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ENERGY OF A MAXIMAL GRAPH

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Abstract In this paper we design a MATLAB program to obtain maximal graphs associated to rings $\mathbb{Z}_n$ and compute their energy. This program expresses an intrinsic relationship between the elements of the ring $\mathbb{Z}_n$ and its structural properties of graphs. In addition, the maximal graph gives a class of graphs whose line graph is hyperenergetic.

1 Introduction

All rings considered below are commutative and unital. For any ring R, R' denote the set of all nonunits of R. Maximal graph associated to a ring R was introduced in [4], and is defined as the simple graph with vertices the elements of R, and two distinct vertices x, y are adjacent if and only if there is a maximal ideal of R containing both x and y. It is denoted by $G(R)$. The restriction of maximal graph to nonunit elements of R is considered in [5], and is denoted by $\Gamma(R)$. Since unit elements of R are just isolated vertices in $G(R)$, the authors continued to call $\Gamma(R)$ also the maximal graph associated to R.

In [10], we study the structure of maximal graph theoretically. Note that sketching the maximal graphs of high order is not always easy as it may be the union of many complete graphs such that intersection of any two of these has at least two vertices. Thus, in this paper, we design a MATLAB program to obtain maximal graph associated to ring $\mathbb{Z}_n$. Also we discuss the concept of energy of a graph which was first introduced by Gutman in [6].

For a simple graph G with n vertices, the adjacency matrix is a $n \times n$ matrix defined as $A = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, A is real, symmetric matrix. Since diagonal entries of A are zero and hence the sum of all the eigenvalues is equal to zero. The eigenvalues of a graph G is defined as the eigenvalues of the adjacency matrix associated with G. The spectrum of a graph G is the set of eigenvalues of G together with their multiplicity. It is denoted as Spec(G). Let $\lambda_1,\ \lambda_2, \dots,\ \lambda_k$ be distinct eigenvalues of a graph G having multiplicities $m_1,\ m_2, \dots,\ m_k$. Then the spectrum of the graph G is written as

$$\text{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \\ m_1 & m_2 & \dots & m_k \end{pmatrix}$$

By order of a graph G we mean the number of vertices in G; and by $m(G)$ we mean the number of edges in G. Note that the order of $\Gamma(\mathbb{Z}_n)$ is $n - \phi(n)$, where ϕ is Euler's phi function. The energy of a graph G of order n is denoted by E_G and is defined as the sum of the absolute values of the eigenvalues of G, that is,

$$E_G = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_i' s$ are the eigenvalues of G. A graph G of order n having energy $E_G > 2(n-1)$ is called hyperenergetic and graphs having energy $E_G \leq 2(n-1)$ is called non-hyperenergetic.

In Section 2, we prove that for any finite ring R with k maximal ideals and $|R'| \geq 4$, $\Gamma(R)$ has an eigenvalue -1 of multiplicity $n - (2^k - 1)$, where n is the order of $\Gamma(R)$. In addition, we prove that for any finite ring R with two maximal ideal and $|R'| \geq 4$, $\Gamma(R)$ has four distinct eigenvalues. We are used a MATLAB program to draw the maximal graph corresponding to ring $\mathbb{Z}_n$ and to compute the energy of $\Gamma(\mathbb{Z}_n)$. In Section 3, we prove that $L(\Gamma(R))$ is hyperenergetic except for few listed rings.

2 Energy of Maximal Graph

Recall that the energy of a graph G is the sum of absolute values of all the eigenvalues of G. For the maximal graph $G(R)$ associated to a ring R, it can be easily verified that the energy of $G(R)$ and the energy of $\Gamma(R)$ are same as the isolated vertices do not affect the energy of a graph. Since the adjacency matrix of any undirected, simple graph is real symmetric, the eigenvalues of $\Gamma(R)$ are real. Thus, $E_{\Gamma(R)}$ is a real number. However, for any local ring R the energy of $\Gamma(R)$ is an integer. This we show in the first result of this section.

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a finite local ring. Then the energy of $\Gamma(R)$ is an integer.*

Proof. Since any finite local ring have p^n elements for some prime p and $n \in \mathbb{N}$, we may assume that $|R| = p^m$ for some prime p and $m \in \mathbb{N}$. Thus, for some $k \in \mathbb{N}$ we have $|\mathfrak{m}| = p^k$ as $|\mathfrak{m}|$ divides $|R|$. Clearly, $\Gamma(R)$ is a complete graph of order p^k and hence by [1, Section 11.3], the eigenvalues of $\Gamma(R)$ are $p^k - 1$ and -1 of multiplicity 1 and $p^k - 1$, respectively. Therefore, energy of $\Gamma(R)$ is

$$E_{\Gamma(R)} = |p^k - 1| + \underbrace{|-1| + |-1| + \cdots + |-1|}_{(p^k-1)\ times} = 2(p^k - 1).$$

□

As a companion to Theorem 2.1 we have the following corollary.

Corollary 2.2. *For $n = p^k$, where p is a prime and $k \geq 1$, the energy of $\Gamma(\mathbb{Z}_n)$ is an integer.*

Proof. Since $\mathbb{Z}_n$ is a local ring with maximal ideal of cardinality p^{k-1}, the result follows by Theorem 2.1. □

Spectral graph theory drew lot of attention because of its wide application in the fields of chemistry, biology, and graph coloring. Thus, spectrum of a graph are of particular interest. Note that if R is a ring with n nonunits and k maximal ideals, then $n \geq 2^k - 1$ as the smallest ring with k maximal ideals is $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{k\ times}$. In the next theorem, we discuss the multiplicity of eigenvalue -1 of $\Gamma(R)$ for any non-local ring R.

Theorem 2.3. *Let R be a finite ring with k maximal ideals and n nonunits. If $n > 2^k - 1$, then $\Gamma(R)$ has an eigenvalue -1 of multiplicity $n - (2^k - 1)$; otherwise -1 is not the eigenvalue of $\Gamma(R)$.*

Proof. Since R is a finite ring with k maximal ideals, say, $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_k$. Then by [9, Remark 2.9], $R \cong R_1 \times R_2 \times \cdots \times R_k$ such that

$$|\mathfrak{m}_i| = p_i^{(m_i-1)\alpha_1} \prod_{\substack{j=1 \\ i \neq j}}^{k} p_j^{m_j \alpha_j},\ |\mathfrak{m}_i \cap \mathfrak{m}_j| = p_i^{(m_i-1)\alpha_i} p_j^{(m_j-1)\alpha_j} \prod_{\substack{l=1 \\ i,j \neq l}}^{k} p_l^{m_l \alpha_l}, \ldots$$

$$|\cap_{i=1}^{k} \mathfrak{m}_i| = \prod_{i=1}^{k} p_i^{(m_i-1)\alpha_i},$$

where m_i is the length of R_i as a module over itself and $|R_i| = p_i^{m_i \alpha_i}$ for all i. Thus, by [10, Theorem 3.1], $\Gamma(R) \cong K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$ where $n_i = p_i^{(m_i-1)\alpha_i} \prod_{\substack{j=1 \\ i \neq j}}^{k} p_j^{m_j \alpha_j}$. Let A be the adjacency matrix of $\Gamma(R)$. Then A is a $n \times n$ matrix, where n is the order of $\Gamma(R)$. Since $\Gamma(R)$

is a simple graph and $\Gamma(R) \cong K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$, we have the rows in $A+I$ corresponding to the vertices of $K_{n_i} \setminus \cup_{\substack{j=1 \\ i \neq j}}^{k} K_{n_j}$, $(K_{n_i} \cap K_{n_j}) \setminus \cup_{\substack{l=1 \\ i,j \neq l}}^{k} K_{n_l}, \ldots, \cap_{i=1}^{k} K_{n_i}$, respectively, are same. Thus,

$$A + I \sim \begin{pmatrix} A_1 & O \\ O & O \end{pmatrix}, \text{ where } A_1 \text{ is a non-singular } 2^k - 1 \times 2^k - 1 \text{ matrix.}$$

From this, we conclude that $\text{rank}(A+I) = 2^k - 1$. If $n > 2^k - 1$, then -1 is the eigenvalue of $\Gamma(R)$ of multiplicity $n - (2^k - 1)$. If $n = 2^k - 1$, then $\text{rank}(A+I) = n$, and hence -1 is not an eigenvalue of $\Gamma(R)$. □

Theorem 2.4. *Let R be a finite ring with two maximal ideals and $|R'| \geq 4$. Then $\Gamma(R)$ has four distinct eigenvalues with -1 an eigenvalue of multiplicity $n-3$, where n is the order of $\Gamma(R)$.*

Proof. By Theorem 2.3, -1 is the eigenvalue of $\Gamma(R)$ of multiplicity $n-3$. Thus, $\Gamma(R)$ has at most four distinct eigenvalues. Also, by [2, Proposition 1.3.3], $\Gamma(R)$ has at least 3 distinct eigenvalues. Suppose $\Gamma(R)$ has exactly 3 distinct eigenvalues, say, $-1, \lambda_1, \lambda_2$. Let λ_1 be the largest among them. Then by [2, Theorem 2.2.1], the multiplicity of λ_1 is one. This implies that multiplicity of λ_2 is 2. As the sum of all eigenvalues of $\Gamma(R)$ is equal to trace of A, we have

$$-1(n-3) + \lambda_1 + 2\lambda_2 = 0$$
$$\lambda_1 + 2\lambda_2 = n - 3$$

Now by [1, Corollary 11.5.2], $\lambda_1 < n-1$. This implies that $\lambda_2 > -1$. Thus, all three eigenvalues of $\Gamma(R)$ are greater than -2, which is a contradiction by [11, Proposition 5]. This shows that $\Gamma(R)$ has four distinct eigenvalues. □

As an immediate consequence of Theorem 2.4 we have the following corollary.

Corollary 2.5. *Let $n = p^r q^s$, where p, q are distinct primes and $r, s \geq 1$. Then $\Gamma(\mathbb{Z}_n)$ has an eigenvalue -1 of multiplicity $m-3$, where m is the order of $\Gamma(\mathbb{Z}_n)$. In particular, $\Gamma(\mathbb{Z}_n)$ has four distinct eigenvalues.*

Proof. As $\mathbb{Z}_{p^r q^s}$ is a ring with two maximal ideals of cardinality $p^{r-1}q^s$ and $p^r q^{s-1}$, the result follows by Theorem 2.4. □

Note that it may not be easy to find out the energy of a maximal graph $\Gamma(R)$ associated to any ring R. Naturally one may think of a computer program to compute the same. Thus, we now used a MATLAB program to draw a maximal graph and find out the energy of maximal graph associated to the ring $\mathbb{Z}_n$.
The maximal graphs of $\mathbb{Z}_{20}$ and $\mathbb{Z}_{60}$ obtained from the MATLAB program are shown in Figures 1 and 2.

Remark 2.6. From the definition of maximal graph we can easily observe that the maximal graph $\Gamma(\mathbb{Z}_n), n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, is same as the maximal graph $\Gamma(R)$ for all the rings R with exactly k maximal ideals, say $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k$, such that $|\mathfrak{m}_i| = p_1^{\alpha_1} \cdots p_i^{\alpha_i - 1} \cdots p_k^{\alpha_k}$ for all i.

By using the MATLAB programs, we computed eigenvalues and energy of $\Gamma(\mathbb{Z}_n)$ for various values of n and observe the following:

(i) Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $p_1,\ p_2$ are distinct primes, $\alpha_1,\ \alpha_2$ are positive integers and at least one α_i is greater than one. Then $\Gamma(\mathbb{Z}_n)$ is hyperenergetic except $n = 12$.

(ii) For $n = pq$, where p and q are distinct primes, $p < q$, the value of largest eigenvalue, say λ, of $\Gamma(\mathbb{Z}_{pq})$ lies between $q-1$ and q, that is, $q - 1 < \lambda < q$. Also, we observe that the spectrum of $\Gamma(\mathbb{Z}_{pq})$ is the following:

$$\text{Spec}(\Gamma(\mathbb{Z}_{pq})) = \begin{pmatrix} -1-a & -1 & p-2+b & q-1+a-b \\ 1 & p+q-4 & 1 & 1 \end{pmatrix}$$

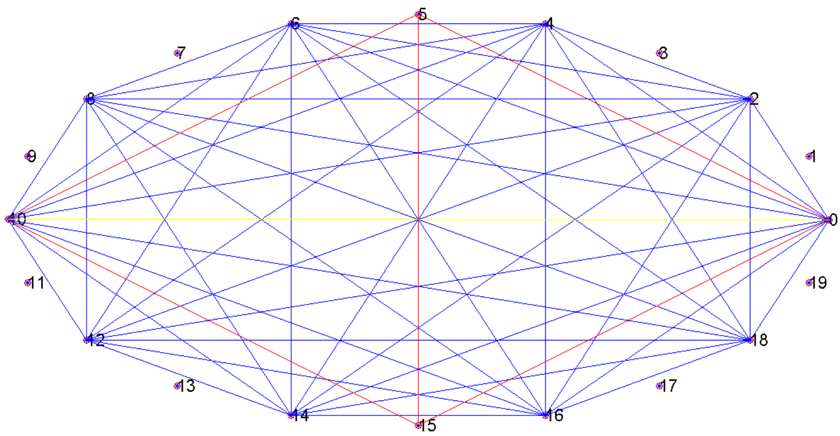

Figure 1. The maximal graph $G(\mathbb{Z}_{20})$

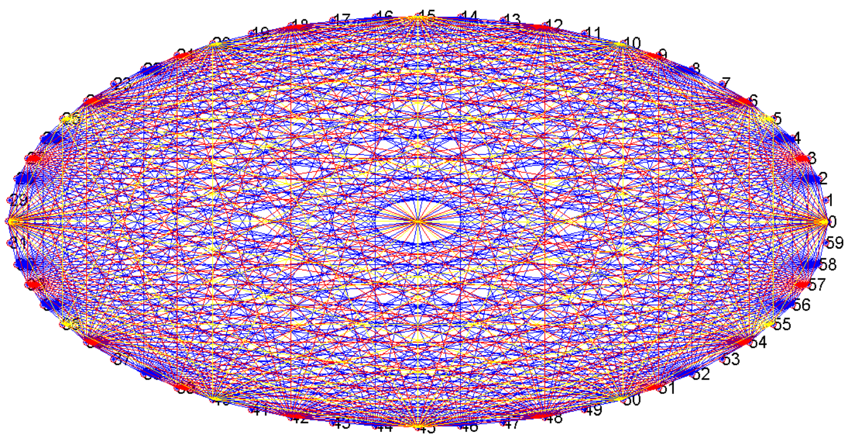

Figure 2. The maximal graph $G(\mathbb{Z}_{60})$

where $0 < a, b < 1$, and the maximal graph $\Gamma(\mathbb{Z}_n)$ is non-hyperenergetic. In particular, $E_{\Gamma(\mathbb{Z}_n)} < 2(p+q-2)$.

(iii) For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, where p_1, p_2, p_3 are distinct primes and $\alpha_1, \alpha_2, \alpha_3$ are positive integers, the maximal graph $\Gamma(\mathbb{Z}_n)$ has an eigenvalue -1 of multiplicity $m-7$, where m is the order of $\Gamma(\mathbb{Z}_n)$. Also, $\Gamma(\mathbb{Z}_n)$ is hyperenergetic graph having 8 distinct eigenvalues.

(iv) For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$, where p_1, p_2, p_3, p_4 are distinct primes, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are positive integers, the maximal graph $\Gamma(\mathbb{Z}_n)$ has an eigenvalue -1 of multiplicity $m-15$, where m is the order of $\Gamma(\mathbb{Z}_n)$. Also, $\Gamma(\mathbb{Z}_n)$ is hyperenergetic having 16 distinct eigenvalues.

(v) For $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} p_5^{\alpha_5}$, where p_1, p_2, p_3, p_4, p_5 are distinct primes and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are positive integers, the maximal graph $\Gamma(\mathbb{Z}_n)$ has an eigenvalue -1 of multiplicity $m-31$, where m is the order of $\Gamma(\mathbb{Z}_n)$. Also, $\Gamma(\mathbb{Z}_n)$ is hyperenergetic having 32 distinct eigenvalues.

Note that in all the cases listed above $\Gamma(\mathbb{Z}_n)$ has 2^k distinct eigenvalues with -1 an eigenvalue of multiplicity $m-(2^k-1)$, where m is the order of $\Gamma(\mathbb{Z}_n)$ and k is the number of distinct primes in the factorization of n. Because of the limited computational efficiency of the computer we could not verify the above observations for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for $k \geq 6$, where $p_1, p_2, \ldots, p_k$

are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers. In particular, we could not find any number for which $\Gamma(\mathbb{Z}_n)$ do not satisfy above points. This forces us to list the following conjecture.

Conjecture 2.7. (i) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots \alpha_k$ are positive integers. Then $\Gamma(\mathbb{Z}_n)$ has 2^k distinct eigenvalues with -1 an eigenvalue of multiplicity $m - (2^k - 1)$, where m is the order of $\Gamma(\mathbb{Z}_n)$.

(ii) For $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are distinct primes, $\alpha_1, \alpha_2, \dots \alpha_k$ are positive integers, and $k \geq 3$. Then $\Gamma(\mathbb{Z}_n)$ is hyperenergetic.

3 Energy of $L(\Gamma(R))$

Recall from Introduction that a graph G of order n having energy $E_G > 2(n-1)$ is called hyperenergetic, and graphs having energy $E_G \leq 2(n-1)$ is called non-hyperenergetic. In this section, we list the rings R such that $L(\Gamma(R))$ is hyperenergetic or non-hypenergetic.

Theorem 3.1. *Let R be a finite ring with $|R'| \geq 5$. Then $L(\Gamma(R))$ is hyperenergetic unless R is isomorphic to one of the following rings:*

$$\mathbb{F}_2 \times \mathbb{Z}_4,\ \mathbb{F}_2 \times \mathbb{F}_2[x]/(x^2),\ \mathbb{F}_2 \times \mathbb{F}_4,\ \mathbb{F}_3 \times \mathbb{F}_3,\ \mathbb{F}_3 \times \mathbb{F}_4,\ \mathbb{F}_4 \times \mathbb{F}_4.$$

Proof. Let R be a finite ring with n maximal ideals and $|R'| \geq 5$. To show that $L(\Gamma(R))$ is hyperenergetic, by [8, Theorem 1], it is enough to show that

$$m(\Gamma(R)) \geq 2V(\Gamma(R)) = 2|R'|.$$

Since R is a finite ring, we have by [9, Remark 2.9(i)],

$$|R'| = |J(R)| \left\{ \prod_{i=1}^{n} p_i^{\alpha_i} - \prod_{i=1}^{n} (p_i^{\alpha_i} - 1) \right\}$$

Also by [10, Theorem 4.2] and [9, Remark 2.9(i)], we have

$$m(\Gamma(R)) = \frac{1}{2} \left[|J(R)|^2 \left\{ \prod_{i=1}^{n} p_i^{2\alpha_i} - \prod_{i=1}^{n} (p_i^{2\alpha_i} - 1) \right\} - |J(R)| \left\{ \prod_{i=1}^{n} p_i^{\alpha_i} - \prod_{i=1}^{n} (p_i^{\alpha_i} - 1) \right\} \right]$$

If possible, suppose that $m(\Gamma(R)) < 2|R'|$. Then

$$|J(R)| \left\{ \prod_{i=1}^{n} p_i^{2\alpha_i} - \prod_{i=1}^{n} (p_i^{2\alpha_i} - 1) \right\} < 5 \left\{ \prod_{i=1}^{n} p_i^{\alpha_i} - \prod_{i=1}^{n} (p_i^{\alpha_i} - 1) \right\} \tag{3.1}$$

Note that even for $|J(R)| = 1$,

$$\prod_{i=1}^{n} p_i^{2\alpha_i} - \prod_{i=1}^{n} (p_i^{2\alpha_i} - 1) \not< 5 \left\{ \prod_{i=1}^{n} p_i^{\alpha_i} - \prod_{i=1}^{n} (p_i^{\alpha_i} - 1) \right\},$$

for all $n \geq 3$. Thus, for $n \geq 3$, that is, for ring R with at least three maximal ideals, $L(\Gamma(R))$ is hyperenergetic.

Now assume that $n = 1$, that is, R is a local ring. In this case, (3.1) reduces to

$$|J(R)| < 5,$$

which is also a contradiction as $|R'| \geq 5$. Thus, for all local rings R such that $|R'| \geq 5$, $L(\Gamma(R))$ is hyperenergetic.

Now assume that $n = 2$. Then (3.1) reduces to

$$|J(R)|\{p_1^{2\alpha_1} + p_2^{2\alpha_2} - 1\} < 5\{p_1^{\alpha_1} + p_2^{\alpha_2} - 1\} \tag{3.2}$$

Now, we have the following three cases:
Case (i). Let $|J(R)| \geq 3$. Since for all primes p_1, p_2 and $\alpha_1, \alpha_2 \in \mathbb{N}$, we have

$$p_1^{2\alpha_1} + p_2^{2\alpha_2} - 1 \geq 2\{p_1^{\alpha_1} + p_2^{\alpha_2} - 1\}$$

Thus,

$$6\{p_1^{\alpha_1} + p_2^{\alpha_2} - 1\} \leq |J(R)|\{p_1^{2\alpha_1} + p_2^{2\alpha_2} - 1\} < 5\{p_1^{\alpha_1} + p_2^{\alpha_2} - 1\},$$

which is a contradiction. Therefore, for rings R with two maximal ideals such that $|J(R)| \geq 3$, $L(\Gamma(R))$ is hyperenergetic.
Case (ii). Let $|J(R)| = 2$. In this case, (3.2) reduces to

$$p_1^{\alpha_1}(2p_1^{\alpha_1} - 5) + p_2^{\alpha_2}(2p_2^{\alpha_2} - 5) < -3$$

which is true only for $\alpha_1 = 1, \alpha_2 = 1$ and $p_1 = 2, p_2 = 2$, that is, for $R \cong \mathbb{F}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{F}_2 \times \mathbb{F}_2[x]/(x^2)$. Since $\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4) = \Gamma(\mathbb{F}_2 \times \mathbb{F}_2[x]/(x^2))$, it is enough to check whether $L(\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4))$ is hyperenergetic or not. For this, we construct the adjacency matrix A of $L(\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4))$ is given as follows:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Thus, the spectrum of $L(\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4))$ is given as

$$\text{Spec}(L(\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4))) = \begin{pmatrix} -2 & 0 & 2 & 6 \\ 5 & 3 & 2 & 1 \end{pmatrix}$$

and hence $E_{L(\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4))} = 20 = 2 \cdot 11 - 2 = E_{K_{11}}$. Thus, we conclude that $L(\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4))$ is non-hyperenergetic. Therefore, $L(\Gamma(R))$ is hyperenergetic, for all rings with two maximal ideals and $|J(R)| = 2$ except $\mathbb{F}_2 \times \mathbb{Z}_4$ and $\mathbb{F}_2 \times \mathbb{F}_2[x]/(x^2)$ (up to isomorphism).
Case (iii). Let $|J(R)| = 1$. In this case, (3.2) reduces to

$$p_1^{\alpha_1}(p_1^{\alpha_1} - 5) + p_2^{\alpha_2}(p_2^{\alpha_2} - 5) < -4 \tag{3.3}$$

which is satisfied only for the following values of $\alpha_1,\ \alpha_2,\ p_1,\ p_2,$

(a) $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 2,\ p_2 = 2;$

(b) $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 2,\ p_2 = 3;$

(c) $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 2,\ p_2 = 5;$

(d) $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 3,\ p_2 = 3;$

(e) $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 3,\ p_2 = 5;$

(f) $\alpha_1 = 1,\ \alpha_2 = 2,\ p_1 = 2,\ p_2 = 2;$

(g) $\alpha_1 = 1,\ \alpha_2 = 2,\ p_1 = 3,\ p_2 = 2;$

(h) $\alpha_1 = 2,\ \alpha_2 = 2,\ p_1 = 2,\ p_2 = 2;$

Among these (a) and (b) are ruled out as the corresponding rings do not satisfies the condition $|R'| \geq 5$.

Subcase (a). Let $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 2,\ p_2 = 5$. In this case, $R \cong \mathbb{F}_2 \times \mathbb{F}_5$. Now adjacency matrix A of $L(\Gamma(\mathbb{F}_2 \times \mathbb{F}_5))$ is given as follows:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Thus, the spectrum of $L(\Gamma(\mathbb{F}_2 \times \mathbb{F}_5))$ is given as

$$\operatorname{Spec}(L(\Gamma(\mathbb{F}_2 \times \mathbb{F}_5))) = \begin{pmatrix} -2 & -1.2749 & 1 & 2 & 6.2749 \\ 5 & 1 & 3 & 1 & 1 \end{pmatrix}$$

and hence $E_{L(\Gamma(\mathbb{F}_2 \times \mathbb{F}_5))} = 22.5498 > 2 \cdot 11 - 2 = E_{K_{11}}$. Thus, $L(\Gamma(\mathbb{F}_2 \times \mathbb{F}_5))$ is hyperenergetic.

Subcase (b). Let $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 3,\ p_2 = 3$. In this case, $R \cong \mathbb{F}_3 \times \mathbb{F}_3$. Since $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_3))$ is of order 6, we have by [7, Theorem 3.4], $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_3))$ is non-hyperenergetic.

Subcase (c). Let $\alpha_1 = 1,\ \alpha_2 = 1,\ p_1 = 3,\ p_2 = 5$. In this case, $R \cong \mathbb{F}_3 \times \mathbb{F}_5$. Now adjacency matrix A of $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_5))$ is given as follows:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Thus, the spectrum of $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_5))$ is given as

$$\operatorname{Spec}(L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_5))) = \begin{pmatrix} -2 & -1 & 0.2984 & 1 & 3 & 6.7016 \\ 6 & 1 & 1 & 3 & 1 & 1 \end{pmatrix}$$

and hence $E_{L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_5))} = 26 > 2 \cdot 13 - 2 = E_{K_{13}}$. Thus $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_5))$ is hyperenergetic.

Subcase (d). Let $\alpha_1 = 1,\ \alpha_2 = 2,\ p_1 = 2,\ p_2 = 2$. In this case, $R \cong \mathbb{F}_2 \times \mathbb{F}_4$. Since $L(\Gamma(\mathbb{F}_2 \times \mathbb{F}_4))$ is of order 7, we have by [7, Theorem 3.4], $L(\Gamma(\mathbb{F}_2 \times \mathbb{F}_4))$ is non-hyperenergetic.

Subcase (e). Let $\alpha_1 = 1,\ \alpha_2 = 2,\ p_1 = 3,\ p_2 = 2$. In this case, $R \cong \mathbb{F}_3 \times \mathbb{F}_4$. Now adjacency matrix A of $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_4))$ is given as follows:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Thus, the spectrum of $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_4))$ is given as

$$\mathrm{Spec}(L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_4))) = \begin{pmatrix} -2 & -1 & 0 & 2 & 5 \\ 3 & 1 & 3 & 1 & 1 \end{pmatrix}$$

and hence $E_{L(\Gamma(\mathbb{F}_3\times\mathbb{F}_4))} = 14 < 2 \cdot 9 - 2 = E_{K_9}$. Thus, $L(\Gamma(\mathbb{F}_3 \times \mathbb{F}_4))$ is non-hyperenergetic.

Subcase (f). Let $\alpha_1 = 2,\ \alpha_2 = 2,\ p_1 = 2,\ p_2 = 2$. In this case, $R \cong \mathbb{F}_4 \times \mathbb{F}_4$. Now adjacency matrix A of $L(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))$ is given as follows:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Thus, the spectrum of $L(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))$ is given as

$$\mathrm{Spec}(L(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))) = \begin{pmatrix} -2 & 0 & 1 & 3 & 6 \\ 5 & 4 & 1 & 1 & 1 \end{pmatrix}$$

and hence $E_{L(\Gamma(\mathbb{F}_4\times\mathbb{F}_4))} = 20 < 2 \cdot 12 - 2 = E_{K_{12}}$. Thus, $L(\Gamma(\mathbb{F}_4 \times \mathbb{F}_4))$ is non-hyperenergetic. □

Theorem 3.2. *Let R be a finite ring with $|R'| \le 4$. Then $L(\Gamma(R))$ is non-hyperenergetic.*

Proof. Let $|R'| \le 4$. Then $|V(\Gamma(R))| \le 4$. This implies that $|V(L(\Gamma(R)))| \le 6$. Thus, by [7, Theorem 3.4], $L(\Gamma(R))$ is non-hyperenergetic. □

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