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GENERALIZATIONS OF δ-PRIMARY ELEMENTS IN MULTIPLICATIVE LATTICES

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Abstract. We define a weakly δ-primary element in a compactly generated multiplicative lattice L and obtain some properties of these elements. We also introduce an almost δ-primary element which unifies an almost prime element and an almost primary element and obtain its properties. Some characterizations of these elements are proved.

1 Introduction

Zhao [15] studied an expansion of ideals and δ-primary ideals for a commutative ring, where δ is a mapping with some additional properties. Manjarekar and Bingi [9] introduced expansion of element and δ-primary elements in a multiplicative lattice. The study of a δ-zero-divisor, a δ-nilpotent element in a commutative ring is done by Atani et.al. [3].

In this paper we introduce these concepts in a multiplicative lattice. We define a weakly δ-primary element with respect to such an expansion. Also we define a δ-twin-zero and prove some results based on δ-twin-zero. Manjrekar and Bingi [10] introduced almost prime, almost primary, 2-potent prime and 2-potent primary elements in a compactly generated multiplicative lattice. In this paper we introduce an almost δ-primary element which unifies an almost prime element and an almost primary element. A number of results about almost prime elements and almost primary elements are proved in this general framework. Also we define a 2-potent δ-primary element. Also we define a ϕ-δ-primary element and a ω-δ-primary element. we prove some characterization for ϕ-δ-primary element.

Throughout in this paper L denotes a compactly generated multiplicative lattice.

2 Preliminaries

The following definitions are from Callialp et. al. [6].

Definition 2.1. A multiplicative lattice L is a complete lattice with a commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity.

Definition 2.2. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leq \bigvee X$ implies the existence of a finite number of elements $a_1, a_2, \ldots a_n \in X$ such that $a \leq a_1 \vee a_2 \vee \ldots a_n$.
The set of compact elements of L will be denoted by L_*. A multiplicative lattice is said to compactly generated if every element of it is a join of compact elements.

Definition 2.3. An element $a \in L$ is said to be proper if $a < 1$.

Definition 2.4. A proper element $p \in L$ is called a prime element if $ab \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$.

Definition 2.5. The radical of $a \in L$ is defined as, $\sqrt{a} = \vee\{x \in L_* | x^n \leq a \text{ for some, } n \in Z_+\}$.

Definition 2.6. A proper element $p \in L$ is called a primary element if $ab \leq p$ implies $a \leq p$ or $b \leq \sqrt{p}$ where $a, b \in L$.

For $a, b \in L$ we denote $(a : b) = \vee\{x \in L | bx \leq a\}$.

Definition 2.7. An element $a \in L$ is called semi primary if $\sqrt{a}$ is a prime element and is called semi prime if $\sqrt{a} = a$.

Definition 2.8. An element $a \in L$ is called p-primary if a is primary and $\sqrt{a} = p$ is a prime element of L.

Definition 2.9. Let $n \geq 2$. A proper element $p \in L$ is called an n-almost prime element if $xy \leq p$ and $xy \nleq p^n$ implies either $x \leq p$ or $y \leq p$, for $x, y \in L$. If $n = 2$, then p is called an almost prime element of L.

Definition 2.10. [11] A proper element $p \in L$ is called an almost primary element if $xy \leq p$ and $xy \nleq p^2$ implies either $x \leq p$ or $y \leq \sqrt{p}$, for $x, y \in L$.

Definition 2.11. [11] An element $p \in L$ is said to be 2-potent prime if $ab \leq p^2$ implies $a \leq p$ or $b \leq p$.

Definition 2.12. A proper element $m \in L$ is said to be a maximal element if $m \nleq a$ for any other proper element $a \in L$.

Definition 2.13. An element $a \in L$ is said to be nilpotent if $a^n = 0$ for some $n \in N$.

The concept of a 2-absorbing ideal in a commutative ring is due to Badawi [4].
Calliap et. al. [6] introduced an analogue of this concept in a multiplicative lattice as follows.

Definition 2.14. A proper element p of L is called a 2-absorbing element of L if whenever $a, b, c \in L$ and $abc \leq p$ implies $ab \leq p$ or $bc \leq p$ or $ac \leq p$.

Definition 2.15. A proper element p of L is called a 2-absorbing primary element of L if whenever $a, b, c \in L$ and $abc \leq p$ implies $ab \leq p$ or $bc \leq \sqrt{p}$ or $ac \leq \sqrt{p}$.

The following definition and Lemma is from Calliap et. al. [6].

Definition 2.16. A multiplicative lattice is called a C-lattice, if it is generated under joins by a multiplicatively closed subset C of compact elements.

Lemma 2.17. *Let L be a C-lattice. Let $x_1, x_2 \in L$. Suppose $y \in L$ satisfies the following property:
(*) If $p \in L$ is compact with $p \leq y$, then either $p \leq x_1$ or $p \leq x_2$. Then either $y \leq x_1$ or $y \leq x_2$.*

3 Weakly δ-primary elements

The following definitions are from Manjarekar and Bingi [9].

Definition 3.1. An expansion of elements, or an expansion function, is a function $\delta : L \to L$, such that the following conditions are satisfied:
(i) $a \leq \delta(a)$ for all $a \in L$.
(ii) $a \leq b$ implies $\delta(a) \leq \delta(b)$ for all $a, b \in L$.

Example 3.2. (1) The identity function $\delta_0 : L \to L$, where $\delta_0(a) = a$ for every $a \in L$, is an expansion of elements.
(2) For a proper element $a \in L$, define $\mathbf{M} : L \to L$, by
$\mathbf{M}(a) = \wedge\{m \in L | a \leq m, m$ is a maximal element$\}$ and $\mathbf{M}(1) = 1$. Then $\mathbf{M}$ is an expansion of elements.
(3) For each element a define $\delta_1 : L \to L$ as $\delta_1(a) = \sqrt{a}$, the radical of a. Then $\delta_1(a)$ is an expansion of elements.

Definition 3.3. Let δ be an expansion of elements of L. An element $p \in L$ is called δ-primary if $ab \leq p$ implies that either $a \leq p$ or $b \leq \delta(p)$ for all $a, b \in L$.

Lemma 3.4. *Let δ be an expansion of element. If p is a semi prime element and 2-potent prime element of L, then p is a δ-primary element.*

Proof. Let $xy \leq p$. Then $(xy)^2 = x^2y^2 \leq p^2$. As p is a 2-potent prime element of L, either $x^2 \leq p$ or $y^2 \leq p$ implies that either $x \leq \sqrt{p}$ or $y \leq \sqrt{p}$ but p is semi prime. Thus we get $x \leq p$ or $y \leq p (\leq \delta(p))$. Hence p is a δ-primary element of L. □

Definition 3.5. Let δ be an expansion of element, an element w of L is called weakly δ-primary if for all $p, q \in L$, $0 \neq pq \leq w$, then either $p \leq w$ or $q \leq \delta(w)$.

The definition of a weakly δ-primary element can also be stated as: if $0 \neq pq \leq w$ then either $p \leq \delta(w)$ or $q \leq w$ for all $p, q \in L$.

Example 3.6. Consider the lattice L of ideals of the ring $R = \langle Z_{24}, +_{24}, \times_{24} \rangle$. Then

$L = \{(0), (2), (3), (4), (6), (8), (12), (1)\}$ is a multiplicative lattice. Its lattice structure is shown in Figure 1 and multiplication in Table 1.

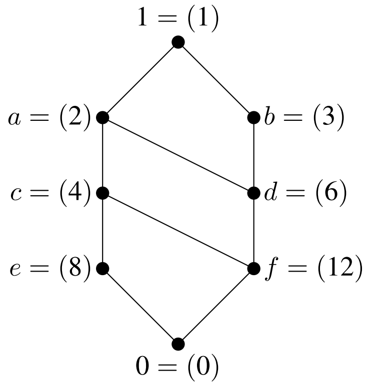

Figure 1

Table 1. Multiplication Table

·	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	d	e	f	e	0	a
b	0	d	b	f	d	0	f	b
c	0	e	f	e	0	e	0	c
d	0	f	d	0	f	0	0	d
e	0	e	0	e	0	e	0	e
f	0	0	f	0	0	0	0	f
1	0	a	b	c	d	e	f	1

From Example 3.2 and multiplication Table 1, we note the following.
(i)For the elements a, b, c, d, e, f, $\mathbf{M}(a) = \delta_1(a) = a$, $\mathbf{M}(b) = \delta_1(b) = b$, $\mathbf{M}(c) = \delta_1(c) = a$, $\mathbf{M}(d) = \delta_1(d) = d$, $\mathbf{M}(e) = \delta_1(e) = a$, $\mathbf{M}(f) = \delta_1(f) = d$, $\mathbf{M}(0) = \delta_1(0) = d$.
(ii) The elements, a, b, c, e are weakly δ_1-primary, weakly $\mathbf{M}$-primary.
(iii)The elements, a, b are weakly δ_0-primary but c, e are not weakly δ_0-primary. Since $ad = f \leq c$ but neither $a \leq c$ nor $d \leq \delta_0(c) = c$. Since $ac = e \leq e$ but neither $a \leq e$ nor $c \leq \delta_0(e) = e$.
(iv) The element d is not a weakly δ_0-primary, not a weakly δ_1-primary, not a weakly $\mathbf{M}$-primary. Since $ab = d \leq d$ but neither $a \leq d$ nor $b \leq \delta_0(d) = \mathbf{M}(d) = \sqrt{d} = d$.

Lemma 3.7. *Every δ-primary element of L is a weakly δ-primary element.*

Proof. Suppose that d is a δ-primary element of L. Let $p, q \in L$, $0 \neq pq \leq d$ and $p \nleq d$. As d is a δ-primary element of L, we have $q \leq \delta(d)$. Thus d is a weakly δ-primary element of L. □

Remark 3.8. The following example shows that the converse of Lemma 3.7 does not hold.

Example 3.9. Consider the multiplicative lattice shown in Figure 1. Here for the element 0, $\delta_0(0) = 0$, $\delta_1(0) = d$, $\mathbf{M}(0) = d$. 0 is a weakly δ_0-primary, weakly δ_1-primary, weakly $\mathbf{M}$-primary element but it is not a δ_0-primary, not a δ_1-primary, not a $\mathbf{M}$-primary element, as $be = 0 \leq 0$ neither $b \leq 0$ nor $e \leq \delta_0(0) = 0$ nor $e \leq \delta_1(0) = \mathbf{M}(0) = d$.

Definition 3.10. Let w be a weakly δ-primary element of L. Then (p,q) is called as a δ-twin-zero of w, if $pq=0$, $p\nleq w$ and $q\nleq \delta(w)$.

Remark 3.11. If w is a weakly δ-primary element of L that is not a δ-primary element of L, then w has a δ-twin-zero (p,q), for some $p,q\in L$.

Proof. Let $p,q\in L$ and $pq\leq w$. If $pq\neq 0$, then as w is a weakly δ-primary element of L, either $p\leq w$ or $q\leq\delta(w)$. But w is not a δ-primary element of L. Hence neither $p\leq w$ nor $q\leq\delta(w)$ is possible. Thus $pq=0$ and so (p,q) is a δ-twin-zero of w. □

Lemma 3.12. *Let w be a weakly δ-primary element of L and suppose that for some $p,q\in L$, (p,q) is a δ-twin-zero of w. Then $pw=qw=0$.*

Proof. Since (p,q) is a δ-twin-zero of w, $pq=0$, $p\nleq w$ and $q\nleq\delta(w)$. Suppose that $pw\neq 0$. Then $0\neq(pq)\vee(pw)=p(q\vee w)\leq w$. As $p\nleq w$, we conclude that $q\vee w\leq\delta(w)$. Hence $q\leq\delta(w)$, a contradiction. Thus $pw=0$.
Similarly, we can show that $qw=0$. □

Lemma 3.13. *Let w be a weakly δ-primary element of L. If w is not a δ-primary element of L, then $w^2=0$.*

Proof. Since w is not a δ-primary element. By Remark 3.11, there exist $p,q\in L$ such that (p,q) is a δ-twin-zero of w. Then $pq=0$, $p\nleq w$ and $q\nleq\delta(w)$. Suppose that $w^2\neq 0$. We have

$$(p\vee w)(q\vee w)=pq\vee pw\vee qw\vee w^2\leq w.$$

Since (p,q) is δ-twin-zero and by Lemma 3.12, we conclude that $pq=pw=qw=0$. Thus $(p\vee w)(q\vee w)=w^2\leq w$. As w is a weakly δ-primary element of L, we get either $p\vee w\leq w$ or $q\vee w\leq\delta(w)$. Thus we get either $p\leq w$ or $q\leq\delta(w)$, a contradiction. Thus $w^2=0$. □

The following example shows that converse of the Lemma 3.13 does not holds.

Example 3.14. Consider the multiplicative lattice shown in Figure 1. From Example 3.6 and Table 1, clearly, for the element f, $f^2=0$. Since $ad=f\leq f$ but neither $a\leq f$ nor $d\leq\delta_0(f)=f$. Since $cb=f\leq f$ but neither $c\leq f$ nor $b\leq\mathbf{M}(f)=\delta_1(f)=d$, we conclude that f is not weakly δ_0-primary, not a weakly δ_1-primary, not a weakly $\mathbf{M}$-primary element of L.

Lemma 3.15. *Let w be a weakly δ-primary element of L. Suppose that for some $p,q\in L$, (p,q) is a δ-twin-zero of w. If $ps\leq w$ for some $s\in L$, then $ps=0$.*

Proof. Suppose that $0\neq ps\leq w$ for some $s\in L$. Then $(pq)\vee(ps)\neq 0$ implies that
$0\neq p(q\vee s)\leq w$, as $p\nleq w$ and w be a weakly δ-primary element of L, we get $q\leq\delta(w)$, which is a contradiction to (p,q) is a δ-twin-zero of w. Thus $ps=0$. □

Lemma 3.16. *Let $w,p,q\in L$ be such that $pq\leq w$ and w be a weakly δ-primary element of L. If w has a δ-twin zero (p_1,q_1) for some $p_1\leq p$ and $q_1\leq q$ then $pq=0$.*

Proof. Suppose that $pq\neq 0$. Since $pq\leq w$, so $0\neq pq\leq w$, as w be weakly δ-primary element, we get $p\leq w$ or $q\leq\delta(w)$. But $p_1\leq p$ and $q_1\leq q$ it gives $p_1\leq w$ and $q_1\leq\delta(w)$, a contradiction to (p_1,q_1) is a δ-twin zero of w. Thus $pq=0$. □

Notation: For $x,y\in L$ we denote $(x:y)=\vee\{z\in L|yz\leq x\}$, see Calliap et. al. [6].

Theorem 3.17. *Let L be a C-lattice and $p\in L$. Then the following statements are equivalent:*
(i) w is a weakly δ-primary element of L;
(ii) Either $(w:a)=w$ or $(w:a)=(0:a)$, for every $a\nleq\delta(w)$.

Proof. (i)⇒(ii): Let $b\in L$ be a compact element such that $b\leq(w:a)$ for some $a\nleq\delta(w)$. Then $ab\leq w$. If $ab=0$, then $b\leq(0:a)$. If $ab\neq 0$, $a\nleq\delta(w)$, then $b\leq w$, as w is weakly δ-primary. Hence by Lemma 3.2 either $(w:a)\leq w$ or $(w:a)\leq(0:a)$. On the other hand, let $h\in L$ be a compact element such that $h\leq w$ or $h\leq(0:a)$. If $h\leq(0:a)$, then $ah\leq 0\leq w$ implies that $h\leq(w:a)$, So we get $(0:a)\leq(w:a)$. If $h\leq w$ then $ah\leq aw\leq w$ implies that $h\leq(w:a)$, So we get $w\leq(w:a)$. Hence either $w\leq(w:a)$ or $(0:a)\leq(w:a)$. Thus either $(w:a)=w$ or $(w:a)=(0:a)$ for every $a\nleq\delta(w)$.
(ii)⇒(i): Let $a,b\in L$ be such that $0\neq ab\leq w$. Suppose that $a\nleq\delta(w)$. Since $ab\leq w$, we have $b\leq(w:a)$. Hence $(w:a)=w$, then $b\leq w$. If $(w:a)=(0:a)$, then $ab=0$, a contradiction. Thus w is a weakly δ-primary element of L. □

4 Almost δ-primary and 2-potent δ-primary elements

In this section, we introduce the concepts of an almost δ-primary element and a 2-potent δ-primary element in multiplicative lattice. We prove some results on almost δ-primary elements.

Definition 4.1. A proper element $d \in L$ is called an n-almost δ-primary element if $pq \leq d$ and $pq \nleq d^n$ implies either $p \leq d$ or $q \leq \delta(d)$, for $p, q \in L$ and $n \geq 2$. If $n = 2$, then d is called an almost δ-primary element of L.

Definition 4.2. An element $d \in L$ is said to be 2-potent δ-primary if $pq \leq d^2$ implies $p \leq d$ or $q \leq \delta(d)$.

Example 4.3. Consider the lattice shown in Figure 1.
(i) Here a is an almost δ_1-primary and almost $\mathbf{M}$-primary element and almost δ_0-primary.
(ii) c is almost δ_1-primary and almost $\mathbf{M}$-primary element.
(iii) d is neither almost δ_1-primary, nor almost $\mathbf{M}$-primary nor almost δ_0-primary. Since $ab = d \leq d$, $ab = d \nleq d^2 = f$ neither $a \leq d$ nor $b \leq \delta_0(d) = d$ nor $b \leq \delta_1(d) = \mathbf{M}(d) = d$.
(iv) Here a, b are 2-potent δ_1-primary and 2-potent $\mathbf{M}$-primary element and 2-potent δ_0-primary.
(v) c, e are 2-potent δ_1-primary and 2-potent $\mathbf{M}$-primary element.
(vi) d is neither 2-potent δ_1-primary, nor 2-potent $\mathbf{M}$-primary nor 2-potent δ_0-primary. Since $bc = f \leq d^2 = f$ neither $b \leq d$ nor $c \leq \delta_0(d) = d$ nor $c \leq \delta_1(d) = \mathbf{M}(d) = d$.

We prove the following characterization.

Proposition 4.4. *Let d be a 2-potent δ-primary element of L. An element $d \in L$ is an almost δ-primary element of L if and only if d is a δ-primary element of L.*

Proof. Suppose that d is an almost δ-primary element of L. Let $p, q \in L$ be such that $pq \leq d$ but $p \nleq d$. If $pq \leq d^2$, then as d is 2-potent δ-primary, we get $q \leq \delta(d)$. If $pq \nleq d^2$, then as d is an almost δ-primary element of L, we get $q \leq \delta(d)$. Thus d is a δ-primary element of L.
Conversely, suppose that d is a δ-primary element of L. Let $x, y \in L$ be such that $xy \leq d$ and $xy \nleq d^2$. As d is δ-primary and $x \nleq d$ or $y \leq \delta(d)$. Hence d is an almost δ-primary element of L. □

The following theorem gives a characterization of an n-almost δ-primary element of L.

Theorem 4.5. *Let L be a C-lattice and $d \in L$. The following statements are equivalent:*
(i) d is an n-almost δ-primary element of L;
(ii) Either $(d : p) = d$ or $(d : p) = (d^n : p)$ for every $p \nleq \delta(d)$.

Proof. (i)⇒(ii): Let $r \in L$ be a compact element such that $r \leq (d : p)$ for some $p \nleq \delta(d)$. Then $pr \leq d$. If $pr \leq d^n$, then $r \leq (d^n : p)$. If $pr \nleq d^n$, then $r \leq d$, as d is an n-almost δ-primary. Hence by Lemma 3.2 either $(d : p) \leq d$ or $(d : p) \leq (d^n : p)$.
On the other hand, if $q \leq d$, then $pq \leq pd \leq d$, so we get $pq \leq d$ implies that $q \leq (d : p)$. If $q \leq (d^n : p)$ then $pq \leq d^n \leq d$. We get $q \leq (d : p)$. Hence by Lemma 3.2, either $d \leq (d : p)$ or $(d^n : p) \leq (d : p)$. Therefore, we get either $(d : p) = d$ or $(d : p) = (d^n : p)$ for every $p \nleq \delta(d)$.
(ii)⇒(i): Let $x, y \in L$ be such that $xy \leq d$ and $xy \nleq d^n$. Suppose that $x \nleq \delta(d)$. Since $xy \leq d$, we have $y \leq (d : x)$. Hence $(d : x) = d$, then $y \leq d$. If $(d : x) = (d^n : x)$, then $xy \leq d^n$, a contradiction. Thus d is a n almost δ-primary element of L □

Theorem 4.6. *Let L be a C-lattice and $d \in L$. The following statements are equivalent:*
(i) d is an n-almost δ-primary element of L;
(ii) Either $(d : y) \leq \delta(d)$ or $(d : y) = (d^n : y)$ for every $y \nleq d$.

Proof. (i)⇒(ii): Let $x \in L$ be a compact element such that $x \leq (d : y)$ for some $y \nleq d$. Then $xy \leq d$. If $xy \leq d^n$, then $x \leq (d^n : y)$. If $xy \nleq d^n$, then $x \leq \delta(d)$, as d is an n-almost δ-primary. Hence by Lemma 3.2 either $(d : y) \leq d$ or $(d : y) \leq (d^n : y)$.
On the other hand, If $z \leq (d^n : y)$ then $yz \leq d^n \leq d$. We get $z \leq (d : y)$. Hence, we get $(d^n : y) \leq (d : y)$. Therefore, we get either $(d : y) \leq \delta(d)$ or $(d : y) = (d^n : y)$ for every $y \nleq d$.
(ii)⇒(i): Let $x, y \in L$ be such that $xy \leq d$ and $xy \nleq d^n$. Suppose that $x \nleq d$. Since $xy \leq d$, we have $y \leq (d : x)$. Hence $(d : x) = \delta(d)$, then $y \leq \delta(d)$. If $(d : x) = (d^n : x)$, then $xy \leq d^n$, a contradiction. Thus d is a n almost δ-primary element of L □

It is known (see [6]) that for any $t \in L$, $L/t = \{s \in L | t \leq s\}$ is a multiplicative lattice with multiplication $p \circ q = pq \vee t$.

Proposition 4.7. *Let L be a multiplicative lattice and d be a almost δ-primary element. If $t \in L$ with $t \leq d$ then d is a almost δ-primary element of L/t.*

Proof. Let $p \circ q \leq d$, $p \circ q \nleq d^2$, for some $p, q \in L/t$ then $pq \vee t \leq d$, so $pq \leq d$ and $pq \nleq d^2$ and d is a almost δ-primary element. So we get either $p \leq d$ or $q \leq \delta(d)$. Hence d is an almost δ-primary element of L/t. □

Next results gives a relation between an almost δ-primary element and a weakly δ-primary element of L.

Theorem 4.8. *Let $d \in L$. Then d is an almost δ-primary element if and only if d is a weakly δ-primary element of L/d^2.*

Proof. Assume that d is an almost δ-primary element. Let $0_{L/d^2} \neq p \circ q \leq d$, $0_{L/d^2} \neq p \circ q \nleq d^2$, for some $p, q \in L/d^2$ then $pq \vee d^2 \leq d$, so $pq \leq d$ and $pq \nleq d^2$ and d is a almost δ-primary. Hence either $p \leq d$ or $q \leq \delta(d)$. Hence d is a weakly δ-primary element of L/d^2.
Conversely, Assume that d is a weakly δ-primary element of L/d^2. Let $rs \leq d$ and $rs \nleq d^2$. Then $r \circ s \leq d$ and $r \circ s \nleq d^2$, as d is a weakly δ-primary element of L/d^2, we get either $r \leq d$ or $s \leq \delta(d)$. Therefore d is an almost δ-primary element of L. □

Proposition 4.9. *Let L be a multiplicative lattice. Let $a, b, d \in L$ be such that $a \leq b \leq d$. If d is an almost δ-primary element of L/b. Then d is an almost δ-primary element of L/a.*

Proof. Let $r \circ s \leq d$, $r \circ s \nleq d^2$ in L/a. Then $rs \vee a \leq d$ and $rs \vee a \nleq d^2$. Hence for $a \leq b \leq d$, $rs \vee b \leq d$ and $rs \vee b \nleq d^2$. Therefore $r \circ s \leq d$, $r \circ s \nleq d^2$ in L/b. As d is an almost δ-primary element of L/b, so we get either $r \leq d$ or $s \leq \delta(d)$. Hence d is an almost δ-primary element of L/a. □

5 ϕ-δ-primary elements

Anderson and Bataineh [1] and Darani [7] have studied ϕ-prime and ϕ-primary ideals for commutative rings. We extend these concepts to multiplicative lattices using an expansion of element. We introduce the notion of a ϕ-δ-primary and an ω-δ-primary element.

Definition 5.1. Let L be a multiplicative lattice such that δ is an expansion of elements of L. Let $\phi : L \to L$ be a function such that $\phi(d) \leq d$. A proper element $d \in L$ is called ϕ-δ-primary if $ab \leq d$, $ab \nleq \phi(d)$ implies either $a \leq d$ or $b \leq \delta(d)$, for $a, b \in L$.

Definition 5.2. Let L be a multiplicative lattice and $\phi : L \to L$ be a function such that $\phi(d) \leq d$. Let δ be an expansion of elements of L. A proper element $d \in L$ is called ϕ_ω-δ-primary
(ω-δ-primary) if $ab \leq d$, $ab \nleq \bigwedge_{n=1}^{\infty} d^n = \phi_\omega(d)$ implies either $a \leq d$ or $b \leq \delta(d)$, for $a, b \in L$.

Theorem 5.3. *Let d be a proper element of L.*
Consider the following statements:
(i) If d is δ-primary, then d is weakly δ-primary.
(ii) If d is weakly δ-primary, then d is ω-δ-primary.
(iii) If d is ω-δ-primary, then d is n-almost δ-primary.
(iv) If d is n-almost δ-primary, then d is almost δ-primary.

Proof. (i) Obviously, d is δ-primary implies d is weakly δ-primary.
(ii) Assume that d is weakly δ-primary but not ω-δ-primary. Then there exist $p, q \in L$ such that $pq \leq d$, $pq \nleq \bigwedge_{n=1}^{\infty} d^n$ and $p \nleq d$ or $q \nleq \delta(d)$. Since d is weakly δ-primary, it follows that $p \leq d$ or $q \leq \delta(d)$, a contradiction. Hence $pq = 0$ this contradicts to $pq \nleq \bigwedge_{n=1}^{\infty} d^n$. Hence d is ω-δ-primary.
(iii) Now we show that if d is ω-δ-primary, then d is n-almost δ-primary $(n \geq 2)$. Assume that d is ω-δ-primary and $(n \geq 2)$. Let $pq \leq d$ and $pq \nleq d^n$ for some $p, q \in L$ then $pq \leq d$, $pq \nleq \bigwedge_{n=1}^{\infty} d^n$ for some $p, q \in L$, since d is ω-δ-primary it follows that either $p \leq d$ or $q \leq \delta(d)$. Hence d is n-almost δ-primary $(n \geq 2)$.
(iv) The last implication is obvious for $n = 2$. □

From this theorem, we get the following characterization of a ω-δ-primary element in L.

Corollary 5.4. *Let $d \in L$ be a proper element. Then d is ω-δ-primary if and only if d is n-almost δ-primary for every $n \geq 2$.*

Proof. Let $d \in L$ be an n-almost δ-primary element, for every $n \geq 2$. Suppose that $ab \leq d$ $ab \nleq \bigwedge_{n=1}^{\infty} d^n$ for some $a, b \in L$, then $ab \leq d$, $ab \nleq d^m$ for some $m \geq 2$ but for every $n \geq 2$, d is n-almost δ-primary, we get either $a \leq d$ or $b \leq \delta(d)$. Hence d is ω-δ-primary.
The converse follows from Theorem 5.3. □

Next we show that the radical of a ϕ-δ-primary element of L is again a ϕ-δ-primary element.

Theorem 5.5. *Let $d \in L$ be a ϕ-δ-primary such that $\sqrt{\phi(d)} = \phi(\sqrt{d})$ and $\sqrt{\delta(d)} = \delta(\sqrt{d})$. Then $\sqrt{d}$ is a ϕ-δ-primary element in L.*

Proof. Let $p, q \in L$ be such that $pq \leq \sqrt{d}$ and $pq \nleq \phi(\sqrt{d})$ Assume that $p \nleq \sqrt{d}$. Then there exist a positive integer n such that $(pq)^n \leq d$. Since $pq \nleq \phi(\sqrt{d}) = \sqrt{\phi(d)}$, so $(pq)^n \nleq \phi(d)$. If $(pq)^n \leq \phi(d)$, then by hypothesis $pq \leq \sqrt{\phi(d)} = \phi(\sqrt{d})$, a contradiction. So assume that $(pq)^n \nleq \phi(d)$ and $p^n \nleq d$ then we get $q^n \leq \delta(d)$, as d is ϕ-δ-primary. Hence
$q \leq \sqrt{\delta(d)} = \delta(\sqrt{d})$. Therefore $\sqrt{d}$ is a ϕ-δ-primary element in L. □

Lemma 5.6. *Let $\phi_1, \phi_2 : L \to L$ be function with $\phi_1 \leq \phi_2$. If d is ϕ_1-δ-primary, then d is also ϕ_2-δ-primary too.*

Proof. Let $p, q \in L$ be such that $pq \leq d$ and $pq \nleq \phi_2(d)$ implies $pq \nleq \phi_1(d)$. Since d is ϕ_1-δ-primary then we get $p \leq d$ or $q \leq \delta(d)$. Thus d is
ϕ_2-δ-primary. □

Proposition 5.7. *Let d be a proper element of L. Suppose that $\phi(d)$ is a δ-primary element of L. If d is a ϕ-δ-primary element of L, then d is a δ-primary element of L.*

Proof. Assume that $pq \leq d$ for some $p, q \in L$ and $p \nleq d$. Suppose that $pq \leq \phi(d)$. Since $\phi(d) \leq d$ and $p \nleq d$ so $p \nleq \phi(d)$. As $\phi(d)$ is δ-primary element of L, we get $q \leq \delta(\phi(d)) \leq \delta(d)$. If $pq \nleq \phi(d)$, then as d is ϕ-δ-primary element of L, we get $q \leq \delta(d)$. Hence d is a δ-primary element of L. □

Next theorem gives a characterization of ϕ-δ-primary elements.

Theorem 5.8. *Let $\phi : L \to L$ be a function. Then the following statements are equivalent:*
(i) d is ϕ-δ-primary.
(ii) For every $p \nleq \delta(p)$, either $(d : p) = d$ or $(d : p) = (\phi(d) : p)$.

Proof. (i)⇒(ii): Let $q \in L$ be a compact element such that $q \leq (d : p)$ for some $p \nleq \delta(d)$. Then $pq \leq d$. If $pq \leq \phi(d)$, then $q \leq (\phi(d) : p)$. If $pq \nleq \phi(d)$, then $q \leq d$, as d is ϕ-δ-primary. Hence by Lemma 3.2 either $(d : p) \leq d$ or $(d : p) \leq (\phi(d) : p)$. On the otherhand, if $r \leq d$ then $pr \leq pd \leq d$. We get $r \leq (d : p)$. If $r \leq (\phi(d) : p)$ then $pr \leq \phi(d) \leq d$. So $r \leq (d : p)$. Hence by Lemma 3.2, either $d \leq (d : p)$ or $(\phi(d) : p) \leq (d : p)$. Therefore, we get either $(d : p) = d$ or $(d : p) = (\phi(d) : p)$, for every $p \nleq \delta(d)$.
(ii)⇒(i): Let $x, y \in L$ be such that $xy \leq d$ and $xy \nleq \phi(d)$. Suppose that $y \nleq \delta(d)$. Since $xy \leq d$, we have $x \leq (d : y)$. If $(d : y) = (\phi(d) : y)$, then $xy \leq \phi(d)$, a contradiction. Hence we assume that $x \leq (d : y) = d$, then $x \leq d$. Thus d is ϕ-δ-primary. □

The following theorem gives some condition so that a ϕ-δ-primary element of L is a δ-primary element of L.

Theorem 5.9. *Let L be a multiplicative lattice and $\phi : L \to L$ be a function such that $\phi(d) \leq d$, Let d be a ϕ-δ-primary element of L.*
(i) If $d^2 \nleq \phi(d)$, then d is δ-primary.
(ii) If d is not a δ-primary element of L and $\delta(d^2) = \delta(d)$, then $\delta(d) = \delta(\phi(d))$.

Proof. (i) Assume that $a, b \in L$ and $ab \leq d$. If $ab \nleq \phi(d)$, since d is a ϕ-δ-primary, then either $a \leq d$ or $b \leq \delta(d)$. Hence we may assume that $ab \leq \phi(d)$. If $ad \nleq \phi(d)$, then there exist an element $d_1 \leq d$ such that $ad_1 \nleq \phi(d)$. Now $a(d_1 \vee b) = ad_1 \vee ab \leq d$, $a(d_1 \vee b) \nleq \phi(d)$. As d is ϕ-δ-primary, we get either $a \leq d$ or $b \leq (d_1 \vee b) \leq \delta(d)$. Similarly, if $bd_2 \nleq \phi(d)$. We can show that either $a \leq d$ or $b \leq \delta(d)$. So we may assume that $ad_1 \leq \phi(d)$ and $bd_2 \leq \phi(d)$. Since $d^2 \nleq \phi(d)$, there exist $p, q \leq d$ with $pq \nleq \phi(d)$. Now $(a \vee p)(b \vee q) = ab \vee aq \vee pb \vee pq \leq d$, $(a \vee p)(b \vee q) \nleq \phi(d)$, implies that either $(a \vee p) \leq d$ or$(b \vee q) \leq \delta(d)$. Therefore either $a \leq d$ or $b \leq \delta(d)$.
(ii) Since $\phi(d) \leq d$, we have $\delta(\phi(d)) \leq \delta(d)$. It follows that from part (1) that $d^2 \leq \phi(d)$. Hence $\delta(d) = \delta(d^2) \leq \delta(\phi(d))$, so $\delta(d) = \delta(\phi(d))$. □

Theorem 5.10. *Let L be a multiplicative lattice and let $\phi : L \to L$ be a function such that $\phi(d) \leq d$. Suppose that $d = \bigwedge_{j\in\Delta} d_j$ is a chain of ϕ-δ-primary elements of L such that for every $j \in \Delta$, $\phi(d_j) \leq \phi(d)$ and $\delta(d_j^2) = \delta(d_j)$. If for every $j \in \Delta$, d_j is a ϕ-δ-primary element of L that is not δ-primary, then $d = \bigwedge_{j\in\Delta} d_j$ is a ϕ-δ-primary element of L.*

Proof. Since d_j is a ϕ-δ-primary element of L that is not δ-primary, then for every $j \in \Delta$, $\delta(d_j) = \delta(\phi(d_j))$, by Theorem 5.9. On the otherhand, we have $\phi(d_j) \leq \phi(d) \leq d$ for every $j \in \Delta$, so $\delta(\phi(d_j)) \leq \delta(\phi(d)) \leq \delta(d)$. We have $d \leq d_j$ implies $\delta(d) \leq \delta(d_j) = \delta(\phi(d_j))$. Hence we get $\delta(d) = \delta(d_j) = \delta(\phi(d_j))$. Let $ab \leq d$, $ab \nleq \phi(d)$ and $a \nleq d$, $a, b \in L$, then there is a $j \in \Delta$ such that $ab \leq d_j$, $ab \nleq \phi(d_j)$ and $a \nleq d_j$. Since d_j is a ϕ-δ-primary element of L we conclude that $b \leq \delta(d_j) = \delta(d)$. Thus d is a ϕ-δ-primary element of L. □

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