

Invariance Properties of Matrix Powers

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Abstract A few years ago, Peter Larcombe discovered an amazing property regarding two by two matrices. For any such 2×2 matrix A , the ratios of the two anti-diagonal entries is the same for all powers of A . We discuss extensions to higher dimensions, and give a short bijective proof of Larcombe and Eric Fennessey’s elegant extension to tri-diagonal matrices of *arbitrary* dimension. This article is accompanied by a Maple package.

Peter Larcombe’s Surprising Discovery

In [2], Peter Larcombe gave four proofs of a seemingly new and amazing property of a 2×2 matrix, for any such matrix (we denote the (i, j) entry of a matrix B by B_{ij})

$$A_{12} \cdot (A^m)_{21} = A_{21} \cdot (A^m)_{12} \quad ,$$

for *all* positive integers m .

We first observe that, in hindsight (but only in hindsight!) this is not that surprising. More generally, for a general $n \times n$ matrix A , and any subset S of cardinality $n + 1$ of the set of n^2 entries $\{(i, j) \mid 1 \leq i, j \leq n\}$, there exist polynomials $q_s(A)$ in the entries of A (independent of m) such that

$$\sum_{s \in S} q_s \cdot (A^m)_s = 0 \quad , \tag{1}$$

for all $m > 1$.

This fact follows from the **Cayley-Hamilton** equation that says that If $P_A(x) := \det(A - xI)$ is the **characteristic polynomial** of A , then the $n \times n$ matrix $P_A(A)$ equals the **zero matrix $\mathbf{0}$** . Writing

$$P_A(x) = \sum_{k=0}^n p_k x^k \quad ,$$

we have

$$\sum_{k=0}^n p_k A^k = \mathbf{0} \quad ,$$

where $\mathbf{0}$ is the all-zero matrix. Multiplying by A^m we get

$$\sum_{k=0}^n p_k A^{m+k} = \mathbf{0} \quad ,$$

for all m .

Taking the ij entry, we have that each of the n^2 sequences $(A^m)_{ij}$ satisfy the **same** n^{th} -order linear recurrence equation with **constant coefficients**

$$\sum_{k=0}^n p_k (A^{m+k})_{ij} = 0 \quad .$$

It is well-known and easy to see ([1][8]) that any $n+1$ sequences that satisfy the **same** recurrence of order n , must be **linearly dependent**. Also, in order to find the relation, it is enough to find $n+1$ values of the q_s for which (1) holds for the $m = 1, 2, \dots, n$. Then this linear combination also satisfies that very same linear recurrence, and since it **vanishes** at the first n **initial conditions** it must be **identically zero**.

If our subset S of entries only consists of non-diagonal entries, we can do even better, Eq. (1) is true for any set S of n non-diagonal entries.

Note that the Cayley-Hamilton equation implies that

$$\sum_{k=1}^n p_k A^k$$

is a **diagonal** matrix (namely $-\det(A)\mathbf{I}$), hence for a non-diagonal entry ij ($i \neq j$), $(A^m)_{ij}$ always satisfies the **same** linear recurrence equation (with constant coefficients) of order $n-1$, hence any n such non-diagonal entries must be **linearly dependent**. In the original case of a 2×2 matrix discussed in [2], it follows that the $(1, 2)$ and $(2, 1)$ entries of A^m always satisfy the **same** relation as those of A , hence we have yet-another-proof (without equations!) of Larcombe's amazing discovery.

But what about higher dimensions? Now things get much more complicated, and we need a computer algebra system (in our case Maple). For example, we have the following

Theorem: Let $A = (a_{ij})_{1 \leq i, j \leq 3}$ be a 3×3 matrix, then for **all** $m \geq 1$, we have

$$\begin{aligned} & (a_{12}a_{21}a_{23} - a_{13}a_{21}a_{22} + a_{13}a_{21}a_{33} - a_{13}a_{23}a_{31}) \cdot (A^m)_{12} \\ & + (a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}) \cdot (A^m)_{13} \\ & + (-a_{12}^2a_{23} + a_{12}a_{13}a_{22} - a_{12}a_{13}a_{33} + a_{13}^2a_{32}) \cdot (A^m)_{21} = 0 \quad . \end{aligned}$$

There are three more such theorems (up to trivial isomorphism) for the $n = 3$ case, while there are 27 inequivalent cases for $n = 4$. They can all be found in the following output file

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/oLarcombe1.txt>

We were unable to find the corresponding relations for $n = 5$, they got too complicated!

A bijective proof of the Larcombe-Fennessey theorem about Tridiagonal matrices

Any matrix identity for a fixed dimension is essentially *high-school algebra* and can be verified by a computer algebra system, even if the power m is arbitrary. But the following theorem of Larcombe and Fennessey, regarding **tridiagonal** matrices of *arbitrary* dimension is *university algebra* and is more interesting.

Theorem (Larcombe and Fennessey [3][6]) : Let A be a general $n \times n$ tridiagonal matrix (for any $n \geq 2$), then for all $1 \leq i < n$, and all $m \geq 1$, we have

$$a_{i,i+1} \cdot (A^m)_{i+1,i} = a_{i+1,i} \cdot (A^m)_{i,i+1} \quad . \quad (2)$$

For a concrete example see [3].

We will give a **combinatorial proof**. Fix n and let $a_{i,j}$ ($1 \leq i, j \leq n$) be n^2 **commuting** indeterminates. It follows immediately from the definition of matrix multiplication that the (i, j) entry of A^m is the **weight-enumerator** of the set of $(m + 1)$ -letter words in the alphabet $\{1, 2, \dots, n\}$ whose first letter is i and last letter is j , with the weight

$$Weight(w_1 \dots w_{m+1}) := a_{w_1 w_2} \cdot a_{w_2 w_3} \cdot \dots \cdot a_{w_{m-1} w_m} \cdot a_{w_m w_{m+1}} \dots$$

For example $Weight(123123) = a_{12}a_{23}a_{31}a_{12}a_{23}$.

If our matrix is *tridiagonal*, then all the words are *continuous* i.e. after the letter i can only come one of the (up to) three letters $\{i - 1, i, i + 1\}$. For example if $n = 5$ then the following is a legal word

$$2333234333221122112234455443 \quad ,$$

but the following one is **not**

$$233312 \quad ,$$

because after the fourth letter, that is a ‘3’, comes the letter ‘1’.

Let $\mathcal{W}_m(i, j)$ be the set of legal $(m + 1)$ -letter words in the alphabet $\{1, 2, \dots, n\}$ that start with the letter i and end with the letter j . Its weight-enumerator (i.e. sum of the weights of its members) is $(A^m)_{ij}$, where now A is a generic $n \times n$ **tridiagonal** matrix. For ease of type-setting let $i' := i + 1$.

Note that:

- The left side of (2) is the weight-enumerator of the set of words, $i \mathcal{W}_m(i', i)$, which is the set of **legal** $(m + 2)$ -letter words that start and end with the letter i , and whose **second** letter is i' .
- The right side of (2) is the weight-enumerator of $i' \mathcal{W}_m(i, i')$ which is the set of **legal** $(m + 2)$ -letter words that start and end with the letter i' , and whose **second** letter is i .

We claim that The mapping

$$T_i : i \mathcal{W}_m(i', i) \rightarrow i' \mathcal{W}_m(i, i') \quad ,$$

to be defined next, is a **weight-preserving** bijection.

Let $w = w_1 w_2 \dots w_{m+2}$ be a member of $i \mathcal{W}_m(i', i)$, then of course $w_1 = i$ and $w_2 = i'$, and $w_{m+2} = i$. Let k be the **smallest** index such that $w_k = i'$, $w_{k+1} = i$. Of course it exists (by “*continuity*”).

Case I: $k = 2$.

If $m = 1$ then the word must be $i i' i$ and we map it to $i' i i'$.

Otherwise we can write

$$w = i i' i u i \quad ,$$

for some $(m - 2)$ -letter word u , and we define

$$T_i(w) := i' i u i i' \quad ,$$

that of course belongs to $i' \mathcal{W}_m(i, i')$.

Case II: $k = m + 1$, then we can write

$$w = i' i' u i' i \quad ,$$

for some $(m - 2)$ -letter word u , and we map it to the

$$T_i(w) := i' i i' u i' \quad ,$$

that of course belongs to $i' \mathcal{W}_m(i, i')$.

Case III: $2 < k < m + 1$. Then we can write

$$w = i' i' u i' i v i \quad ,$$

for some words u and v , whose total length is $m - 2$, and we define

$$T_i(w) := i' i v i i' u i' \quad ,$$

that of course belongs to $i' \mathcal{W}_m(i, i')$.

Let's state the inverse mapping

$$U_i : i' \mathcal{W}_m(i, i') \rightarrow i \mathcal{W}_m(i', i) \quad .$$

Let $w = w_1 w_2 \dots w_{m+2}$ be a member of $i' \mathcal{W}_m(i, i')$, then of course $w_1 = i'$ and $w_2 = i$, and $w_{m+2} = i'$. Let k be the **largest** index such that $w_k = i, w_{k+1} = i'$. Of course it exists (by "continuity").

Case I: $k = m + 1$. If $m = 2$ the $w = i' i i'$ and we let $U_i(w)$ be $i i' i$. Otherwise we can write

$$w = i' i u i i' \quad ,$$

for some $(m - 2)$ -letter word u , and we define

$$U_i(w) := i i' i u i \quad .$$

Case II: $k = 2$. We can write

$$w = i' i i' u i'$$

and we define

$$U_i(w) := i i' u i' i \quad .$$

Case III: $2 < k < m + 1$. We can write

$$w = i' i v i i' u i' \quad ,$$

for some words u and v whose lengths add-up to $m - 2$, and we define

$$U_i(w) := i i' u i' i v i \quad .$$

Readers are welcome to play with the Maple package

<https://sites.math.rutgers.edu/~zeilberg/tokhniot/Larcombe.txt>

that contains procedure REL to discover generalized Larcombe relations (mentioned above) and also implements the above bijection (procedures Ti and Ui, and CheckTi verifies it empirically).

In order to use the Maple package, one should have Maple, of course. Then start a Maple session, and type read 'Larcombe.txt'. For on-line help, type

```
ezra();
```

Darij Grinberg's Extension

Darij Grinberg discovered that our argument proves a bit more. Here is what he wrote to us:

"I'd like to remark that (2) holds not only if A is tridiagonal, but more generally if A has the property that

(*) $a_{u,v} = a_{v,u} = 0$ whenever $u \leq i$ and $v > i$ satisfy $v - u > 1$.

(That is, there is a "tridiagonal bottleneck" between i and $i + 1$ in A .) Your proof still applies to this generalization, except that the words should not be "continuous" but rather need to pass the $(i, i + 1)$ checkpoint whenever they cross the border between " $\leq i$ " and " $> i$ ". Instead of continuity, you thus need to make a "what goes up must come down" argument when constructing the bijection."

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