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3-divisor cordiality of some product related graphs

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Abstract Let G be a (p, q) graph and $2 \leq k \leq p$. Let $f : V(G) \to \{1, 2, \dots, k\}$ be a map. For each edge uv, assign the label 1 if either $f(u)$ or $f(v)$ divides the other and 0 otherwise. f is called a k-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ $i, j \in \{1, 2, ..., k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, where $x \in \{1, 2, \dots, k\}$, $e_f(i)$ denote the number of edges labeled with i, $i \in \{0, 1\}$. A graph with a k-divisor cordial labeling is called a k-divisor cordial graph. In this paper, we discuss 3-divisor cordial labeling behavior of ladder, prism and book graphs.

1 Introduction

Graphs considered here are finite, undirected and simple. In 1980, Cahit [1] introduced the cordial labeling of graphs. In [4], Varatharajan, Navanaeethakrishnan, and Nagarajan introduced a notion, called divisor cordial labeling and proved the standard graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial. Sathish Narayanan introduced the notion of k-divisor cordial labeling in [5]. In [6], 3-divisor cordiality of wheel and $\overline{K_n} + 2K_2$ have been studied. In this paper we studied the 3-divisor cordial labeling behavior of ladder, prism and book graphs. Terms and definitions not defined here are used in the sense of Harary [3] and Gallian [2].

2 3-divisor cordial labeling

Definition 2.1. Let G be a (p, q) graph and $2 \leq k \leq p$. Let $f : V(G) \to \{1, 2, \dots, k\}$ be a map. For each edge xy, assign the label 1 if either $f(x)$ or $f(y)$ divides the other and 0 otherwise. f is called a k-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ $i, j \in \{1, 2, ..., k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, where $x \in \{1, 2, \dots, k\}$, $e_f(i)$ denote the number of edges labeled with i, $i \in \{0, 1\}$. A graph with a k-divisor cordial labeling is called a k-divisor cordial graph.

Definition 2.2. The *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \square G_2$ with the vertex set $V_1 \square V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$.

First we consider the graph ladder $L_n = P_n \square P_2$. Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. Note that L_n consists of $2n$ vertices and $3n - 2$ edges.

Theorem 2.3. The ladder $L_n = P_n \square P_2$ is 3-divisor cordial.

Proof. The proof is divided into twelve cases.
Case 1. $n \equiv 0 \pmod{12}$.

Let $n = 12t$ where $t \geq 1$. Here we define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$\begin{array}{lllll} f(u_{2i-1}) & = & f(v_{2i}) & = & 2, \quad 1 \leq i \leq 3t \\ f(u_{2i}) & = & f(v_{2i-1}) & = & 3, \quad 1 \leq i \leq 3t \\ f(u_{6t+i}) & = & f(v_{6t+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+i}) & = & f(v_{7i+i}) & = & 1, \quad 1 \leq i \leq 4t \\ f(u_{11t+i}) & = & f(v_{11t+i}) & = & 3, \quad 1 \leq i \leq t \end{array}$$

Case 2. $n \equiv 1 \pmod{12}$.
Let $n = 12t + 1$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$\begin{array}{lllll} f(u_{2i-1}) & = & 2, & & 1 \leq i \leq 3t+1 \\ f(u_{2i}) & = & 3, & & 1 \leq i \leq 3t \\ f(v_{2i-1}) & = & 3, & & 1 \leq i \leq 3t+1 \\ f(v_{2i}) & = & 2, & & 1 \leq i \leq 3t \\ f(u_{6t+2+i}) & = & f(v_{6t+2+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+3+i}) & = & f(v_{7t+3+i}) & = & 3, \quad 1 \leq i \leq t \\ f(u_{8t+3+i}) & = & f(v_{8t+3+i}) & = & 1, \quad 1 \leq i \leq 4t-2 \end{array}$$

and $f(u_{6t+2}) = f(v_{6t+2}) = f(u_{7t+3}) = f(v_{7t+3}) = 1$.
Case 3. $n \equiv 2 \pmod{12}$.
For $n = 2$, Figure 1 shows that L_2 is a 3-divisor cordial graph.

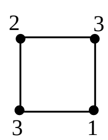

Figure 1.

Let $n = 12t + 2$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows: assign the labels to the vertices u_i, v_i $(1 \leq i \leq 6t + 1)$ as in case 2.

$$\begin{array}{lllll} f(u_{6t+1+i}) & = & f(v_{6t+1+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+1+i}) & = & f(v_{7t+1+i}) & = & 1, \quad 1 \leq i \leq 4t+1 \\ f(u_{11t+2+i}) & = & f(v_{11t+2+i}) & = & 3, \quad 1 \leq i \leq t. \end{array}$$

Case 4. $n \equiv 3 \pmod{12}$.
The vertex labelings given in Figure 2 establish that L_3 and L_{15} are 3-divisor cordial graphs.

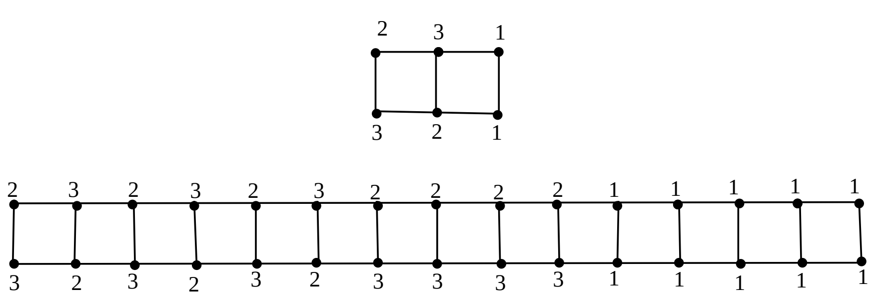

Figure 2.

Let $n = 12t + 3$ where $t \geq 2$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows: assign the labels to the vertices u_i, v_i $(1 \leq i \leq 6t+1)$ as in case 2.

$$\begin{array}{lllll} f(u_{6t+1+i}) & = & 2, & & 1 \leq i \leq 3 \\ f(v_{6t+1+i}) & = & 3, & & 1 \leq i \leq 3 \\ f(u_{6t+5+i}) & = & f(v_{6t+5+i}) & = & 2, \quad 1 \leq i \leq t-1 \\ f(u_{7t+5+i}) & = & f(v_{7t+5+i}) & = & 3, \quad 1 \leq i \leq t-1 \\ f(u_{8t+4+i}) & = & f(v_{8t+4+i}) & = & 1, \quad 1 \leq i \leq 4t-1 \end{array}$$

and $f(u_{6t+5}) = f(v_{6t+5}) = f(u_{7t+5}) = f(v_{7t+5}) = 1$.
Case 5. $n \equiv 4 \pmod{12}$.
Figure 3 shows that L_4 is a 3-divisor cordial graph.

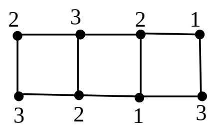

Figure 3.

Let $n = 12t+4$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1,2,3\}$ as follows: assign the labels to the vertices u_i, v_i $(1 \leq i \leq 6t+1)$ as in case 2. Then put the number 3 to the vertex u_{6t+2} and 2 to v_{6t+2}. Then

$$\begin{array}{lllll} f(u_{6t+2+i}) & = & f(v_{6t+2+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+3+i}) & = & f(v_{7t+3+i}) & = & 1, \quad 1 \leq i \leq 4t \\ f(u_{11t+4+i}) & = & f(v_{11t+4+i}) & = & 3, \quad 1 \leq i \leq t \end{array}$$

$f(v_{7t+3}) = f(v_{11t+4}) = 1$, $f(u_{7t+3}) = 2$ and $f(u_{11t+4}) = 3$.
Case 6. $n \equiv 5 \pmod{12}$.
From Figure 4, we observe that L_5 is 3-divisor cordial.

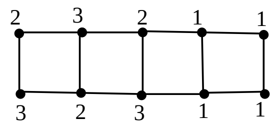

Figure 4.

Let $n = 12t+5$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1,2,3\}$ as follows:

$$\begin{array}{lllll} f(u_{2i-1}) & = & 2, & & 1 \leq i \leq 3t+2 \\ f(u_{2i}) & = & 3, & & 1 \leq i \leq 3t+1 \\ f(v_{2i-1}) & = & 3, & & 1 \leq i \leq 3t+2 \\ f(v_{2i}) & = & 2, & & 1 \leq i \leq 3t+1 \\ f(u_{6t+4+i}) & = & f(v_{6t+4+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+5+i}) & = & f(v_{7t+5+i}) & = & 3, \quad 1 \leq i \leq t \\ f(u_{8t+5+i}) & = & f(v_{8t+5+i}) & = & 1, \quad 1 \leq i \leq 4t \end{array}$$

and $f(u_{6t+4}) = f(v_{6t+4}) = f(u_{7t+5}) = f(v_{7t+5}) = 1$.
Case 7. $n \equiv 6 \pmod{12}$.
For L_6, Figure 5 establish that L_6 is 3-divisor cordial.
Let $n = 12t+6$ where $t \geq 1$. Consider $f : V(L_n) \to \{1,2,3\}$ as follows: assign the labels to the vertices u_i, v_i $(1 \leq i \leq 6t+3)$ as in case 6. Then

$$\begin{array}{lllll} f(u_{6t+3+i}) & = & f(v_{6t+3+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+4+i}) & = & f(v_{7t+4+i}) & = & 1, \quad 1 \leq i \leq 4t+1 \\ f(u_{11t+6+i}) & = & f(v_{11t+6+i}) & = & 3, \quad 1 \leq i \leq t \end{array}$$

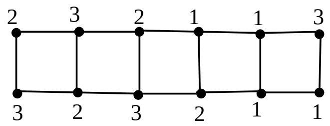

Figure 5.

$f(v_{7t+4}) = f(v_{11t+6}) = 1$, $f(u_{7t+4}) = 2$ and $f(u_{11t+6}) = 3$.
Case 8. $n \equiv 7 \pmod{12}$.
Figure 6 shows that L_7 is a 3-divisor cordial graph.

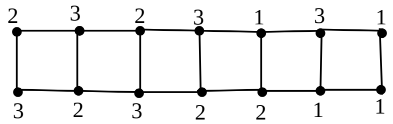

Figure 6.

Let $n = 12t + 7$ where $t \geq 1$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$\begin{array}{lllll} f(u_{2i-1}) & = & f(v_{2i}) & = & 2, \quad 1 \leq i \leq 3t+2 \\ f(u_{2i}) & = & f(v_{2i-1}) & = & 3, \quad 1 \leq i \leq 3t+2 \\ f(u_{6t+5+i}) & = & f(v_{6t+5+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+7+i}) & = & f(v_{7t+7+i}) & = & 3, \quad 1 \leq i \leq t \\ f(u_{8t+8+i}) & = & f(v_{8t+8+i}) & = & 1, \quad 1 \leq i \leq 4t-1 \end{array}$$

$f(v_{7t+6}) = f(u_{7t+7}) = f(v_{7t+7}) = f(v_{8t+8}) = 1$, $f(u_{7t+6}) = 2$ and $f(u_{8t+8}) = 3$.
Case 9. $n \equiv 8 \pmod{12}$.
Let $n = 12t + 8$ where $t \geq 0$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$\begin{array}{lllll} f(u_{2i-1}) & = & f(v_{2i}) & = & 2, \quad 1 \leq i \leq 3t+2 \\ f(u_{2i}) & = & f(v_{2i-1}) & = & 3, \quad 1 \leq i \leq 3t+2 \\ f(u_{6t+4+i}) & = & f(v_{6t+4+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+5+i}) & = & f(v_{7t+5+i}) & = & 1, \quad 1 \leq i \leq 4t+2 \\ f(u_{11t+8+i}) & = & f(v_{11t+8+i}) & = & 3, \quad 1 \leq i \leq t \end{array}$$

$f(v_{7t+5}) = f(v_{11t+8}) = 1$, $f(u_{7t+5}) = 2$ and $f(u_{11t+8}) = 3$.
Case 10. $n \equiv 9 \pmod{12}$.
Let $n = 12t + 9$ where $t \geq 0$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$\begin{array}{lllll} f(u_{2i-1}) & = & 2, & & 1 \leq i \leq 3t+3 \\ f(u_{2i}) & = & 3, & & 1 \leq i \leq 3t+2 \\ f(v_{2i-1}) & = & 3, & & 1 \leq i \leq 3t+3 \\ f(v_{2i}) & = & 2, & & 1 \leq i \leq 3t+2 \\ f(u_{6t+6+i}) & = & f(v_{6t+6+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+8+i}) & = & f(v_{7t+8+i}) & = & 3, \quad 1 \leq i \leq t \\ f(u_{8t+9+i}) & = & f(v_{8t+9+i}) & = & 1, \quad 1 \leq i \leq 4t \end{array}$$

$f(u_{6t+6}) = f(v_{6t+6}) = f(v_{7t+7}) = f(u_{7t+8}) = f(v_{7t+8}) = f(v_{8t+9}) = 1$, $f(u_{7t+7}) = 2$ and $f(u_{8t+9}) = 3$.
Case 11. $n \equiv 10 \pmod{12}$.
Let $n = 12t + 10$ where $t \geq 0$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows: assign the labels to the vertices u_i, v_i $(1 \leq i \leq 6t+5)$ as in case 10. Then

$$\begin{array}{lllll} f(u_{6t+5+i}) & = & f(v_{6t+5+i}) & = & 2, \quad 1 \leq i \leq t+1 \\ f(u_{7t+6+i}) & = & f(v_{7t+6+i}) & = & 1, \quad 1 \leq i \leq 4t+3 \\ f(u_{11t+9+i}) & = & f(v_{11t+9+i}) & = & 3, \quad 1 \leq i \leq t+1 \end{array}$$

Case 12. $n \equiv 11 \pmod{12}$.
Let $n = 12t + 11$ where $t \geq 0$. We define a map $f : V(L_n) \to \{1, 2, 3\}$ as follows:

$$\begin{array}{lllll} f(u_{2i-1}) & = & f(v_{2i}) & = & 2, \quad 1 \leq i \leq 3t+3 \\ f(u_{2i}) & = & f(v_{2i-1}) & = & 3, \quad 1 \leq i \leq 3t+3 \\ f(u_{6t+7+i}) & = & f(v_{6t+7+i}) & = & 2, \quad 1 \leq i \leq t \\ f(u_{7t+9+i}) & = & f(v_{7t+9+i}) & = & 3, \quad 1 \leq i \leq t \\ f(u_{8t+10+i}) & = & f(v_{8t+10+i}) & = & 1, \quad 1 \leq i \leq 4t+1 \end{array}$$

$f(u_{6t+7}) = f(v_{6t+7}) = f(v_{7t+8}) = f(u_{7t+9}) = f(v_{7t+9}) = f(v_{8t+10}) = 1$, $f(u_{7t+8}) = 2$ and $f(u_{8t+10}) = 3$.
The vertex and edge conditions are given in Table 1 and Table 2.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$
$n \equiv 1 \pmod 3$	$\frac{2n-2}{3}$	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$
$n \equiv 2 \pmod 3$	$\frac{2n+2}{3}$	$\frac{2n-1}{3}$	$\frac{2n-1}{3}$

Table 1.

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$
$n \equiv 1 \pmod 2$	$\frac{3n-1}{2}$	$\frac{3n-3}{2}$

Table 2.

Hence L_n is 3-divisor cordial. □

Next we consider prism graph $C_n \square P_2$. Let $V(C_n \square P_2) = V(L_n)$ and $E(C_n \square P_2) = E(L_n) \cup \{u_n u_1, v_n v_1\}$.

Theorem 2.4. Prism $C_n \square P_2$ is 3-divisor cordial.

Proof. Assign the labels to the vertices as in Theorem 2.3. The vertex condition given in Table 1 and the edge condition given in Table 3 shows that $C_n \square P_2$ is 3-divisor cordial.

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 2$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 2$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 3.

This completes the proof. □

Next we discuss 3-divisor cordial labeling behavior of book graph. A book graph is a cartesian product of star graph with K_2 and we denote it by $B_n = K_{1,n} \square K_2$. Let $V(B_n) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_n) = \{uv, uu_i, vv_i : 1 \leq i \leq n\}$. Note that B_n consists of $2n + 2$ vertices and $3n + 1$ edges.

Theorem 2.5. Book $B_n = K_{1,n} \square K_2$ is 3-divisor cordial, for all n.

Proof. We construct a vertex labeling f from the set of vertices of B_n to the set $\{1, 2, 3\}$ by $f(u) = 2$, $f(v) = 3$ and we consider the following cases for the labeling of other vertices.
Case 1. $n \equiv 0 \pmod 6$.
Let $n = 6t$ where $t \geq 1$. Here

$$f(u_i) = \begin{cases} 3 & \text{if } 1 \leq i \leq 3t \\ 2 & \text{if } 3t+1 \leq i \leq 4t \\ 1 & \text{if } 4t+1 \leq i \leq 6t \end{cases} \qquad f(v_i) = \begin{cases} 2 & \text{if } 1 \leq i \leq 3t \\ 1 & \text{if } 3t+1 \leq i \leq 5t \\ 3 & \text{if } 5t+1 \leq i \leq 6t \end{cases}$$

In this case $v_f(1) = 4t$, $v_f(2) = v_f(3) = 4t+1$, $e_f(0) = 9t+1$, and $e_f(1) = 9t$.

Case 2. $n \equiv 1 \pmod 6$.

Since $B_1 \cong L_2$, Figure 1 shows that B_1 is 3-divisor cordial. Let $n = 6t+1$ where $t > 0$. Assign the labels to the vertices of u_i, v_i $(1 \leq i \leq 6t)$ as in Case 1. Then put the labels 1, 3 respectively to the vertices u_{6t+1}, v_{6t+1}. Now relabel the vertices v_{3t+1}, v_{5t+1}, v_{5t+2} by 3, 1, 1 respectively. Here $v_f(1) = 4t+2$, $v_f(2) = v_f(3) = 4t+1$, and $e_f(0) = e_f(1) = 9t+2$.

Case 3. $n \equiv 2 \pmod 6$.

Let $n = 6t+2$ where $t \geq 0$. As in case 2, assign the labels to u_i, v_i $(1 \leq i \leq 6t)$. Then assign 1, 3 respectively to the vertices u_{6t+2}, v_{6t+2}. Now relabel the vertices u_{3t+1}, u_{4t+1}, v_{3t+1} by 3, 2, 2 respectively. Note that $v_f(1) = v_f(2) = v_f(3) = 4t+2$, $e_f(0) = 9t+4$, and $e_f(1) = 9t+3$.

Case 4. $n \equiv 3 \pmod 6$.

Let $n = 6t+3$ where $t \geq 0$. As in case 3, assign the labels to u_i, v_i $(1 \leq i \leq 6t)$. Then assign 1, 3 to the vertices u_{6t+3}, v_{6t+3} respectively. Now relabel the vertices u_{4t+2}, v_{3t+2}, v_{5t+3} by 2, 3, 1 respectively. Clearly $v_f(1) = 4t+2$, $v_f(2) = v_f(3) = 4t+3$, and $e_f(0) = e_f(1) = 9t+5$.

Case 5. $n \equiv 4 \pmod 6$.

Let $n = 6t+4$ where $t \geq 0$. Assign the labels to u_i, v_i $(1 \leq i \leq 6t)$ as in case 4. Then assign 1, 3 to the vertices u_{6t+4}, v_{6t+4} respectively. Finally relabel the vertices u_{3t+2}, v_{3t+2}, v_{5t+4} by 2, 2, 1 respectively. In this case $v_f(1) = 4t+4$, $v_f(2) = v_f(3) = 4t+3$, $e_f(0) = 9t+7$, and $e_f(1) = 9t+6$.

Case 6. $n \equiv 5 \pmod 6$.

Let $n = 6t+5$ where $t \geq 0$. As in case 5, assign the labels to u_i, v_i $(1 \leq i \leq 6t)$. Then assign 1, 3 to the vertices u_{6t+5}, v_{6t+5} respectively. Now relabel the vertices u_{4t+3}, v_{3t+3}, v_{5t+5} by 2, 3, 1 respectively. Clearly $v_f(1) = v_f(2) = v_f(3) = 4t+4$, and $e_f(0) = e_f(1) = 9t+8$.

Hence B_n is 3-divisor cordial, for all values of n. □

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