 1980, Cahit [1] introduced the cordial labeling of graphs. In [4], Varatharajan, Navaneethakrishnan, and Nagarajan introduced a notion, called divisor cordial labeling and proved the standard graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial. Sathish Narayanan introduced the notion of $k$-divisor cordial labeling in [5]. In [6], 3-divisor cordiality of wheel and $K_n + 2K_2$ have been studied. In this paper we studied the 3-divisor cordial labeling behavior of ladder, prism and book graphs. Terms and definitions not defined here are used in the sense of Harary [3] and Gallian [2].

2 3-divisor cordial labeling

Definition 2.1. Let $G$ be a $(p, q)$ graph and $2 \leq k \leq p$. Let $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a map. For each edge $xy$, assign the label 1 if either $f(x)$ or $f(y)$ divides the other and 0 otherwise. $f$ is called a $k$-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ if $i, j \in \{1, 2, \ldots, k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with $x$, where $x \in \{1, 2, \ldots, k\}$, $e_f(i)$ denote the number of edges labeled with $i$, $i \in \{0, 1\}$. A graph with a $k$-divisor cordial labeling is called a $k$-divisor cordial graph.

Definition 2.2. The Cartesian product of two graphs $G_1$ and $G_2$ is the graph $G_1 \square G_2$ with the vertex set $V_1 \square V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever $[u_1 = v_1$ and $u_2 \text{ adj } v_2]$ or $[u_2 = v_2$ and $u_1 \text{ adj } v_1]$.

First we consider the graph ladder $L_n = P_n \square P_2$. Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i, v_{i+1}, v_i, v_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. Note that $L_n$ consists of $2n$ vertices and $3n - 2$ edges.

Theorem 2.3. The ladder $L_n = P_n \square P_2$ is 3-divisor cordial.

Proof. The proof is divided into twelve cases.

Case 1. $n \equiv 0 \pmod{12}$.
Let $n = 12t$ where $t \geq 1$. Here we define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$
\begin{align*}
  f(u_{2i-1}) &= f(v_{2i}) = 2, & 1 \leq i \leq 3t \\
  f(u_{2i}) &= f(v_{2i-1}) = 3, & 1 \leq i \leq 3t \\
  f(u_{6t+i}) &= f(v_{6t+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7t+i}) &= f(v_{7t+i}) = 1, & 1 \leq i \leq 4t \\
  f(u_{11t+i}) &= f(v_{11t+i}) = 3, & 1 \leq i \leq t 
\end{align*}
$$

**Case 2.** $n \equiv 1 \pmod{12}$.

Let $n = 12t + 1$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$
\begin{align*}
  f(u_{2i-1}) &= 2, & 1 \leq i \leq 3t + 1 \\
  f(u_{2i}) &= 3, & 1 \leq i \leq 3t \\
  f(v_{2i-1}) &= 3, & 1 \leq i \leq 3t + 1 \\
  f(v_{2i}) &= 2, & 1 \leq i \leq 3t \\
  f(u_{6t+2+i}) &= f(v_{6t+2+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7t+3+i}) &= f(v_{7t+3+i}) = 3, & 1 \leq i \leq t \\
  f(u_{8t+3+i}) &= f(v_{8t+3+i}) = 1, & 1 \leq i \leq 4t - 2 
\end{align*}
$$

and $f(u_{6t+2}) = f(v_{6t+2}) = f(u_{7t+3}) = f(v_{7t+3}) = 1$.

**Case 3.** $n \equiv 2 \pmod{12}$.

For $n = 2$, Figure 1 shows that $L_2$ is a 3-divisor cordial graph.

![Figure 1](image1.png)

**Figure 1.**

Let $n = 12t + 2$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows: assign the labels to the vertices $u_i, v_i$ ($1 \leq i \leq 6t + 1$) as in case 2.

$$
\begin{align*}
  f(u_{6t+1+i}) &= f(v_{6t+1+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7t+1+i}) &= f(v_{7t+1+i}) = 1, & 1 \leq i \leq 4t + 1 \\
  f(u_{11t+2+i}) &= f(v_{11t+2+i}) = 3, & 1 \leq i \leq t 
\end{align*}
$$

**Case 4.** $n \equiv 3 \pmod{12}$.

The vertex labelings given in Figure 2 establish that $L_3$ and $L_{15}$ are 3-divisor cordial graphs.

![Figure 2](image2.png)

**Figure 2.**

Let $n = 12t + 3$ where $t \geq 2$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows: assign the
labels to the vertices $u_i, v_i$ ($1 \leq i \leq 6t + 1$) as in case 2.

\[
\begin{align*}
  f(u_{6t+1+i}) &= 2, & 1 \leq i \leq 3 \\
  f(v_{6t+1+i}) &= 3, & 1 \leq i \leq 3 \\
  f(u_{6t+5+i}) &= f(v_{6t+5+i}) = 2, & 1 \leq i \leq t - 1 \\
  f(u_{7t+5+i}) &= f(v_{7t+5+i}) = 3, & 1 \leq i \leq t - 1 \\
  f(u_{8t+4+i}) &= f(v_{8t+4+i}) = 1, & 1 \leq i \leq 4t - 1 \\
\end{align*}
\]

and $f(u_{6t+5}) = f(v_{6t+5}) = f(u_{7t+5}) = f(v_{7t+5}) = 1.$

**Case 5.** $n \equiv 4 \pmod{12}.$

Figure 3 shows that $L_4$ is a 3-divisor cordial graph.

![Figure 3](image_url)

Let $n = 12t + 4$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows: assign the labels to the vertices $u_i, v_i$ ($1 \leq i \leq 6t + 1$) as in case 2. Then put the number 3 to the vertex $u_{6t+2}$ and 2 to $v_{6t+2}$. Then

\[
\begin{align*}
  f(u_{6t+2+i}) &= f(v_{6t+2+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7t+3+i}) &= f(v_{7t+3+i}) = 1, & 1 \leq i \leq 4t \\
  f(u_{11t+4+i}) &= f(v_{11t+4+i}) = 3, & 1 \leq i \leq t \\
\end{align*}
\]

$f(v_{7t+3}) = f(v_{11t+4}) = 1, f(u_{7t+3}) = 2$ and $f(u_{11t+4}) = 3.$

**Case 6.** $n \equiv 5 \pmod{12}.$

From Figure 4, we observe that $L_5$ is 3-divisor cordial.

![Figure 4](image_url)

Let $n = 12t + 5$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

\[
\begin{align*}
  f(u_{2i-1}) &= 2, & 1 \leq i \leq 3t + 2 \\
  f(u_{2i}) &= 3, & 1 \leq i \leq 3t + 1 \\
  f(v_{2i-1}) &= 3, & 1 \leq i \leq 3t + 2 \\
  f(v_{2i}) &= 2, & 1 \leq i \leq 3t + 1 \\
  f(u_{6t+4+i}) &= f(v_{6t+4+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7t+5+i}) &= f(v_{7t+5+i}) = 3, & 1 \leq i \leq t \\
  f(u_{8t+5+i}) &= f(v_{8t+5+i}) = 1, & 1 \leq i \leq 4t \\
\end{align*}
\]

and $f(u_{6t+4}) = f(v_{6t+4}) = f(u_{7t+5}) = f(v_{7t+5}) = 1.$

**Case 7.** $n \equiv 6 \pmod{12}.$

For $L_6,$ Figure 5 establish that $L_6$ is 3-divisor cordial.

Let $n = 12t + 6$ where $t \geq 1.$ Consider $f : V(L_n) \to \{1, 2, 3\}$ as follows: assign the labels to the vertices $u_i, v_i$ ($1 \leq i \leq 6t + 3$) as in case 6. Then

\[
\begin{align*}
  f(u_{6t+3+i}) &= f(v_{6t+3+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7t+4+i}) &= f(v_{7t+4+i}) = 1, & 1 \leq i \leq 4t + 1 \\
  f(u_{11t+6+i}) &= f(v_{11t+6+i}) = 3, & 1 \leq i \leq t \\
\end{align*}
\]
shows that 

\[ f(v_{\ell+t}) = f(v_{11t+6}) = 1, f(u_{\ell+t}) = 2 \text{ and } f(u_{11t+6}) = 3. \]

**Case 8.** \(n \equiv 7 \pmod{12}\).

Figure 6 shows that \(L_7\) is a 3-divisor cordial graph.

![Figure 6](image)

Let \(n = 12t + 7\) where \(t \geq 1\). We define a map \(f : V(L_n) \to \{1, 2, 3\}\) as follows:

\[
\begin{align*}
  f(u_{2i-1}) &= f(v_{2i}) = 2, & 1 \leq i \leq 3t + 2 \\
  f(u_{2i}) &= f(v_{2i-1}) = 3, & 1 \leq i \leq 3t + 2 \\
  f(u_{6i+5+i}) &= f(v_{6i+5+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7i+7+i}) &= f(v_{7i+7+i}) = 3, & 1 \leq i \leq t \\
  f(u_{8i+8+i}) &= f(v_{8i+8+i}) = 1, & 1 \leq i \leq 4t - 1
\end{align*}
\]

Case 9. \(n \equiv 8 \pmod{12}\).

Let \(n = 12t + 8\) where \(t \geq 0\). We define a map \(f : V(L_n) \to \{1, 2, 3\}\) as follows:

\[
\begin{align*}
  f(u_{2i-1}) &= f(v_{2i}) = 2, & 1 \leq i \leq 3t + 2 \\
  f(u_{2i}) &= f(v_{2i-1}) = 3, & 1 \leq i \leq 3t + 2 \\
  f(u_{6i+4+i}) &= f(v_{6i+4+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7i+5+i}) &= f(v_{7i+5+i}) = 1, & 1 \leq i \leq 4t + 2 \\
  f(u_{11i+8+i}) &= f(v_{11i+8+i}) = 3, & 1 \leq i \leq t
\end{align*}
\]

f(u_{7i+5}) = f(v_{11i+8}) = 1, f(u_{7i+5}) = 2 \text{ and } f(u_{11i+8}) = 3.

Case 10. \(n \equiv 9 \pmod{12}\).

Let \(n = 12t + 9\) where \(t \geq 0\). We define a map \(f : V(L_n) \to \{1, 2, 3\}\) as follows:

\[
\begin{align*}
  f(u_{2i-1}) &= 2, & 1 \leq i \leq 3t + 3 \\
  f(u_{2i}) &= 3, & 1 \leq i \leq 3t + 2 \\
  f(v_{2i-1}) &= 3, & 1 \leq i \leq 3t + 3 \\
  f(u_{2i}) &= 2, & 1 \leq i \leq 3t + 2 \\
  f(u_{6i+6+i}) &= f(v_{6i+6+i}) = 2, & 1 \leq i \leq t \\
  f(u_{7i+8+i}) &= f(v_{7i+8+i}) = 3, & 1 \leq i \leq t \\
  f(u_{8i+9+i}) &= f(v_{8i+9+i}) = 1, & 1 \leq i \leq 4t
\end{align*}
\]

f(u_{6i+6}) = f(v_{6i+6}) = f(v_{7i+7}) = f(u_{7i+8}) = f(v_{7i+8}) = f(v_{8i+9}) = 1, f(u_{7i+7}) = 2 \text{ and } f(u_{8i+9}) = 3.

Case 11. \(n \equiv 10 \pmod{12}\).

Let \(n = 12t + 10\) where \(t \geq 0\). We define a map \(f : V(L_n) \to \{1, 2, 3\}\) as follows: assign the labels to the vertices \(u_i, v_i\) (\(1 \leq i \leq 6t + 5\)) as in case 10. Then

\[
\begin{align*}
  f(u_{6i+5+i}) &= f(v_{6i+5+i}) = 2, & 1 \leq i \leq t + 1 \\
  f(u_{7i+6+i}) &= f(v_{7i+6+i}) = 1, & 1 \leq i \leq 4t + 3 \\
  f(u_{11i+9+i}) &= f(v_{11i+9+i}) = 3, & 1 \leq i \leq t + 1
\end{align*}
\]
Case 12. \( n \equiv 1 \pmod{12} \).
Let \( n = 12t + 11 \) where \( t \geq 0 \). We define a map \( f : V(L_n) \rightarrow \{1, 2, 3\} \) as follows:

\[
\begin{align*}
  f(u_{2t-1}) &= f(v_{2t}) = 2, \quad 1 \leq i \leq 3t + 3 \\
  f(u_{2t}) &= f(v_{2t-1}) = 3, \quad 1 \leq i \leq 3t + 3 \\
  f(u_{6t+7+i}) &= f(v_{6t+7+i}) = 2, \quad 1 \leq i \leq t \\
  f(u_{7t+9+i}) &= f(v_{7t+9+i}) = 3, \quad 1 \leq i \leq t \\
  f(u_{8t+10+i}) &= f(v_{8t+10+i}) = 1, \quad 1 \leq i \leq 4t + 1
\end{align*}
\]

\( f(u_{6t+7}) = f(v_{6t+7}) = f(u_{7t+9}) = f(v_{7t+9}) = f(v_{8t+10}) = 1, f(u_{7t+8}) = 2 \) and \( f(u_{8t+10}) = 3 \).

The vertex and edge conditions are given in Table 1 and Table 2.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(2) )</th>
<th>( v_f(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{3} )</td>
<td>( \frac{2n}{3} )</td>
<td>( \frac{2n}{3} )</td>
<td>( \frac{2n}{3} )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{3} )</td>
<td>( \frac{2n+1}{3} )</td>
<td>( \frac{2n+1}{3} )</td>
<td>( \frac{2n+1}{3} )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{3} )</td>
<td>( \frac{2n+2}{3} )</td>
<td>( \frac{2n+2}{3} )</td>
<td>( \frac{2n+2}{3} )</td>
</tr>
</tbody>
</table>

Table 1.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{2} )</td>
<td>( \frac{3n-1}{2} )</td>
<td>( \frac{3n-1}{2} )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{2} )</td>
<td>( \frac{3n-2}{2} )</td>
<td>( \frac{3n-2}{2} )</td>
</tr>
</tbody>
</table>

Table 2.

Hence \( L_n \) is 3-divisor cordial. □

Next we consider prism graph \( C_n \Box P_2 \). Let \( V(C_n \Box P_2) = V(L_n) \) and \( E(C_n \Box P_2) = E(L_n) \cup \{ u_n, u_1, v_n, v_1 \} \).

**Theorem 2.4.** Prism \( C_n \Box P_2 \) is 3-divisor cordial.

**Proof.** Assign the labels to the vertices as in Theorem 2.3. The vertex condition given in Table 1 and the edge condition given in Table 3 shows that \( C_n \Box P_2 \) is 3-divisor cordial.

<table>
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<td>( \frac{3n+2}{2} )</td>
</tr>
</tbody>
</table>

Table 3.

This completes the proof. □

Next we discuss 3-divisor cordial labeling behavior of book graph. A book graph is a cartesian product of star graph with \( K_2 \) and we denote it by \( B_n = K_{1,n} \Box K_2 \). Let \( V(B_n) = \{ u, v, u_i, v_i : 1 \leq i \leq n \} \) and \( E(B_n) = \{ uv, uu_i, vv_i : 1 \leq i \leq n \} \). Note that \( B_n \) consists of \( 2n + 2 \) vertices and \( 3n + 1 \) edges.

**Theorem 2.5.** Book \( B_n = K_{1,n} \Box K_2 \) is 3-divisor cordial, for all \( n \).

**Proof.** We construct a vertex labeling \( f \) from the set of vertices of \( B_n \) to the set \( \{1, 2, 3\} \) by \( f(u) = 2, f(v) = 3 \) and we consider the following cases for the labeling of other vertices.

**Case 1.** \( n \equiv 0 \pmod{6} \).
Let \( n = 6t \) where \( t \geq 1 \). Here
shows that \( v_1, v_2 \) respectively. Clearly \( u_1, 3 \) to the vertices 

Hence Case 6.

Let the labels to the vertices of \( u_1, v_1 (1 \leq i \leq 6t) \) as in Case 1. Then put the labels 1, 3 respectively to the vertices \( u_{6t+1}, v_{6t+1} \). Now relabel the vertices \( v_{3t+1}, v_{5t+1}, v_{5t+2} \) by 3, 1 respectively. Here \( v_f(1) = 4t + 2, v_f(2) = v_f(3) = 4t + 1, \) and \( e_f(0) = e_f(1) = 9t + 2 \).

Case 3. \( n \equiv 2 \) (mod 6).

Let \( n = 6t + 2 \) where \( t \geq 0 \). As in case 2, assign the labels to \( u_1, v_1 (1 \leq i \leq 6t) \). Then assign 1, 3 respectively to the vertices \( u_{6t+2}, v_{6t+2} \). Now relabel the vertices \( u_{3t+1}, u_{4t+1}, v_{3t+1} \) by 3, 2, 2 respectively. Note that \( v_f(1) = v_f(2) = v_f(3) = 4t + 2, e_f(0) = 9t + 4, \) and \( e_f(1) = 9t + 3 \).

Case 4. \( n \equiv 3 \) (mod 6).

Let \( n = 6t + 3 \) where \( t \geq 0 \). As in case 3, assign the labels to \( u_1, v_1 (1 \leq i \leq 6t) \). Then assign 1, 3 to the vertices \( u_{6t+3}, v_{6t+3} \) respectively. Now relabel the vertices \( u_{3t+1}, u_{4t+1}, v_{3t+1} \) by 3, 2, 1 respectively. Clearly \( v_f(1) = 4t + 2, v_f(2) = v_f(3) = 4t + 3, \) and \( e_f(0) = e_f(1) = 9t + 5 \).

Case 5. \( n \equiv 4 \) (mod 6).

Let \( n = 6t + 4 \) where \( t \geq 0 \). Assign the labels to \( u_1, v_1 (1 \leq i \leq 6t) \) as in case 4. Then assign 1, 3 to the vertices \( u_{6t+4}, v_{6t+4} \) respectively. Finally relabel the vertices \( u_{3t+1}, u_{4t+1}, v_{3t+1} \) by 2, 2, 1 respectively. In this case \( v_f(1) = 4t + 4, v_f(2) = v_f(3) = 4t + 3, e_f(0) = 9t + 7, \) and \( e_f(1) = 9t + 6 \).

Case 6. \( n \equiv 5 \) (mod 6).

Let \( n = 6t + 5 \) where \( t \geq 0 \). As in case 5, assign the labels to \( u_1, v_1 (1 \leq i \leq 6t) \). Then assign 1, 3 to the vertices \( u_{6t+5}, v_{6t+5} \) respectively. Now relabel the vertices \( u_{4t+3}, v_{4t+3}, v_{5t+3} \) by 2, 3, 1 respectively. Clearly \( v_f(1) = v_f(2) = v_f(3) = 4t + 4, \) and \( e_f(0) = e_f(1) = 9t + 8 \).

Hence \( B_n \) is 3-divisor cordial, for all values of \( n \).

\[ f(u_i) = \begin{cases} 
3 & \text{if } 1 \leq i \leq 3t \\
2 & \text{if } 3t + 1 \leq i \leq 4t \\
1 & \text{if } 4t + 1 \leq i \leq 6t 
\end{cases} 
\]

\[ f(v_i) = \begin{cases} 
2 & \text{if } 1 \leq i \leq 3t \\
1 & \text{if } 3t + 1 \leq i \leq 5t \\
3 & \text{if } 5t + 1 \leq i \leq 6t 
\end{cases} 
\]

In this case \( v_f(1) = 4t, v_f(2) = v_f(3) = 4t + 1, e_f(0) = 9t + 1, \) and \( e_f(1) = 9t \).

**References**


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