Palestine Journal of Mathematics

Vol. 10(1)(2021) , 8–16

COMPUTATION OF THE SK INDEX OVER DIFFERENT CORONA PRODUCTS OF GRAPHS

P.G. Sheeja, P. S. Ranjini, V. Lokesha and A. Sinan Cevik

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 05C05; Secondary 05C12, 05C90.

Keywords and phrases: Topological indices, Corona products, SK index.

Abstract. It is known that the SK index ([10]) of a graph G is defined by

$$SK(G) = \sum_{uv \in E(G)} \frac{d_u + d_v}{2}.$$

In this paper, the explicit expressions for the SK index over different types of corona products on graphs are presented.

1 Introduction and Terminologies

Throughout this paper all graphs G will be assumed undirected and without any self loops or parallel edges having the vertex set $V(G)$ and the edge set $E(G)$. Unless stated otherwise the cardinality of $V(G)$ will be considered n while the cardinality of $E(G)$ will be considered m. Recall that the degree of any vertex v in G is denoted by $d_G(v)$ (or shortly d_v) which is the number of edges incident to v.

A topological index is defined as a real valued function, which maps each molecular graph to a real number and is necessarily invariant under automorphism of graphs. There are various topological indices having strong correlation with physio-chemical characteristics and have been found to be useful in isomer discrimination, quantitative structure activity relationship (QSAR) and structure property relationship (QSPR). A topological index of a chemical compound is an integer, derived following a rule, which can be used to characterize the chemical compound and predict certain physio-chemical properties like boiling point, molecular weight, density, refractive index, and so forth [2, 5]. Molecules and molecular compounds are often modeled by molecular graph. A molecular graph is a representation of the structural formula of a chemical compound in terms of graph theory, whose vertices correspond to the atoms of the compound and edges correspond to chemical bonds. Note that hydrogen atoms are often omitted.

Among the various degree-based topological indices, the first and second Zagreb indices of a graph G are one of the oldest and most studied topological indices that are firstly introduced by Gutman and Trinajstic in [6] which are defined respectively as

$$M_1(G) = \sum_{u \in V(G)} d_u^2 \quad \text{and} \quad M_2(G) = \sum_{u,v \in V(G)} d_u d_v \,.$$

In fact these Zagreb indices have extensively studied both with respect to mathematical and chemical point of view. After these degree-based indices, there have been introduced so many same based indices and it still keep going. For example, in [10], Shegahalli et al. introduced new topological indices; Arithmetic-Geometric AG_1 index, SK index, SK_1 index and SK_2 index for a graph G, and further presented their formulas. Among these four indices, we wil give our attention to the SK index that is defined by

$$SK(G) = \sum_{uv \in E(G)} \frac{d_u + d_v}{2} \,,$$

where d_u and d_v are the degrees of the vertices u and v in G.

On the other hand, to define some graph products over simple graphs always imply interesting results over these structures in applied sciences. For example, the corona product of graphs appears in chemical literature as plerographs of the hydrogen suppressed molecular graphs known as kenographs. Also, by specializing the components of corona products of graphs, different type of graphs can be obtained such as t-thorny graph, sunlet graph, suspension graphs and some classes of bridge graphs (we may refer [1, 3, 4, 7, 15, 11] for the details). Because of these consequences, people interested to study on indices for the corona products. For instance, in [8], Liu studied the F-index of different type of corona product of graphs, and [15] the authors computed the Szeged, vertex PI and the first and second Zagreb indices of corona products.

With a similar thought as in [8, 15], in this paper, we will express the SK index of different types of corona products within different sections as the main results.

2 SK index of corona products

Our main results will be given in this section via separate subsections under the name of classical corona product, subdivision-vertex corona product, subdivision-edge corona product, subdivision-vertex neighborhood corona product, subdivision-edge neighborhood corona product and finally the vertex-edge corona product.

2.1 The Case: Classical Corona Product

Let G_1 and G_2 be two simple connected graphs with n_j number of vertices and m_j number of edges respectively, for $j \in \{1, 2\}$. The *(classical) corona product* $G_1 \circ G_2$ of these two graphs is obtained by taking one copy of G_1 and n_1 copies of G_2; and then by joining each vertex of the i-th copy G_2 to the i-th vertex of G_1, where $1 \leq i \leq n_1$ (see, for instance, [13, 16]). From the definition, it is clear that the product $G_1 \circ G_2$ has total $n_1 + n_2 n_1$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges. With a similar approximation as in the paper [15], to obtain a detailed calculation, let us partition the edges of $G_1 \circ G_2$ into the three subsets E_1, E_2 and E_3 as follows:

(i) $E_1 = \{e \in E(G_1 \circ G_2) \mid e \in E(G_2)\}$ such that the cardinality $|E_1| = n_1 m_2$. For an edge $e = uv \in E_1$, there always exist $d_{G_1 \circ G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1 \circ G_2}(v) = d_{G_2}(v) + 1$.

(ii) $E_2 = \{e \in E(G_1 \circ G_2) \mid e \in E(G_1)\}$ such that the cardinality $|E_2| = m_1$. For an edge $e = uv \in E_2$, there always exist $d_{G_1 \circ G_2}(u) = d_{G_1}(u) + n_2$ and $d_{G_1 \circ G_2}(v) = d_{G_2}(v) + n_2$.

(iii) $E_3 = \{e \in E(G_1 \circ G_2) \mid e = uv,\ u \in E(G_2),\ v \in E(G_1)\}$ such that the cardinality $|E_3| = n_1 n_2$. For an edge $e = uv \in E_3$, there always exist $d_{G_1 \circ G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1 \circ G_2}(v) = d_{G_1}(v) + n_2$.

We then have the following main theorem for this section.

Theorem 2.1. *The SK index of the classical corona product $G_1 \circ G_2$ is presented by*

$$SK(G_1 \circ G_2) = n_1 SK(G_2) + SK(G_1) + 2n_1 m_2 + 2n_2 m_1 + \frac{n_1 n_2 (1 + n_2)}{2}. \tag{2.1}$$

Proof. From the definitions of SK index and corona product, we obtain

$$SK(G_1 \circ G_2) = \sum_{uv \in G_1 \circ G_2} \frac{d_u + d_v}{2} = \sum_{i=1}^{3} \left(\sum_{uv \in E_i} \frac{d_u + d_v}{2} \right),$$

where

$$\left.\begin{array}{l} \sum_{uv \in E_1} \frac{d_u + d_v}{2} = n_1 SK(G_2) + n_1 m_2\,, \\ \sum_{uv \in E_2} \frac{d_u + d_v}{2} = SK(G_1) + n_2 m_1\,, \\ \sum_{uv \in E_3} \frac{d_u + d_v}{2} = m_2 n_1 + n_2 m_1 + \frac{n_1 n_2 (1 + n_2)}{2}\,. \end{array}\right\} \tag{2.2}$$

Adding three equations in (2.2) gives the equality given in (2.1).

Hence the result. □

Corollary 2.2. *Let us assume that G_1 is any graph of order n_1 while G_2 is a complete graph of order n_2. Then we have*

$$SK(G_1 \circ G_2) = SK(G_1) + 2n_2m_1 + \frac{n_1n_2[(n_2)^2 + n_2]}{2}.$$

Proof. Substituting $SK(G_2) = \frac{n_2(n_2-1)^2}{2}$ and $m_2 = \frac{n_2(n_2-1)}{2}$ in Equation (2.1), we get the result. □

Another consequence of Theorem 2.1 is the following.

Corollary 2.3. *Let P_n and C_n denote the path and cycle on n vertices, respectively. Then the following equalities always hold.*

$$\begin{aligned} SK(P_{n_1} \circ P_{n_2}) &= 6n_1n_2 - 3n_1 - 2n_2 - 3 + \frac{n_1n_2(n_2+1)}{2}, \\ SK(C_{n_1} \circ C_{n_2}) &= 6n_1n_2 + 2n_1 + \frac{n_1n_2(n_2+1)}{2}, \\ SK(P_{n_1} \circ C_{n_2}) &= 6n_1n_2 + 2n_1 - 2n_2 - 3 + \frac{n_1n_2(n_2+1)}{2}. \end{aligned}$$

2.2 The Case: Subdivision-Vertex Corona

Recall that, for a simple graph G, the subdivision graph $S = S(G)$ is obtained from G by replacing each of its edges with a path of length two or, equivalently, by inserting an additional vertex into each edge of G.

Considering the definition of subdivision graphs, the *subdivision-vertex corona product* $G_1 \odot G_2$ of G_1 and G_2 is actually obtained from the $S(G_1)$ and n_1 copies of G_2 such that for all disjoint vertices joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2 ([7, 9]). It is easy to see that $G_1 \odot G_2$ has $n_1(1+n_2) + m_1$ vertices and $2m_1 + n_1(n_2+m_2)$ edges.

As in Section 2.1, we can partition the edges of $G_1 \odot G_2$ into the three subsets as in the following:

(a) $E_1 = \{e \in E(G_1 \odot G_2) \mid e \in E(G_2)\}$ such that the cardinality $|E_1| = n_1m_2$. For an edge $e = uv \in E_1$, there do exist $d_{G_1 \odot G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1 \odot G_2}(v) = d_{G_2}(v) + 1$.

(b) $E_2 = \{e \in E(G_1 \odot G_2) \mid e \in E(S(G_1))\}$ such that the cardinality $|E_2| = 2m_1$. For an edge $e = uv \in E_2$, there do exist $d_{G_1 \odot G_2}(u) = d_{G_1}(u) + n_2$ and $d_{G_1 \odot G_2}(v) = d_{S(G_1)}(v) = 2$.

(c) $E_3 = \{e \in E(G_1 \odot G_2) \mid e = uv,\ u \in E(G_2),\ v \in E(G_1)\}$ such that the cardinality $|E_3| = n_1n_2$. For an edge $e = uv \in E_3$, there do exist $d_{G_1 \odot G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1 \odot G_2}(v) = d_{G_1}(v) + n_2$.

Therefore, we obtain the following result.

Theorem 2.4. *The SK index of subdivision-vertex corona product graph $G_1 \odot G_2$ is given by*

$$SK(G_1 \odot G_2) = n_1SK(G_2) + SK(G_1) + 2n_1m_2 + (n_2+1)(2m_1 + \frac{n_1n_2}{2}).$$

Proof. We clearly have

$$SK(G_1 \odot G_2) = \sum_{uv \in E_1} \frac{d_u + d_v}{2} + \sum_{uv \in E_2} \frac{d_u + d_v}{2} + \sum_{uv \in E_3} \frac{d_u + d_v}{2},$$

where each part of the sum is equal to the $n_1SK(G_2) + n_1m_2$, $SK(G_1) + m_1(n_2+2)$ and $m_2n_1 + n_2m_1 + \frac{n_1n_2(1+n_2)}{2}$, respectively. In fact, the addition of all those values imply the equality of the index $SK(G_1 \circ G_2)$ stated in the theorem. □

The similar consequences as in Corollaries 2.2 and 2.3 are the following.

Corollary 2.5. *Let G_1 be any graph of order n_1 and G_2 be a complete graph of order n_2. Then the SK index of subdivision-vertex corona product of these two graphs is*

$$SK(G_1 \odot G_2) = SK(G_1) + \frac{n_1n_2[(n_2)^2 - 1]}{2} + (n_2+1)(m_1 + \frac{n_1n_2}{2}).$$

Proof. Replacing $SK(G_2) = \frac{n_2(n_2-1)^2}{2}$ and $m_2 = \frac{n_2(n_2-1)}{2}$ in the statement of Theorem 2.4, we get the result. □

Corollary 2.6. *For the SK index, we have the following equalities:*

$$\begin{aligned} SK(P_{n_1} \odot P_{n_2}) &= 5n_1n_2 - 2n_1 - n_2 - 4 + \frac{n_1n_2(n_2+1)}{2}, \\ SK(C_{n_1} \odot C_{n_2}) &= 4n_1n_2 + 2n_1 + (n_2+1)(n_1 + \frac{n_1n_2}{2}), \\ SK(P_{n_1} \odot C_{n_2}) &= 5n_1n_2 + 3n_1 - n_2 - 4 + \frac{n_1n_2(n_2+1)}{2}. \end{aligned}$$

2.3 The Case: Subdivision-Edge Corona

Again by considering the definition of subdivision graphs, the *subdivision-edge corona product* $G_1 \Theta G_2$ of G_1 and G_2 is obtained from the $S(G_1)$ and m_1 copies of G_2 such that for all disjoint vertices joining the i-th vertex of $S(G_1)$ to every vertex in the i-th copy of G_2 ([7, 9]). Clearly, the product $G_1 \Theta G_2$ has $m_1(1+n_2)+n_1$ vertices and $m_1(n_2+m_2+2)$ edges. Similarly as in above sections, if we partition the edge set of $G_1 \Theta G_2$ then we get the following three subsets.

(1) $E_1 = \{e \in E(G_1 \Theta G_2) \mid e \in E(G_2)\}$ such that the cardinality $|E_1| = m_1m_2$. For an edge $e = uv \in E_1$, there exist $d_{G_1 \Theta G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1 \Theta G_2}(v) = d_{G_2}(v) + 1$.

(2) $E_2 = \{e \in E(G_1 \Theta G_2) \mid e \in E(S(G_1))\}$ such that the cardinality $|E_2| = 2m_1$. For an edge $e = uv \in E_2$, there exist $d_{G_1 \Theta G_2}(u) = d_{G_1}(u)$ and $d_{G_1 \Theta G_2}(v) = d_{S(G_1)}(v) = 2 + n_2$.

(3) $E_3 = \{e \in E(G_1 \Theta G_2) \mid e = uv,\ u \in E(G_2),\ v \in E(S(G_1))\}$ such that the cardinality $|E_3| = m_1n_2$. For an edge $e = uv \in E_3$, there exist $d_{G_1 \Theta G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1 \Theta G_2}(v) = d_{S(G_1)}(v) + n_2$.

So, the result of this section is the following:

Theorem 2.7. *The SK index of subdivision-edge corona product $G_1 \Theta G_2$ is defined by*

$$SK(G_1 \Theta G_2) = m_1 SK(G_2) + SK(G_1) + m_1(2m_2 + 2 + n_2) + \frac{m_1n_2(n_2+3)}{2}.$$

Proof. Similarly with the previous theorems, if we consider the definitions of SK index and the product $G_1 \Theta G_2$, then we get

$$\begin{aligned} SK(G_1 \Theta G_2) &= \sum_{uv \in G_1 \Theta G_2} \frac{d_u + d_v}{2} \\ &= \sum_{uv \in E_1} \frac{d_u + d_v}{2} + \sum_{uv \in E_2} \frac{d_u + d_v}{2} + \sum_{uv \in E_3} \frac{d_u + d_v}{2} \\ &= (m_1 SK(G_2) + m_1m_2) + (SK(G_1) + m_1(n_2+2)) + \\ &\quad + \left(m_1m_2 + \frac{m_1n_2(3+n_2)}{2}\right) \\ &= m_1 SK(G_2) + SK(G_1) + m_1(2m_2 + 2 + n_2) + \frac{m_1n_2(n_2+3)}{2}, \end{aligned}$$

as required. □

In the statement of Theorem 2.7, by substituting $SK(G_2)$ by $\frac{n_2(n_2-1)^2}{2}$ and m_2 by $\frac{n_2(n_2-1)}{2}$, we obtain the next result.

Corollary 2.8. *Suppose the graph G_2 is as in Corollaries 2.2 and 2.5. Then*

$$SK(G_1 \Theta G_2) = SK(G_1) + \frac{m_1n_2[(n_2)^2 - n_2 + 4]}{2} + [(n_2)^2 + 1]m_1.$$

Additionally,

Corollary 2.9. *We obtain*

$$\begin{aligned} SK(P_{n_1}\Theta P_{n_2}) &= 5n_1n_2 - n_1 - 5n_2 + \frac{(n_1-1)n_2(n_2+3)}{2}, \\ SK(C_{n_1}\Theta C_{n_2}) &= 5n_1n_2 + 4n_1 + \frac{n_1n_2(n_2+3)}{2}, \\ SK(P_{n_1}\Theta C_{n_2}) &= 5n_1n_2 + 4n_1 - 5n_2 - 5 + \frac{(n_1-1)n_2(n_2+3)}{2}. \end{aligned}$$

2.4 The Case: Subdivision-Vertex Neighborhood Corona

The *subdivision-vertex neighborhood corona product* $G_1 \boxdot G_2$ of G_1 and G_2 is obtained from the $S(G_1)$ and n_1 copies of G_2 such that for all disjoint vertices joining the neighbors of the i-th vertex of $S(G_1)$ to every vertex in the i-th copy of G_2 ([7, 9]). Thus, $G_1 \boxdot G_2$ has $m_1(1+n_2)+n_1$ vertices and $2m_1+n_1n_2+2m_1n_2$ edges. By partitioning the edge set of $G_1 \boxdot G_2$, we again obtain three subsets E_1, E_2 and E_3 which are

$(1a)$ $E_1 = \{e \in E(G_1 \boxdot G_2) \mid e \in E(G_2)\}$ such that the cardinality $|E_1| = n_1m_2$. For an edge $e = uv \in E_1$, there exist $d_{G_1 \boxdot G_2}(u) = d_{G_2}(u) + d_{G_1}(w)$ and $d_{G_1 \boxdot G_2}(v) = d_{G_2}(v) + d_{G_1}(w)$.

$(2b)$ $E_2 = \{e \in E(G_1 \boxdot G_2) \mid e \in E(S(G_1))\}$ such that the cardinality $|E_2| = 2m_1$. For an edge $e = uv \in E_2$, there exist $d_{G_1 \boxdot G_2}(u) = d_{G_1}(u)$ and $d_{G_1 \boxdot G_2}(v) = d_{S(G_1)}(v)+2n_2 = 2+2n_2$.

$(3c)$ $E_3 = \{e \in E(G_1 \boxdot G_2) \mid e = uv,\ u \in E(G_2),\ v \in E(S(G_1))\}$ such that the cardinality $|E_3| = 2m_1n_2$. For an edge $e = uv \in E_3$, there exist $d_{G_1 \boxdot G_2}(u) = d_{G_2}(u) + d_{G_1}(w)$ and $d_{G_1 \boxdot G_2}(v) = d_{S(G_1)}(v) + 2n_2 = 2 + 2n_2$.

Therefore the corresponding result for the related product under these material is the following:

Theorem 2.10. *The SK index of subdivision-vertex neighborhood corona product $G_1 \boxdot G_2$ for the given graphs G_1 and G_2 is stated by*

$$SK(G_1 \boxdot G_2) = n_1SK(G_2) + (n_2+1)SK(G_1) + 2m_1n_2(2+n_2) + 2m_1(n_2+m_2+1).$$

Proof. An easy calculation shows that

$$\begin{aligned} SK(G_1 \boxdot G_2) &= \sum_{uv \in G_1 \boxdot G_2} \frac{d_u + d_v}{2} = \sum_{i=1}^{3} \left(\sum_{uv \in E_i} \frac{d_u + d_v}{2} \right) \\ &= (n_1SK(G_2) + 2m_1n_2) + (SK(G_1) + m_1(2n_2+2)) \\ &\quad + (2m_1m_2 + n_2SK(G_1) + m_1n_2(2+2n_2)) \\ &= n_1SK(G_2) + (n_2+1)SK(G_1) + 2m_1n_2(2+n_2) + 2m_1(n_2+m_2+1). \end{aligned}$$

Hence the result. □

The first part of the proof of the following corollary can be obtained by taking $SK(G_2)$ as $\frac{n_2(n_2-1)^2}{2}$ and m_2 as $\frac{n_2(n_2-1)}{2}$ in the statement of Theorem 2.10. Moreover the next part of it can be seen easily as in the similar versions stated in previous sections.

Corollary 2.11. *Let G_2 be a complete graph of order n_2 while G_1 be as in the general idea. Then*

$$SK(G_1 \boxdot G_2) = (n_2+1)SK(G_1) + \frac{n_1n_2(n_2-1)^2}{2} + 5m_1n_2 + 3m_1(n_2)^2 + 2m_1.$$

Furthermore, by considering P_n and C_n, the following equalities are hold for the subdivision-vertex neighborhood corona product $SK(G_1 \boxdot G_2)$.

$$\begin{aligned} SK(P_{n_1} \boxdot P_{n_2}) &= 12n_1n_2 - n_1 - 11n_2 - 3 + 2n_2(n_1n_2 - n_2), \\ SK(C_{n_1} \boxdot C_{n_2}) &= 12n_1n_2 + 4n_1 + 2n_1(n_2)^2, \\ SK(P_{n_1} \boxdot C_{n_2}) &= 12n_1n_2 + 4n_1 - 11n_2 - 5 + 2n_2(n_1n_2 - n_2). \end{aligned}$$

2.5 The Case: Subdivision-Edge Neighborhood Corona

For the graphs G_1 and G_2, the *subdivision-edge neighborhood corona product* $G_1 \diamond G_2$ is obtained from the $S(G_1)$ and n_1 copies of G_2 such that for all disjoint vertices joining the neighbors of the i-th vertex of G_1 to every vertex in the i-th copy of G_2 ([7, 9]). From the definition, $G_1 \diamond G_2$ has $m_1(1+n_2)+n_1$ vertices and $2m_1+m_1m_2+2m_1n_2$ edges. Moreover, by partition the edge set of $G_1 \diamond G_2$, we again obtain three subsets E_1, E_2 and E_3 which are

$(i-a)$ $E_1 = \{e \in E(G_1 \diamond G_2) \mid e \in E(G_2)\}$ such that the cardinality $|E_1| = m_1m_2$. For an edge $e = uv \in E_1$, there exist $d_{G_1 \diamond G_2}(u) = d_{G_2}(u) + 2$ and $d_{G_1 \diamond G_2}(v) = d_{G_2}(v) + 2$.

$(ii-b)$ $E_2 = \{e \in E(G_1 \diamond G_2) \mid e \in E(S(G_1))\}$ such that the cardinality $|E_2| = 2m_1$. For an edge $e = uv \in E_2$, there exist $d_{G_1 \diamond G_2}(u) = d_{G_1}(u)(n_2+1)$ and $d_{G_1 \diamond G_2}(v) = d_{S(G_1)} = 2$.

$(iii-c)$ $E_3 = \{e \in E(G_1 \diamond G_2) \mid e = uv,\ u \in E(G_2),\ v \in E(S(G_1))\}$ such that the cardinality $|E_3| = 2m_1n_2$. For an edge $e = uv \in E_3$, there exist $d_{G_1 \diamond G_2}(u) = d_{G_2}(u) + 2$ and $d_{G_1 \diamond G_2}(v) = d_{G_1}(v)(1+n_2)$.

Similarly with the previous cases, we have the following result.

Theorem 2.12. *The SK index of subdivision-edge neighborhood corona product graph $G_1 \diamond G_2$ is expressed by*

$$SK(G_1 \diamond G_2) = m_1 SK(G_2) + (n_2+1)^2 SK(G_1) + 4m_1m_2 + 2m_1(1+n_2)\,.$$

Proof. We have

$$\begin{aligned} SK(G_1 \diamond G_2) &= \sum_{uv \in G_1 \diamond G_2} \frac{d_u + d_v}{2} \\ &= \sum_{uv \in E_1} \frac{d_u + d_v}{2} + \sum_{uv \in E_2} \frac{d_u + d_v}{2} + \sum_{uv \in E_3} \frac{d_u + d_v}{2} \\ &= [m_1 SK(G_2) + 2m_1m_2] + [(n_2+1)SK(G_1) + 2m_1] \\ &\quad + [2m_1m_2 + 2m_1n_2 + n_2(n_2+1)SK(G_1)] \\ &= m_1 SK(G_2) + (n_2+1)^2 SK(G_1) + 4m_1m_2 + 2m_1(1+n_2)\,. \end{aligned}$$

This completes the proof. □

The first part of the proof of the following corollary can be obtained by substituting $SK(G_2) = \frac{n_2(n_2-1)^2}{2}$ and $m_2 = \frac{n_2(n_2-1)}{2}$ in the statement of Theorem 2.12. Moreover the next part of it is standard as before.

Corollary 2.13. *Let G_2 be a complete graph of order n_2 while G_1 be as in the general idea. Then*

$$SK(G_1 \diamond G_2) = (n_2+1)^2 SK(G_1) + \frac{m_1n_2(n_2-1)(n_2+3)}{2} + 2m_1(n_2+1)\,.$$

In addtion, the following equalities are held for $SK(G_1 \diamond G_2)$:

$$\begin{cases} 8n_1n_2 - 5n_1 - 8n_2 + 5 + (n_2+1)^2(2n_1-3) & ; \ \textit{if } G_1 = P_{n_1} \textit{ and } G_2 = P_{n_2} \\ 8n_1n_2 + 2n_1 + 2n_1(n_2+1)^2 & ; \ \textit{if } G_1 = C_{n_1} \textit{ and } G_2 = C_{n_2} \\ 8n_1n_2 + 2n_1 - 8n_2 - 2 + (n_2+1)^2(2n_1-3) & ; \ \textit{if } G_1 = P_{n_1} \textit{ and } G_2 = C_{n_2} \end{cases}.$$

2.6 The Case: The Vertex-Edge Corona

As in all previous sections, let us start by recalling the definition of the related product.

For any two graphs G_1 and G_2, the *vertex-edge corona product* $G_1 \oplus G_2$ of them is obtained by after taking one copy of G_1, n_1 copies of G_2 and also m_1 copies of G_2, then joining the i-th vertex of G_1 to every vertex in the i-th vertex copy of G_2 and also joining the end vertices of j-th edge of G_1 to every vertex in the j-th edge copy of G_2 (we may refer [7, 9]). According to the this definition, it is not hard to see that the vertex-edge corona product graph $G_1 \oplus G_2$ has total $n_1n_2 + n_1 + m_1n_2$ vertices and $m_1 + m_1(m_2 + 2n_2) + n_1(m_2 + n_2)$ edges.

We can partition the edges of $G_1 \oplus G_2$ into five subsets as in the following:

- $E_1 = \{e \in E(G_1 \oplus G_2) \mid e \in E(G_2)\}$ such that the cardinality $|E_1| = m_2n_1$. For an edge $e = uv \in E_1$, there exist $d_{G_1\oplus G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1\oplus G_2}(v) = d_{G_2}(v) + 1$.
- $E_2 = \{e \in E(G_1 \oplus G_2) \mid e \in E(G_1)\}$ such that the cardinality $|E_2| = m_1$. For an edge $e = uv \in E_2$, there exist $d_{G_1\oplus G_2}(u) = d_{G_1}(u)(n_2+1)+n_2$ and $d_{G_1\oplus G_2}(v) = d_{G_1}(v)(n_2+1)+n_2$.
- $E_3 = \{e \in E(G_1 \oplus G_2) \mid e \in E(G_2)\}$ such that the cardinality $|E_2| = m_1m_2$. For an edge $e = uv \in E_3$, there exist $d_{G_1\oplus G_2}(u) = d_{G_2}(u) + 2$ and $d_{G_1\oplus G_2}(v) = d_{G_2}(v) + 2$.
- $E_4 = \{e = uv \in E(G_1 \oplus G_2) \mid u \in E(G_2), v \in E(G_1)\}$ such that the cardinality $|E_4| = n_2n_1$. For an edge $e = uv \in E_4$, there exist $d_{G_1\oplus G_2}(u) = d_{G_2}(u) + 1$ and $d_{G_1\oplus G_2}(v) = d_{G_1}(v)(n_2 + 1) + n_2$.
- $E_5 = \{e = uv \in E(G_1 \oplus G_2) \mid u \in E(G_2), v \in E(G_1)\}$ such that the cardinality $|E_5| = 2n_2m_1$. For an edge $e = uv \in E_5$, there exist $d_{G_1\oplus G_2}(u) = d_{G_2}(u) + 2$ and $d_{G_1\oplus G_2}(v) = d_{G_1}(v)(n_2 + 1) + n_2$.

We note that the sets E_1 and E_4 are the vertex copies of $E(G_2)$, and the sets E_3 and E_5 are the edge copies of $E(G_2)$.

The final main result of this paper is the following.

Theorem 2.14. *The SK index of the vertex-edge corona product $G_1 \oplus G_2$ is expressed by*

$$\begin{aligned} SK(G_1 \oplus G_2) &= (n_1 + m_1)SK(G_2) + (n_2 + 1)^2SK(G_1) + 2n_1m_2 + 2m_1n_2(2 + n_2) \\ &\quad + 4m_1m_2 + \frac{n_1n_2(n_2 + 1)}{2}. \end{aligned}$$

Proof. According to the our case, we have

$$SK(G_1 \oplus G_2) = \sum_{uv\in G_1\oplus G_2} \frac{d_u + d_v}{2} = \sum_{i=1}^{5} \left(\sum_{uv\in E_i} \frac{d_u + d_v}{2} \right),$$

where

$$\left.\begin{aligned} &\text{for } uv \in E_1, \text{ we have} && n_1SK(G_2) + n_1m_2\,, \\ &\text{for } uv \in E_2, \text{ we have} && (n_2 + 1)SK(G_1) + m_1n_2\,, \\ &\text{for } uv \in E_3, \text{ we have} && m_1SK(G_2) + 2m_1m_2\,, \\ &\text{for } uv \in E_4, \text{ we have} && n_1m_2 + m_1n_2(n_2 + 1) + \tfrac{n_1n_2(n_2+1)}{2} \text{ and} \\ &\text{for } uv \in E_5, \text{ we have} && 2m_1m_2 + m_1n_2(n_2 + 2) + n_2(n_2 + 1)SK(G_1)\,. \end{aligned}\right\} \quad (2.3)$$

A simple calculation after adding all these five values in (2.3), we obtain the equality in the statement of theorem, as required. □

Corollary 2.15. *For a complete graph G_2 of order n_2, the SK index of the vertex-edge corona product $G_1 \oplus G_2$*

$$SK(G_1 \oplus G_2) = (n_2 + 1)^2SK(G_1) + \frac{m_1n_2[(n_2)^2 + 6n_2 + 5]}{2} + \frac{n_1n_2[(n_2)^2 + n_2]}{2}.$$

Proof. Substitute $SK(G_2) = \frac{n_2(n_2-1)^2}{2}$ and $m_2 = \frac{n_2(n_2-1)}{2}$ in the statement of Theorem 2.14, we get the result as required. □

Also, the product $SK(G_1 \oplus G_2)$ has the following special cases by considering P_n and C_n.

Corollary 2.16.

$$\begin{cases} 14n_1n_2 - 12n_1 - 10n_2 + 7 + (n_2+1)^2(2n_1-3) & ; \text{ if } G_1 = P_{n_1} \text{ and } G_2 = P_{n_2} \\ +2(n_2)^2(n_1-1) + \frac{n_1n_2(n_2+1)}{2} & \\ 18n_1n_2 + 2n_1 + 4n_1(n_2)^2 + \frac{n_1n_2(n_2+1)}{2} & ; \text{ if } G_1 = C_{n_1} \text{ and } G_2 = C_{n_2} \\ 18n_1n_2 + 2n_1 - 16n_2 - \hat{}3 + (n_2)^2(4n_1-5) & ; \text{ if } G_1 = P_{n_1} \text{ and } G_2 = C_{n_2} \\ +\frac{n_1n_2(n_2+1)}{2} & \end{cases}.$$

References

[1] H. Bian, X. Ma, E. Vumar, The Wiener-type indices of the corona of two graphs, *Ars Combinatoria* **107**, 193-199 (2012).

[2] D. Cvetkovic, *Applications of Graph Spectra: An Introduction to the literature*, in the book “Selected Topics on Applications of Graph Spectra", D. Cvetkovic, I. Gutman(eds), Matematički Institute SANU, Zbornik Radova 14 (22), ISBN 978-86-80593-44-9. (2011).

[3] N. De, S. M. A. Nayeem, A. Pal, Reformulated first first Zagreb index of some graph operations, *Journal of Mathematics* **3**(4), 945-960 (2015).

[4] N. De, S. M. A. Nayeem, A. Pal, Total eccentricity index of the generalized hierarchical product of graphs, *International Journal of Applied and Computational Mathematics* **1**(3), 503-511 (2015).

[5] M. V. Diudea, I. Gutman, J. Lorentz, *Molecular Topology*, Babe-Bolyai University, Cluj-Napoca, Romania, (2001).

[6] I. Gutman, N. Trinajstic, Graph theory and molecular orbitals: Total Π electron energy of alternant hydrocarbons, *Chemical Physics Letters* **17**(4), 535-538 (1972).

[7] P. Lu, Y. Miao, Spectra of subdivision-vertex and subdivision-edge coronae, arXiv preprint arXiv:1302.0457, (2013).

[8] X. Liu, P. Lu, Spectra of subdivision-vertex and subdivision-edge neighborhood coronae, *Linear Algebra and its Applications* **438**(8), 3547-3559 (2013).

[9] N. De, Computing F-Index of Different Corona Product of Graphs, *Bulletin of Mathematical Sciences and Applications* **19**, 24-30 (2017).

[10] V. S. Shegahalli and R. Kanabur, Computation of new degree-based topological indices of graphene, *Journal of Mathematics* **5**, Article Id 4341919 (2016).

[11] B. S. Swetha, V. Lokesha, P. S. Ranjini, On the harmonic index of graph operations, *Transactions on Combinatorics* **4**(4), 5-14 (2015).

[12] D. S. Nandappa, M. R. Rajesh Kanna, R. P. Kunar, Narumi-Katayama and multiplicative Zagreb Indices of Dutch Windmill graph, *International Journal of Scientific and Engineering Research* **7**(5), 2229-5518 (2016).

[13] M. Tavakoli, F. Rahbarnia, A. R. Ashrafi, Studying the corona product of graphs under some graph invariants, *Transactions on Combinatorics* **3**(3), 43-49 (2014).

[14] N. Trinajstic, *Chemical Graph Theory*, Mathematical Chemistry Series, CRC Press; 2nd Edition, (1992).

[15] Z. Yarahmadi, A. R. Ashrafi, The Szeged, vertex PI, first and second Zagreb Indices of corona product of graphs, *Filomat* **26**(3), 467-472 (2012).

[16] Y. N. Yeh, I. Gutman, On the sum of all distances in composite graphs, *Discrete Math.* **135**, 359-365 (1994).

Author information

P.G. Sheeja, P. S. Ranjini, V. Lokesha and A. Sinan Cevik,

P.G. Sheeja and P. S. Ranjini: Department of Mathematics, Don Bosco Institute of Technology, Bangalore-74, INDIA

V. Lokesha: Department of Mathematics, Vijayanagara Sri Krishnadevaraya University, Ballari, Karnataka, INDIA

A. Sinan Cevik: *(Prior)* Department of Mathematics, Faculty of Science, Selcuk University, Campus, 42075, Konya, TURKEY. *(Current)* Department of Mathematics, Science Faculty, King Abdulaziz University, Jeddah, 21589, SAUDI ARABIA,.
E-mail: srgeotherese@gmail.com, ranjini p s@yahoo.com, v.lokesha@gmail.com, sinan.cevik@selcuk.edu.tr, ahmetsinancevik@gmail.com

Received: March 28, 2019.
Accepted: December 14, 2019