Palestine Journal of Mathematics

Vol. 10(1)(2021) , 27–36

GORENSETEIN T_n^d-INJECTIVE AND GORENSETEIN T_n^d-FLAT MODULES WHEN T IS A TILTING MODULE

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Communicated by Salah Kabbaj

MSC 2010 Classifications: : Primary 13D07, Secondary 16D40.

Keywords and phrases: (n, T)-Coherent, Gorensetein T_n^d-Injective, Gorensetein T_n^d-flat, Tilting.

Abstract. Let T be a tilting module. In this paper, Gorenstein T_n^d-injective and Gorenstein T_n^d-flat modules are introduced. If $G \in$ Cogen$T$ (resp; $G \in$ Gen$T$), then G is called Gorenstein T_n^d-injective (resp; Gorenstein T_n^d-flat) if there exists the exact sequence $\mathbf{M} = \cdots \to M_1 \to M_0 \to M^0 \to M^1 \to \cdots$ of $\mathcal{ET}$-modules (resp; $\mathcal{FT}$-modules) with $G = \ker(M^0 \to M^1)$ such that $\mathcal{E}_T^d(U, \mathbf{M})$ (resp; $\Gamma_d^T(U, \mathbf{F})$) leaves this sequence exact whenever $U \in$ F.Pres^nT with T.pdim$(U) < \infty$ (resp; T.fdim$(U) < \infty$). Also, some characterizations of rings over Gorenstein T_n^d-injective and Gorenstein T_n^d-flat modules are given.

1 Introduction

Throughout this paper, R is an associative ring with non-zero identity, all modules are unitary left R-modules. First we recall some known notions and facts needed in the sequel. Let R be a ring and T an R-module. Then

(1) A module M is said to be *cogenerated*, by T, denoted by $M \in CogenT$, (resp; *generated*, denoted $M \in GenT$) by T if there exists an exact sequence $0 \to M \to T^n$ (resp; $T^{(n)} \to M \to 0$), for some positive integer n.

(2) We denote by $ProdT$ (resp; $F.ProdT$), the class of modules isomorphic to direct summands of direct product of copies (resp; finitely many copies) of T.

(3) We denote by $AddT$ (resp; $F.AddT$), the class of modules isomorphic to direct summands of direct sum of copies (resp; finitely many copies) of T.

(4) By $Copres^nT$ (resp; $F.Copres^nT$) and $Copres^{\infty}T$ (resp; $F.Copres^{\infty}T$), we denote the set of all modules M such that there exists exact sequences

$$0 \longrightarrow M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

and

$$0 \longrightarrow M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n \longrightarrow \cdots ,$$

respectively, where $T_i \in$ Prod$T$ (resp. $T_i \in$ F.Prod$T$), for every $i \geq 0$.

(5) By $Pres^nT$ (resp; $F.Pres^nT$) and $Pres^{\infty}T$ (resp; $F.Pres^{\infty}T$), we denote the set of all modules M such that there exists exact sequences

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

and

$$\cdots \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0,$$

respectively, where $T_i \in$ Add$T$ (resp. $T_i \in$ F.Add$T$), for every $i \geq 0$.

(6) Following [1], a module T is called *tilting* if it satisfies the following conditions:
(a) $pd(T) \leq 1$, where $pd(T)$ denotes the *projective dimension of* T.
(b)$Ext^i(T, T^{(\lambda)}) = 0$, for each $i > 0$ and for every cardinal λ.
(c) There exists the exact sequence $0 \to R \to T_0 \to T_1 \to 0$, where $T_0, T_1 \in$ Add$T$.

(7) For any tilting module T, if $N \in \mathrm{Cogen}T$ and $M \in \mathrm{Gen}T$, then [5, Proposition 2.1] implies that $\mathrm{Cogen}T = \mathrm{Copres}^{\infty}T$ and $\mathrm{Gen}T = \mathrm{Pres}^{\infty}T$. This shows that any module cogenerated by T and any module generated by T has an Prod$T$-resolution and Add$T$-resolution.

(8) For any homomorphism f, we denote by $kerf$ and imf, the kernel and image of f, respectively. Let $N \in \mathrm{Cogen}T$ and $M \in \mathrm{Gen}T$ be two modules, where T is tilting module. We define the functors

$$\Gamma_n^T(M,-) := \frac{\ker(\delta_n \otimes 1_B)}{\mathrm{im}(\delta_{n+1} \otimes 1_B)};\ \ \mathcal{E}_T^n(-,N) := \frac{\ker\delta_*^n}{\mathrm{im}\delta_*^{n-1}},$$

where

$$\cdots \longrightarrow T_2 \xrightarrow{\delta_2} T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

is an Add$T$-resolution of M,

and

$$0 \longrightarrow N \xrightarrow{\delta_0} T^0 \xrightarrow{\delta_1} T^1 \xrightarrow{\delta_2} \cdots$$

is an Prod$T$-resolution of N and $\delta_*^n = Hom(\delta_n, id_B)$, for every $i \geq 0$.

Let $M \in \mathrm{Gen}T$ be a module. A similar proof to that of [6, Lemma 2.11] shows that $\mathcal{E}_T^0(M,-) \cong \mathrm{Hom}(M,-)$. Similarly, it is seen that $\Gamma_T^0(M,-) \cong M \otimes -$. Moreover, $\mathcal{E}_T^1(M,-) = 0$ implies that $M \in \mathrm{Add}T$. If $N \in \mathrm{Cogen}T$, then $\mathcal{E}_T^1(-,N) = 0$ implies that $N \in \mathrm{Prod}T$. It is clear that $\mathrm{T.pdim}(M) = n$ if and only if n is the least non-negative integer such that $\mathcal{E}_T^{n+1}(M,B) = 0$, for any module B. Naturally, we say that M has *T-flat dimension* (*T-injective dimension*) n, denoted by $T.fdim(M) = n$ ($T.idim(N) = n$) if n is the least non-negative integer such that $\Gamma_{n+1}^T(M,B) = 0$ ($\mathcal{E}_T^{n+1}(B,N) = 0$), for any module B. A module with zero T-projective (resp., T-injective) dimension is called *T-projective* (*resp., T-injective*) (see [5, 8]).

(9) M is said to be *n-presented* [10, 11] if there is an exact sequence of R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to$, where each F_i is a finitely generated free.

(10) R is said to be *n-coherent* [10, 11] if every n-presented R-module is $(n+1)$-presented.

(11) M is said to be *Gorenstein flat* (resp.,*Gorenstein injective*) [2, 4] if there is an exact sequence $\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$ of flat (resp., injective) modules with $M = \ker(I^0 \to I^1)$ such that $U \otimes_R -$ (resp; $\mathrm{Hom}(U,-)$) leaves the sequence exact whenever U is an injective module.

A module M is called *T_n^d-injective* if $\mathcal{E}_T^{d+1}(U,M) = 0$ for every $U \in \mathrm{F.Pres}^nT$. A module M is called *T_n^d-flat* if $\Gamma_{d+1}^T(U,M) = 0$ for every $U \in \mathrm{F.Pres}^nT$ (see [9]). We denote by $\mathcal{ET}$ and $\mathcal{FT}$ the class of T_n^d-injective modules belong to Cogen$T$ and T_n^d-flat modules belong to Gen$T$, respectivily. In this paper, T is a tilting module. We introduce the *Gorenstein T_n^d-injective* and *Gorenstein T_n^d-flat* modules. A module $G \in \mathrm{Cogen}T$ is called Gorenstein T_n^d-injective if there exists the following exact sequence of $\mathcal{ET}$-modules:

$$\mathbf{M} = \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots$$

with $G = \ker(M^0 \to M^1)$ such that $\mathcal{E}_T^d(U,\mathbf{M})$ leaves this sequence exact whenever $U \in \mathrm{F.Pres}^nT$ with $\mathrm{T.pdim}(U) < \infty$. A module $G \in \mathrm{Gen}T$ is said to be Gorenstein T_n^d-flat if there exists an exact sequence of $\mathcal{FT}$-modules:

$$\mathbf{N} = \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow N^0 \longrightarrow N^1 \longrightarrow \cdots$$

with $G = \ker(N^0 \to N^1)$ such that $\Gamma_d^T(U,\mathbf{N})$ leaves this sequence exact whenever $U \in \mathrm{F.Pres}^nT$ with $\mathrm{T.fdim}(U) < \infty$. Replacing T by R as an R-module, every Gorenstein T_0^0-injective R-module is Gorenstein injective, and every Gorenstein T_1^0-flat is Gorenstein flat.

A ring R is called *(n,T)-coherent* if $\mathrm{F.Pres}^nT = \mathrm{F.Pres}^{n+1}T$. In Section 2, we study some basic properties of the Gorenstein T_n^d-flat and Gorenstein T_n^d-injective modules. Then some characterizations of (n,T)-coherent rings over Gorenstein T_n^d-injective and Gorenstein T_n^d-flat modules are given.

2 Main Results

We start with the following lemma.

Lemma 2.1. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence. Then*

(1) *If* $A \in \text{Pres}^n T$ *and* $C \in \text{Pres}^n T$*, then* $B \in \text{Pres}^n T$.

(2) *If* $A \in \text{Pres}^n T$ *and* $B \in \text{Pres}^{n+1} T$*, then* $C \in \text{Pres}^{n+1} T$.

(3) *If* $B \in \text{Pres}^n T$ *and* $C \in \text{Pres}^{n+1} T$*, then* $A \in \text{Pres}^n T$.

Proof. (1) We prove the assertion by induction on n. If $n = 0$, then the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 \longrightarrow T_0^{'} & \xrightarrow{i_0} & T_0^{'} \oplus T_0^{''} & \xrightarrow{\pi_0} & T_0^{''} \longrightarrow 0 \\
\downarrow h_0^{'} & & \downarrow h_0 & & \downarrow h_0^{''} \\
0 \longrightarrow A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

exists, where $T_0', T_0'' \in \text{Add}T$, i_0 is the inclusion map, π_0 is a canonical epimorphism and $h_0 = fh_0'$ is epimorphism, by Five Lemma. Let $K_1' = \ker h_0'$, $K_1 = \ker h_0$ and $K_1'' = \ker h_0''$. So, $B \in \text{Pres}^0 T$. It is clear that $K_1', K_1'' \in \text{Pres}^n T$; so, the induction hypotises implies that $K_1 \in \text{Pres}^n T$. Hence $B \in \text{Pres}^n T$.

(2) First assume that $n = 0$. If $B \in \text{Pres}^1 T$ and $A \in \text{Pres}^0 T$, then the following commutative diagram with exact rows:

$$
\begin{array}{ccccc}
 & & T_0^{'} & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow \gamma & & \downarrow f \\
T_1 & \xrightarrow{\alpha_2} & T_0 & \xrightarrow{\alpha_1} & B \longrightarrow 0 \\
 & & \| & & \downarrow g \\
T_0^{'} \oplus T_1 & \xrightarrow{h} & T_0 & \xrightarrow{g\alpha_1} & C \longrightarrow 0 \\
 & & & & \downarrow \\
 & & & & 0
\end{array}
$$

in which the existence of γ follows from the exactness of the sequence

$$\text{Hom}(T_0^{'}, T_0) \to \text{Hom}(T_0^{'}, B) \to 0,$$

since $T_0^{'}$ is T-projective. Also, h is defined by $h(t_0^{'}, t_1) = \gamma(t_0^{'}) + \alpha(t_1)$. Therefore, we deduce that $C \in \text{Pres}^1 T$. For $n > 0$, the assertion follows from induction.

(3) This is proved similarly. □

Definition 2.2. Let G be a module.

(1) If $G \in \text{Cogen}T$, then G is called Gorenstein T_n^d-injective if there exists the following exact sequence of $\mathcal{ET}$-modules:

$$\mathbf{M} = \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots$$

with $G = \ker(M^0 \to M^1)$ such that $\mathcal{E}_T^d(U, \mathbf{M})$ leaves this sequence exact whenever $U \in \text{F.Pres}^n T$ with $\text{T.pdim}(U) < \infty$.

(2) If $G \in \text{Gen}T$, then G is called Gorenstein T_n^d-flat if there exists the following exact sequence of $\mathcal{FT}$-modules:

$$\mathbf{N} = \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow N^0 \longrightarrow N^1 \longrightarrow \cdots$$

with $G = \ker(N^0 \to N^1)$ such that $\Gamma_d^T(U, \mathbf{N})$ leaves this sequence exact whenever $U \in \text{F.Pres}^n T$ with $\text{T.fdim}(U) < \infty$.

In the following theorem, we show that in the case of (n,T)-coherent rings, the existence of $\mathcal{FT}$-complex and $\mathcal{ET}$-complex of a module is sufficient to be Gorenstein T_n^d-flat and Gorenstein T_n^d-injective.

Theorem 2.3. *Let R be an (n,T)-coherent. Then*

(1) *$G \in \mathrm{Cogen}T$ is Gorenstein T_n^d-injective if and only if there is an exact sequence*

$$\mathbf{M} = \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots$$

of $\mathcal{ET}$-modules such that $G = \ker(M^0 \to M^1)$.

(2) *$G \in \mathrm{Gen}T$ is Gorenstein T_n^d-flat if and only if there is an exact sequence*

$$\mathbf{N} = \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow N^0 \longrightarrow N^1 \longrightarrow \cdots$$

of $\mathcal{FT}$-modules such that $G = \ker(N^0 \to N^1)$.

Proof. (1) $(\Longrightarrow)$: This is a direct consequence of definition.

$(\Longleftarrow)$: By definition, it suffices to show that $\mathcal{E}_T^d(U,\mathbf{M})$ is exact for every module $U \in$ F.Pres^nT with T.pdim$(U) < \infty$. To prove this, we use the induction on d. Let $d = 0$ and T.pdim$(U) = m$, then we show that Hom$(U,\mathbf{M})$ is exact. To prove this, we use the induction on m. The case $m = 0$ is clear. Assume that $m \geq 1$. Since T.pdim$(U) = m$, there exists an exact sequence $0 \to L \to T_0 \to U \to 0$ with $T_0 \in$ F.Add$T \subseteq$ F.Pres^{n-1}T. Now, from the (n,T)-coherence of R and Lemma 2.1, we deduce that $L, T_0 \in$ F.Pres^nT. Also, T.pdim$(L) \leq m-1$ and T.pdim$(T_0) = 0$. So, the following short exact sequence of complexes exists:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Hom}(U,M_1) & \longrightarrow & \mathrm{Hom}(T_0,M_1) & \longrightarrow & \mathrm{Hom}(L,M_1) & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Hom}(U,M_0) & \longrightarrow & \mathrm{Hom}(T_0,M_0) & \longrightarrow & \mathrm{Hom}(L,M_0) & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Hom}(U,M^0) & \longrightarrow & \mathrm{Hom}(T_0,M^0) & \longrightarrow & \mathrm{Hom}(L,M^0) & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Hom}(U,M^1) & \longrightarrow & \mathrm{Hom}(T_0,M^1) & \longrightarrow & \mathrm{Hom}(L,M^1) & \longrightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & \\
 & \| & & \| & & \| & \\
0 \longrightarrow & \mathrm{Hom}(U,\mathbf{M}) & \longrightarrow & \mathrm{Hom}(T_0,\mathbf{M}) & \longrightarrow & \mathrm{Hom}(L,\mathbf{M}) & \longrightarrow 0.
\end{array}$$

By induction, Hom$(L,\mathbf{M})$ and Hom$(T_0,\mathbf{M})$ are exact, hence Hom$(U,\mathbf{M})$ is exact by [7, Theorem 6.10].

Let $d \geq 1$ and $U \in$ F.Pres^nT. Consider the exact sequence $0 \to K \to T_0 \to U \to 0$, where $T_0 \in$ F.Add$T$. So the following short exact sequence of complexes exists:

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{E}_T^{d-1}(K, M_1) & \longrightarrow & \mathcal{E}_T^{d}(U, M_1) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{E}_T^{d-1}(K, M_0) & \longrightarrow & \mathcal{E}_T^{d}(U, M_0) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{E}_T^{d-1}(K, M^0) & \longrightarrow & \mathcal{E}_T^{d}(U, M^0) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{E}_T^{d-1}(K, M^1) & \longrightarrow & \mathcal{E}_T^{d}(U, M^1) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & \\
 & \| & & \| & \\
0 \longrightarrow & \mathcal{E}_T^{d-1}(K, \mathbf{M}) & \longrightarrow & \mathcal{E}_T^{d}(U, \mathbf{M}) & \longrightarrow 0.
\end{array}$$

By induction, $\mathcal{E}_T^{d-1}(K, \mathbf{M})$ is exact. So, $\mathcal{E}_T^{d}(U, \mathbf{M})$ is exact and hence, G is Gorenstein T_n^d-flat.

(2) A similar proof to that of (1). □

Remark 2.4. (1) If $U \in \text{F.Pres}^n T$, then $U \in \text{F.Pres}^m T$ for any $n \geq m$.

(2) Every T_m^d-injective R-module is T_n^d-injective, for any $n \geq m$.

(3) Direct sum of T_n^d-injective R-modules is T_n^d-injective.

(4) Every T_m^d-flat R-module is T_n^d-flat, for any $n \geq m$.

Corollary 2.5. *Let R be an (n, T)-coherent ring and $G \in \text{Cogen}T$ a module. Then the following assertions are equivalent:*

(1) *G is Gorenstein T_n^d-injective;*

(2) *There is an exact sequence $\cdots \to M_1 \to M_0 \to G \to 0$ of modules, where every $M_i \in \mathcal{ET}$;*

(3) *There is a short exact sequence $0 \to L \to N \to G \to 0$ of modules, where $N \in \mathcal{ET}$ and L is Gorenstein T_n^d-injective.*

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) follow from definition.

(2) $\Longrightarrow$ (1) For any module $G \in \text{Cogen}T$, there is an exact sequence

$$0 \longrightarrow G \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots$$

where any $T^i \in \text{Prod}T \subseteq \mathcal{ET}$. So, the exact sequence

$$\cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots$$

of $\mathcal{ET}$- modules exists, where $G = \ker(T^0 \to T^1)$. Therefore, G is Gorenstein T_n^d-injective, by Theorem 2.3.

(3) $\Longrightarrow$ (2) Assume that the exact sequence

$$0 \longrightarrow L \longrightarrow N \longrightarrow G \longrightarrow 0 \quad (1)$$

exists, where $N \in \mathcal{ET}$ and L is Gorenstein T_n^d-injective. Since L is Gorenstein T_n^d-injective, there is an exact sequence

$$\cdots \longrightarrow M_2' \longrightarrow M_1' \longrightarrow M_0' \longrightarrow L \longrightarrow 0 \quad (2)$$

where every $M_i' \in \mathcal{ET}$. Assembling the sequences (1) and (2), we get the exact sequence

$$\cdots \to M_2' \to M_1' \to M_0' \to N \to G \to 0,$$

where $N, M_i' \in \mathcal{ET}$, as desired. □

Corollary 2.6. *Let R be an (n, T)-coherent ring and $G \in \text{Gen}T$ a module. Then the following assertions are equivalent:*

(1) *G is Gorenstein T_n^d-flat;*

(2) *There is an exact sequence $0 \to G \to N^0 \to N^1 \to \cdots$ of R-modules, where every $N^i \in \mathcal{FT}$;*

(3) *There is a short exact sequence $0 \to G \to M \to K \to 0$ of R-modules, where $M \in \mathcal{FT}$ and K is Gorenstein T_n^d-flat.*

Proof. $(1) \Longrightarrow (2)$ and $(1) \Longrightarrow (3)$ follow from definition.

$(2) \Longrightarrow (1)$ For any R-module $G \in \text{Gen}T$, there is an exact sequence

$$\cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow G \longrightarrow 0,$$

where any $T_i \in \text{Add}T \subseteq \mathcal{FT}$. Thus, the exact sequence

$$\cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow N^0 \longrightarrow N^1 \longrightarrow \cdots$$

of $\mathcal{FT}$-modules exists, where $G = \ker(N^0 \to N^1)$. Therefore by Theorem 2.3, G is Gorenstein T_n^d-flat,

$(3) \Longrightarrow (2)$ Assume that the exact sequence

$$0 \longrightarrow G \longrightarrow M \longrightarrow K \longrightarrow 0 \quad (1)$$

exists, where $M \in \mathcal{FT}$ and K is Gorenstein T_n^d-flat. Since K is Gorenstein T_n^d-flat, there is an exact sequence

$$0 \longrightarrow K \longrightarrow (N^0)' \longrightarrow (N^1)' \longrightarrow (N^2)' \longrightarrow \cdots \quad (2)$$

where every $(N^i)' \in \mathcal{FT}$. Assembling the sequences (1) and (2), we get the exact sequence

$$0 \to G \to M \to (N^0)' \to (N^1)' \to (N^2)' \to \cdots,$$

where $M, (N^i)' \in \mathcal{FT}$, as desired. □

Proposition 2.7. *Let G be a module. Then:*

(1) *If $G \in \text{Cogen}T$ is Gorenstein T_n^d-injective, then $\mathcal{E}_T^i(U, G) = 0$ for any $i > d$ and every $U \in \text{F.Pres}^n T$ with* $\text{T.pdim}(U) < \infty$.

(2) *If $0 \to G \to G_0 \to G_1 \to \cdots \to G_{m-1} \to N \to 0$ is an exact sequence of modules where every G_j is Gorenstein T_n^d-injective and $G_j \in \text{Cogen}T$, then $\mathcal{E}_T^i(U, N) = \mathcal{E}_T^{m+i}(U, G)$ for any $i > d$ with* $\text{T.pdim}(U) < \infty$.

(3) *If $G \in \text{Gen}T$ is Gorenstein T_n^d-flat, then $\Gamma_i^T(U, G) = 0$ for any $i > d$ and every $U \in \text{F.Pres}^n T$ with* $\text{T.fdim}(U) < \infty$.

(4) *If $0 \to N \to G_{m-1} \to G_{m-2} \to \cdots \to G_0 \to G \to 0$ is an exact sequence of modules where every G_i is Gorenstein T_n^d-flat and $G_i \in \text{Gen}T$, then $\Gamma_i^T(U, N) = \Gamma_{m+i}^T(U, G)$ with* $\text{T.fdim}(U) < \infty$.

Proof. (1) Let G be a Gorenstein T_n^d-injective R-module, and $\text{T.pdim}(U) = m < \infty$. Then by hypothesis, the following $\mathcal{ET}$-resolution of G exists:

$$0 \to L \to M_{m-1} \to \cdots \to M_0 \to G \to 0.$$

So, $\mathcal{E}_T^i(U, M_j) = 0$ for every $0 \leq j \leq m-1$ and any $i > d$, since $U \in \text{F.Pres}^n T$ and any $M_j \in \mathcal{ET}$. Thus by [5, Proposition 2.2], we deduce that $\mathcal{E}_T^i(U, G) \cong \mathcal{E}_T^{m+i}(U, L)$. Therefore $\mathcal{E}_T^i(U, G) = 0$, since $\text{T.pdim}(U) = m < \infty$.

(2) Setting $G_m = N$ and $K_{j-1} = \ker(G_{j-1} \to G_j)$, for every $0 \leq j \leq m-1$, the short exact sequence $0 \to K_{j-1} \to G_{j-1} \to K_j \to 0$ exists. Thus by (1), the induced exact sequences

$$0 = \mathcal{E}_R^i(U, G_{j-1}) \to \mathcal{E}_R^i(U, K_j) \to \mathcal{E}_R^{i+1}(U, K_{j-1}) \to \mathcal{E}_R^{i+1}(U, G_{j-1}) = 0$$

exists and so $\mathcal{E}_T^i(U, K_j) \cong \mathcal{E}_R^{i+1}(U, K_{j-1})$. Since $K_0 = G$, we have

$$\mathcal{E}_R^{m+i}(U, N) \cong \mathcal{E}_R^{m+i-1}(U, K_{m-1}) \cong \cdots \cong \mathcal{E}_R^i(U, G),$$

as desired.

(3) and (4) are similar to the proof of (1) and (2). □

Lemma 2.8. *Let* $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ *be an exact sequence. Then*

(1) *If* A *is* T*-injective and* $A, B, C \in \mathrm{Cogen}T$*, then* $B = A \oplus C$*.*

(2) *If* $A \in \mathrm{Copres}^n T$ *and* $C \in \mathrm{Copres}^n T$*, then* $B \in \mathrm{Copres}^n T$*.*

(3) *If* $C \in \mathrm{Copres}^n T$ *and* $B \in \mathrm{Copres}^{n+1} T$*, then* $A \in \mathrm{Copres}^{n+1} T$*.*

(4) *If* $B \in \mathrm{Copres}^n T$ *and* $A \in \mathrm{Copres}^{n+1} T$*, then* $C \in \mathrm{Copres}^n T$*.*

Proof. (1) If A is T-injective and $A, B, C \in \mathrm{Cogen}T$, then we deduce that the sequence

$$0 \longrightarrow \mathrm{Hom}(C, A) \overset{g^*}{\longrightarrow} \mathrm{Hom}(B, A) \overset{f^*}{\longrightarrow} \mathrm{Hom}(A, A) \longrightarrow \mathcal{E}_T^1(C, A) = 0$$

is exact. So, there exists $h : B \to A$ such that $hf = 1_A$.

(2) It is similar to the proof of Lemma 2.1(1).

(3) Let $B \in \mathrm{Copres}^{n+1} T$ and $C \in \mathrm{Copres}^n T$, then the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & T_0 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

exists, where $T_0 \in \mathrm{Prod}T$ and $L \in \mathrm{Copres}^n T$. By (2), $D \in \mathrm{Copres}^n T$. So, we deduce that $A \in \mathrm{Copres}^{n+1} T$.

(4) Let $A \in \mathrm{Copres}^{n+1} T$ and $B \in \mathrm{Copres}^n T$, then the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & T_0^{'} & \longrightarrow & L^{'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & T_0 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

exists, where $T_0, T_0^{'} \in \mathrm{Prod}T$ and $L \in \mathrm{Copres}^{n-1} T$. Since $T_0^{'}$ is T-injective, we have that $T_0 = T_0^{'} \oplus D$ by (1), and $D \in \mathrm{Cogen}T$. Thus for any module N, we have

$$\mathcal{E}_T^1(N, T_0) = \mathcal{E}_T^1(N, T_0^{'} \oplus D) = \mathcal{E}_T^1(N, T_0^{'}) \oplus \mathcal{E}_T^1(N, D) = 0.$$

Hence $D \in \mathrm{Prod}T$. On the other hand, $L \in \mathrm{Copres}^{n-1} T$. Therefore, we conclude that $C \in \mathrm{Copres}^n T$. □

Proposition 2.9. *Let* R *be an* (n, T)*-coherent.*

(1) *Let* $0 \to M \to G \to N \to 0$ *be an exact sequence. If* $N \in \mathrm{Cogen}T$ *is Gorenstein* T_n^d*-injective and* $M \in \mathcal{ET}$*, then* G *is Gorenstein* T_n^d*-injective.*

(2) *Let* $0 \to K \to G \to N \to 0$ *be an exact sequence. If* $K \in \mathrm{Gen}T$ *is Gorenstein* T_n^d*-flat and* $N \in \mathcal{FT}$*, then* G *is Gorenstein* T_n^d*-flat.*

Proof. (1) By Lemma 2.8, $G \in \text{Cogen}T$, since $M, N \in \text{Cogen}T$. N is Gorenstein T_n^d-injective. So by Corollary 2.5, there exists an exact sequence of $0 \to K \to M^{'} \to N \to 0$, where $M^{'} \in \mathcal{ET}$ and K is Gorenstein T_n^d-injective. Now, we consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & D & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The exactness of the middle horizontal sequence with $M, M^{'} \in \mathcal{ET}$, implies that $D \in \mathcal{ET}$. Hence from the middle vertical sequence and Corollary 2.5, we deduce that G is Gorenstein T_n^d-injective.

(2) By Lemma 2.1, $G \in \text{Gen}T$, since $K, N \in \text{Gen}T$. K is Gorenstein T_n^d-flat. So by Corollary 2.6, there exists an exact sequence of $0 \to K \to N^{'} \to L \to 0$, where $N^{'} \in \mathcal{FT}$ and L is Gorenstein T_n^d-flat. Now, we consider the following diagram:

$$
\begin{array}{cccccccc}
 & & 0 & & 0 & & & \\
 & & \downarrow & & \downarrow & & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N' & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The exactness of the middle horizontal sequence with $N, N^{'} \in \mathcal{FT}$, implies that $E \in \mathcal{FT}$. Hence from the middle vertical sequence and Corollary 2.6, we deduce that G is Gorenstein T_n^d-flat. □

In this part, we show that which conditions under every module in Cogen$T$ is Gorenstein T_n^d-injective.

Proposition 2.10. *Let R be a ring. The following assertions are equivalent:*

(1) *Every module in* Cogen$T$ *is Gorenstein T_n^d-injective;*

(2) *The ring satisfies the following two conditions:*

(i) *Every T-projective module is T_n^d-injective.*

(ii) $\mathcal{E}_T^{d+1}(U, N) = 0$ *for any* $N \in \text{Cogen}T$ *and any* $U \in \text{F.Pres}^n T$ *with* $\text{T.pdim}(U) < \infty$.

Proof. (1) $\Longrightarrow$ (2) The condition (i) follows from this fact that every T-projective module M is Gorenstein T_n^d-injective. So, the following $\mathcal{ET}$-resolution of M exists:

$$\cdots \to M_1 \to M_0 \to M \to 0.$$

Since M is T-projective, M is T_n^d-injective as a direct summand of M_0. Also, by Proposition 2.7 and (1), the condition (ii) follows.

(2) $\Longrightarrow$ (1) Since T is tilting, the exact sequence $0 \to R \to T_0 \to T_1 \to 0$ exists, where $T_0, T_1 \in \text{Add}T$. So $T_0, T_1 \in \text{Gen}T$. Hence, the exact sequence $0 \to K \to T^{(m)} \to T_i \to 0$ exists for $i = 0, 1$. On the other hand, $K \subseteq T^{(m)} \subseteq T^m$. So, $K, T^{(m)} \in \text{Cogen}T$. Thus by [5, Proposition 2.1], $K, T^{(m)} \in \text{Copres}^{\infty}T$. Therefore by Lemma 2.8, $T_i, R \in \text{Copres}^k T$ and

hence $R \in \mathrm{Cogen}T$. Let $G \in \mathrm{Cogen}T$. Choose a Prod$T$-resolution $0 \to G \to T^0 \to T^1 \to \cdots$ of G and a free resolution $\cdots \to F_1 \to F_0 \to G \to 0$, where every $F_i \in \mathrm{Cogen}T$. Also by Lemma 2.1 and [5, Proposition 2.1], we get that $F_i \in \mathrm{Gen}T = \mathrm{Pres}^{\infty}T$, since $T_0, T_1 \in \mathrm{Gen}T$. Every projective in Gen$T$ is T-projective. So by (2), every F_i is T_n^d-injective. Assembling these resolutions, by Remark 2.4 and (2)(i), we get the following $\mathcal{ET}$-resolotion:

$$\mathbf{A} = \cdots \to F_1 \to F_0 \to T^0 \to T^1 \to \cdots ,$$

where $G = \ker(T^0 \to T^1)$, $K^i = \ker(T^i \to T^{i+1})$ and $K_i = \ker(F_i \to F_{i-1})$ for any $i \geq 1$. By Lemma 2.8, $K_i, K^i \in \mathrm{Cogen}T$, since $G, T^i, F_i \in \mathrm{Cogen}T$. Let $U \in \mathrm{F.Pres}^n T$ with T.p.dim$(U) < \infty$. Then by (2), $\mathcal{E}_T^{d+1}(U, G) = \mathcal{E}_T^{d+1}(U, F_i) = \mathcal{E}_T^{d+1}(U, T^i) = 0$ for any $i \geq 0$. So, $\mathcal{E}_T^{d}(U, \mathbf{A})$ is exact, and hence G is Gorenstein T_n^d-injective. □

Theorem 2.11. *Let R be an (n, T)-coherent ring. Then the following assertions are equivalent:*

(1) *Every module in* Cogen$T$ *is Gorenstein T_n^d-injective;*

(2) *Every T-projective module is T_n^d-injective;*

(3) *R is T_n^d-injective;*

(4) *Every Gorenstein T_n^d-flat is Gorenstein T_n^d-injective;*

(5) *Every T-flat module is Gorenstein T_n^d-injective;*

(6) *Every T-projective module is Gorenstein T_n^d-injective.*

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) follow from Proposition 2.10.

(3) $\Longrightarrow$ (1) Let $G \in \mathrm{Cogen}T$ be a module and $\cdots \to F_1 \to F_0 \to G \to 0$ be any free resolution of G. Then, similar to proof ((2) $\Longrightarrow$ (1)) of Proposition 2.10, each $F_i \in \mathcal{ET}$. Hence Corollary 2.5 completes the proof.

(3) $\Longrightarrow$ (4) Let $N \in \mathrm{Gen}T$ is Gorenstein T_n^d-flat. Similar to proof ((2) $\Longrightarrow$ (1)) from Proposition 2.10, $N \in \mathrm{Cogen}T$. So, (2) follows immediately from (1).

(4) $\Longrightarrow$ (5) every T-flat is T_n^d-flat and every T_n^d-flat is Gorenstein T_n^d-flat. So by (4), (5) is hold.

(5) $\Longrightarrow$ (6) is clear, since every T-projective is T-flat.

(6) $\Longrightarrow$ (3) Similar to proof ((1) $\Longrightarrow$ (2)) from Proposition 2.10, every T-projective module is T_n^d-injective. Also, the exact sequence $0 \to R \to T_0 \to T_1 \to 0$ exists, where $T_0, T_1 \in \mathrm{Add}T$. So $T_0, T_1 \in \mathrm{Gen}T$. Thus by [5, Proposition 2.1], $T_i \in \mathrm{Gen}T = \mathrm{Pres}^{\infty}T$ for $i = 0, 1$. Hence by Lemma 2.1, $R \in \mathrm{Gen}T$. Therefore R is T-projective and hence, it is T_n^d-injective. □

Example 2.12. Let R be a 1-Gorenstein ring and $0 \to R \to E^0 \to E^1 \to 0$ be the minimal injective resolution of R. Then, $T = E_0 \oplus E_1$ is Gorenstein T_n^d-injective and Gorenstein T_n^d-flat, since by [3], T is a tilting module.

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Received: July 9, 2019.
Accepted: September 22, 2019