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Reciprocal complementary Wiener index in terms of vertex connectivity, independence number, independence domination number and chromatic number of a graph

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Abstract The reciprocal complementary Wiener index is a Wiener like molecular descriptor. It was introduced in the chemical graph theory and has shown to be useful. Reciprocal complementary Wiener index of a graph G is defined as

$$RCW(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{1 + D - d_G(u, v)},$$

where D is the diameter of a graph G and $d_G(u, v)$ is the distance between the vertices u and v. The aim of this paper is to determine new inequalities involving the reciprocal complementary Wiener index (RCW) and characterize graphs extremal with respect to them.

1 Introduction

Let G be a simple, undirected, connected graph with n vertices and m edges and $\bar{G}$ is its complement with $\bar{m}(= \binom{n}{2} - m)$ edges. The distance between the vertices u and v, denoted by $d_G(u, v)$, is the length of the shortest path between them. The eccentricity of v, denoted $e(v)$, is defined to be the greatest distance from v to any other vertex. The diameter of a graph G, denoted by $diam(G) = D$ is the maximum distance between any pair of vertices of G. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and is defined as the number of edges that are incident with v in the graph G. The minimum vertex degree is denoted by δ, the maximum by $\triangle$. As usual, we denote by P_n the path, by C_n the cycle, by S_n the star, by K_n the complete graph, each on n vertices.

The vertex connectivity k of a graph G is defined to be the minimum number of vertices whose removal from G results in to a disconnected or a trivial graph. A subset S of a vertex set $V(G)$ of a graph G is said to be an independent set, if no two vertices of S are adjacent in G. The independence number $\beta_0(G)$ of G is the maximum number of vertices in the independent sets in G. A dominating set for a graph $G = (V, E)$ is a subset $\mathbb{D}$ of V such that every vertex not in $\mathbb{D}$ is adjacent to at least one member of $\mathbb{D}$. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G. A dominating set $\mathbb{D}$ of a graph G is an independent dominating set if the induced subgraph $< \mathbb{D} >$ has no edges. The independent domination number $\gamma_0(G)$ of a graph G is the minimum cardinality of an independent dominating set. A chromatic number of graph G , written $\chi(G)$ is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. Related to the notations, undefined terminologies reader can refer [2, 10].

Topological indices are numerical quantities of molecular graphs (or simple graphs), that are invariant under graph isomorphism and are used to correlate with various physical properties, chemical reactivity or biological activity. There are numerous of topological indices that have found some applications in theorotical chemistry, especially in QSPR/QSAR research. With in all topological indices one of the most important, widely studied and oldest topological index is the Wiener index [32]. In light of Wiener's definition Hosoya formalized it as Wiener number by means of the distance matrix [11]. Wiener index of a graph G is defined as follows

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v). \tag{1.1}$$

It has innumerable applications for designing quantitative structure property relationships (QSPR) [15, 16]. Mathematical research on $W(G)$ started in 1976 [6], since then this distance based quantity was much studied in [4, 5, 17, 20, 22, 23, 31, 33, 34] and see the references cited there in and recent researches concerning this quantity; see for instance [1, 8, 9, 18, 19, 21, 24, 27].

The complementary Wiener index [13, 29]of a graph G denoted by $CW(G)$ is defined as

$$CW(G) = \sum_{\{u,v\}\subseteq V(G)} (1 + D - d_G(u,v)), \tag{1.2}$$

where D is the diameter of graph G. The reciprocal complementary Wiener index [12, 13] of a graph G is denoted by $RCW(G)$ and is defined as follows

$$RCW(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{1 + D - d_G(u,v)}. \tag{1.3}$$

In the family of Wiener-like molecular descriptors [30], RCW index is the newest addition. It has been introduced by Ivanciuc et al. [12, 13, 14, 15]. The RCW index has been successfully applied in the structure-property modeling of the molar heat capacity, standard Gibbs energy of formation and vaporization enthalpy, refractive index, and density of 134 alkanes $C_6 - C_{10}$ [12, 35]. For recent results we refer the reader to see [3, 26, 28, 35] and for a survey paper [34].

The reciprocal complementary distance number of a vertex u of a graph G is denoted by $RCD(u|G)$ and is defined as

$$RCD(u|G) = \sum_{v\in V(G)} \frac{1}{1 + D - d_G(u,v)}. \tag{1.4}$$

From (1.4) we can rewrite (1.3) as

$$RCW(G) = \frac{1}{2}\sum_{u\in V(G)} RCD(u|G). \tag{1.5}$$

In [35] Zhou et al. determined various lower and upper bounds for the RCW index and also given Nordhaus-Gaddum type results for the same which states that

$$RCW(G) + RCW(\bar{G}) \geq RCW(P_n) + RCW(\bar{P_n}).$$

Further more, in [34] Xu et al. obtained that, for any tree on n vertices, the path P_n and $S_n(or K_{1,n-1})$ attains minimum value and maximum value for the RCW index respectively.

$$RCW(P_n) \leq RCW(T) \leq RCW(S_n).$$

In the continuation of the study on RCW index Qi and Zhou [25] characterized the trees of fixed number of vertices and matching number with the smallest RCW index, and the non-caterpillars on $n \geq 7$ vertices with the smallest, the second- smallest and the third-smallest RCW index.

The aim of this paper is to obtain new inequalities involving the reciprocal complementary Wiener index and characterize graphs extremal with respect to them. In particular, we provide lower and upper bounds on RCW of a graph G in terms of some graph parameters like n, m, D, vertex connectivity(k), independence number(β_0), independence domination number(γ_0) and chromatic number(χ).

2 Lower Bounds on RCW

In this section, we establish lower bounds on RCW of a graph G in terms of number of vertices n, number of edges m, diameter D, vertex connectivity (k), independence number(β_0), independence domination number(γ_0), and chromatic number(χ).

Theorem 2.1. *Let G be a non-complete connected graph of order n, size m with $D \geq 3$. Then*

$$RCW(G) \geq \frac{1}{(D-1)}\left[\frac{n(n-1)}{2} + \frac{1}{D-2} - \frac{m}{D}\right]. \tag{2.1}$$

Equality holds if and only if G contains exactly two vertices of eccentricity three and rest are of eccentricity two.

Proof. Let $u \in V$ be an arbitrary vertex in G and $D \geq 3$. Then define two sets A and B as $A = \{u \in V | e(u) = 2\}$, $B = \{u \in V | e(u) \geq 3\}$. Then $|A| + |B| = n$.
Now, consider two cases which are as follows
Case a): If $u \in A$, we define two sets A_1 and A_2 as follows
$A_1 = \{v \in V | 1/1 + D - d_G(u, v) = 1/D\}, A_2 = \{v \in V | 1/1 + D - d_G(u, v) = 1/D - 1\}$.
By (1.4), we get

$$\begin{aligned} RCD(u|G) &= \frac{|A_1|}{D} + \frac{|A_2|}{D-1} \\ &= |A_1| + |A_2| + \frac{|A_1|}{D} - |A_1| + \frac{|A_2|}{D-1} - |A_2| \\ &= n - 1 + \frac{1-D}{D}|A_1| + \frac{(2-D)(n-1-|A_1|)}{D-1} \\ &= \frac{D(n-1) - d_G(u)}{D(D-1)}. \end{aligned} \tag{2.2}$$

Case b): If $u \in B$, we define three sets
$B_1 = \{v \in V | 1/1 + D - d_G(u, v) = 1/D\}, B_2 = \{v \in V | 1/1 + D - d_G(u, v) = 1/D - 1\}$,
$B_3 = \{v \in V | 1/1 + D - d_G(u, v) \geq 1/D - 2\}$.
Clearly, $|B_1| + |B_2| + |B_3| = n - 1$.
Then by (1.4), we get

$$RCD(u|G) = \frac{D^2 n - D^2 - 2Dn + 3D - Dd_G(u) + 2d_G(u)}{D(D-1)(D-2)}. \tag{2.3}$$

Now from (1.5), we have the following

$$\begin{aligned} RCW(G) &= \frac{1}{2}\sum_{u \in V(G)} RCD(u|G) \\ &= \frac{1}{2}\sum_{u \in A} RCD(u|G) + \frac{1}{2}\sum_{u \in B} RCD(u|G) \end{aligned} \tag{2.4}$$

Using (2.2) and (2.3) in (2.4), then we get

$$\begin{aligned} RCW(G) &\geq \frac{1}{2D(D-1)(D-2)}\begin{pmatrix} (D^2 n - D^2)(|A| + |B|) - D\sum_{u \in A} d_G(u) \\ -2Dn(|A| + |B|) + 2D|A| + 3D|B| \\ +2\sum_{u \in B} d_G(u) \end{pmatrix} \\ &= \frac{n(D^2 n - D^2) - 2mD - 2Dn^2 + 2D(|A| + |B|) + D|B| + 4m}{2D(D-1)(D-2)} \\ &\geq \frac{D^2 n(n-1) - 2m(D-2) - 2Dn(n-1) + 2D}{2D(D-1)(D-2)} \qquad \text{since } |B| \geq 2, \\ &= \frac{1}{(D-1)}\left[\frac{n(n-1)}{2} + \frac{1}{D-2} - \frac{m}{D}\right]. \end{aligned}$$

□

Theorem 2.2. *Let G be a non-complete connected graph of order n connectivity k and $H_1, H_2, \ldots, H_t$ be the connected components of $G-S$, where S be any cut set and $|S|=k$. Then*

$$RCW(G) \geq \frac{1}{2D}\left[n(n-1)-\left(\frac{2l(l+k-n)}{D-1}\right)\right],$$

where $l=\min_{1\leq i\leq t}\{|V(H_i)|\}$.
Further, the equality holds if and only if $G=K_l+K_k+K_{n-l-k}$.

Proof. Let G be a graph with n vertices and $|S|=k$, where S be any cut-set of G. Let H_i for $i=1,2,\ldots,t$ be the connected components of $G-S$, with $l=\min_{1\leq i\leq t}\{|V(H_i)|\}$. Without loss of generality, assume that $|V(H_1)|=l$, $G_1=H_1$ and $G_2=\bigcup_{i=2}^{t}H_i$.
Then, $|V(G_1)|=l$ and $|V(G_2)|=n-k-l$. Now we have

$$\begin{aligned} RCW(G) &= \frac{1}{2}\sum_{u\in V(G)} RCD(u|G) \\ &= \frac{1}{2}\left[\sum_{u\in V(G_1)} RCD(u|G)+\sum_{u\in S} RCD(u|G)+\sum_{u\in V(G_2)} RCD(u|G)\right]. \end{aligned} \tag{2.5}$$

The result follows in three cases: which are as follows
Case a): Let $u\in V(G_1)$. Then we have

$$\begin{aligned} RCD(u|G) &= \sum_{v\in V(G)}\frac{1}{1+D-d_G(u,v)} \\ &= \sum_{v\in V(G_1)}\frac{1}{1+D-d_G(u,v)}+\sum_{v\in S}\frac{1}{1+D-d_G(u,v)} \\ &+ \sum_{v\in V(G_2)}\frac{1}{1+D-d_G(u,v)} \\ &\geq \frac{l-1}{D}+\frac{k}{D}+\frac{n-l-k}{D-1} \\ &= \frac{D(n-1)-(l+k-1)}{D(D-1)}. \end{aligned} \tag{2.6}$$

Since $d_G(u,v)\geq 1$, if $v\in V(G_1)$, $v\in S$ and $d_G(u,v)\geq 2$, if $v\in V(G_2)$.
Case b) : Let $u\in S$. Then,

$$\begin{aligned} RCD(u|G) &= \sum_{v\in V(G)}\frac{1}{1+D-d_G(u,v)} \\ &= \sum_{v\in V(G_1)}\frac{1}{1+D-d_G(u,v)}+\sum_{v\in S}\frac{1}{1+D-d_G(u,v)} \\ &+ \sum_{v\in V(G_2)}\frac{1}{1+D-d_G(u,v)} \\ &\geq \frac{l}{D}+\frac{k-1}{D}+\frac{n-l-k}{D} \\ &= \frac{n-1}{D}. \end{aligned} \tag{2.7}$$

Since $d_G(u,v)\geq 1$, if the vertex v is in either sets $V(G_1)$, S and $V(G_2)$.
Case c) : Let $u\in V(G_2)$. Then we have the following

$$\begin{aligned} RCD(u|G) &\geq \frac{l}{D-1}+\frac{k}{D}+\frac{n-l-k-1}{D} \\ &= \frac{D(n-1)-(n-l-1)}{D(D-1)}. \end{aligned} \tag{2.8}$$

Using (2.6), (2.7) and (2.8) in (2.5), then we get the required result. Proof of second part of the Theorem 2.2 holds from the proof of inequality itself. □

Theorem 2.3. *Let G be a non-complete connected graph of order n. Then*

$$RCW(G) \geq \frac{1}{2D}\left[n(n-1)+\frac{\beta_0(\beta_0-1)}{(D-1)}\right].$$

Equality holds if and only if $G=\overline{K}_{\beta_0}+K_{n-\beta_0}$.

Proof. Let S be the maximum independent set with $|S|=\beta_0$ and u be any vertex in S. Then

$$\begin{aligned} RCD(u|G) &= \sum_{v\in V(G)} \frac{1}{1+D-d_G(u,v)} \\ &= \sum_{v\in S} \frac{1}{1+D-d_G(u,v)} + \sum_{v\in V-S} \frac{1}{1+D-d_G(u,v)} \\ &\geq \frac{\beta_0-1}{D-1}+\frac{n-\beta_0}{D} \\ &= \frac{1}{D}\left[n-\left(\frac{D-\beta_0}{D-1}\right)\right]. \end{aligned} \tag{2.9}$$

Since $u\neq v$ and $u\in S$, so that there are (β_0-1) vertices in S which are at distance at least two from u and $d_G(u,v)\leq 1$, for any $v\in V-S$.
Next, Let $u\in V-S$. Then

$$\begin{aligned} RCD(u|G) &= \sum_{v\in V(G)} \frac{1}{1+D-d_G(u,v)} \\ &\geq \frac{n-1}{D}. \end{aligned} \tag{2.10}$$

Therefore, from (1.5), we have the following

$$\begin{aligned} RCW(G) &= \frac{1}{2}\sum_{u\in V(G)} RCD(u|G) \\ &= \sum_{u\in S} RCD(u|G)+\sum_{u\in V-S} RCD(u|G) \\ &\geq \frac{1}{2}\left[\left(\frac{1}{D}\left[n-\left(\frac{D-\beta_0}{D-1}\right)\right]\right)\beta_0+(n-\beta_0)\left(\frac{n-1}{D}\right)\right] \text{ from (2.9) and (2.10)} \\ &= \frac{1}{2D}\left[n(n-1)+\frac{\beta_0(\beta_0-1)}{(D-1)}\right]. \end{aligned}$$

For the equality, one can easily see that equality holds if and only if the graph $G=\overline{K}_{\beta_0}+K_{n-\beta_0}$.

Conversely, suppose the equality holds, then we have to prove $G=\overline{K}_{\beta_0}+K_{n-\beta_0}$. If possible assume that $G\neq\overline{K}_{\beta_0}+K_{n-\beta_0}$. Let S be the maximum independent set with $|S|=\beta_0$ in G. For any two vertices u and v in G, the reciprocal complementary distance is $\frac{1}{D-1}$ and $\frac{1}{D}$ if both u and v are in S and $V-S$ respectively otherwise, it will lead to $RCW(G) > \frac{1}{2D}\left[n(n-1)+\frac{\beta_0(\beta_0-1)}{(D-1)}\right]$, a contradiction. Thus $\langle S\rangle=\overline{K}_{\beta_0}$ and $\langle V-S\rangle=K_{n-\beta_0}$. Further, if $u\in S$ and $v\in V-S$, we claim that reciprocal complementary distance is $\frac{1}{D}$, for, otherwise $RCD(v_i|G)>\frac{n-\beta_0}{D}$ and there by $RCW(G)>\frac{1}{2D}\left[n(n-1)+\frac{\beta_0(\beta_0-1)}{(D-1)}\right]$ holds, a contradiction. Thus $G=\overline{K}_{\beta_0}+K_{n-\beta_0}$. This completes the proof. □

Theorem 2.4. *Let G be any connected graph of order n. Then,*

$$RCW(G) \geq \frac{1}{2D}\left[n(n-1)+\frac{\gamma_0(\gamma_0-1)}{(D-1)}\right].$$

Proof. The proof follows directly from Theorem 2.3. □

Theorem 2.5. *Let G be a non-complete connected graph of order n with chromatic number $\chi(G) = t$. Then*

$$RCW(G) \geq \frac{1}{2D(D-1)} \left[Dn(n-1) - n^2 + \sum_{i=1}^{t} n_i^2 \right]. \tag{2.11}$$

Equality holds if and only if $G = K_{n_1,n_2,\ldots,n_t}$.

Proof. Suppose $\chi(G) = t$, then the vertex set $V(G)$ of G can be partitioned into t color classes $\zeta_1, \zeta_2, \ldots, \zeta_t$ such that no two vertices in any ζ_i adjacent and let $|\zeta_i| = n_i$, for $i = 1, 2, \ldots, t$. Thus, $n = \sum_{1=1}^{t} n_i$. Let $u \in \zeta_i$, for $i = 1, 2, \ldots, t$. Then from (1.4), we have

$$\begin{aligned} RCD(u|G) &= \sum_{v \in V(G)} \frac{1}{1 + D - d_G(u,v)} \\ &= \sum_{v \in \zeta_i} \frac{1}{1 + D - d_G(u,v)} + \sum_{v \in V - \zeta_i} \frac{1}{1 + D - d_G(u,v)} \\ &\geq \frac{n_i - 1}{D - 1} + \frac{n - n_i}{D} \\ &= \frac{D(n-1) - n + n_i}{D(D-1)}. \end{aligned} \tag{2.12}$$

Since $d_G(u,v) \geq 2$, if $v \in \zeta_i$ and $d_G(u,v) \geq 1$, if $v \in V - \zeta_i$. Therefore

$$\begin{aligned} RCW(G) &= \frac{1}{2} \sum_{u \in V(G)} RCD(u|G) \\ &= \frac{1}{2} \left[\sum_{i=1}^{t} \sum_{u_i \in \zeta_i} \frac{1}{1 + D - d_G(u,v)} \right] \\ &\geq \frac{1}{2} \left[\sum_{i=1}^{t} \sum_{u_i \in \zeta_i} \frac{D(n-1) - n + n_i}{D(D-1)} \right] \text{ from Eq. (2.12)} \\ &= \frac{1}{2} \left[\frac{Dn(n-1) - n^2 + \sum_{i=1}^{t} n_i^2}{D(D-1)} \right]. \end{aligned}$$

Further, one can easily see that equality holds in (2.11) for a graph $G = K_{n_1,n_2,\ldots,n_t}$. On the other hand, if the equality holds in (2.11) and $\chi(G) = t$, then the vertex set $V(G)$ can be partitioned into the color classes $\zeta_1, \zeta_2, \ldots, \zeta_t$ such that $|\zeta_i| = n_i$, for $i = 1, 2, \ldots, t$. Now, we claim that any two vertices u and v belonging to two different color classes are adjacent. For if $u \in \zeta_i$, and $v \in \zeta_j$ for $i \neq j$ are not adjacent then $\sum_{v \in V - \zeta_i} \frac{1}{1+D-d_G(u,v)} > \frac{n-n_i}{D}$, which in turn implies that, $RCD(u|G) > \frac{D(n-1)-n+n_i}{D(D-1)}$ and there by it will lead to

$$RCW(G) > \frac{1}{2} \left[\frac{Dn(n-1) - n^2 + \sum_{i=1}^{t} n_i^2}{D(D-1)} \right]$$

a contradiction. Again, if both u and v belongs to the same color class then the reciprocal complementary distance is $\frac{1}{D-1}$, otherwise it leads to the same contradiction. Hence $G = K_{n_1,n_2,\ldots,n_t}$ holds. □

Note: For any graph G, we know that $D \leq (n - \triangle + 1)$ [7]. Suppose $\triangle$ and δ are the maximum and minimum degrees respectively then, diameter is of the form $D \leq n - (\triangle + \delta) + 2$.

Proposition 2.6. *[35] Let G be a non complete connected graph with $n \geq 3$ vertices and m edges. Then*

$$RCW(G) \leq \frac{n(n-1)-m}{2} \tag{2.13}$$

with equality if and only if G has diameter 2.

Theorem 2.7. *Let G be a non complete connected graph with n vertices and m edges, δ and $\triangle$ are the minimum and maximum degree respectively. Then*

$$RCW(G) \geq \left[\frac{1}{n-\triangle}\right]\left[\frac{n(n-1)}{2} - \frac{m}{n-\triangle+1}\right]. \tag{2.14}$$

Equality holds if and only if atleast one vertex should have maximum degree $\triangle = n-1$ in G or $G \cong S_n$.

Proof. Since $D \leq n - \triangle + 1$. Now, we have m – pairs are at distance 1 and $\left[\binom{n}{2} - m\right]$ – pairs are at distance 2. From (1.3), we get the required result. □

Theorem 2.8. *Let G be a graph with order n, size m and $\triangle + \delta \leq n$. Then*

$$RCW(G) \geq \left[\frac{1}{n-(\triangle+\delta)+1}\right]\left[\frac{n(n-1)}{2} - \frac{m}{n-(\triangle+\delta)+2}\right]. \tag{2.15}$$

Equality holds if and only if $D = n - (\triangle + \delta) + 2$.

Proof. We have $D \leq n - (\triangle + \delta) + 2$. Thus the result is follows from Theorem 2.7. □

Remark 2.9. If the graph G has maximum degree $\triangle = n-1$ and $\triangle + \delta = n$ in (2.14) and (2.15), respectively, then both equations reduces to the equality part of (2.13).

3 Upper bounds on RCW(G)

In this section, we give upper bounds for RCW of graph G in terms of n, m and D. The *radius* of graph G is defined as, the minimum eccentricity among all vertices of G and denoted as $rad(G)$.

Theorem 3.1. *Let G be a connected graph of order n, size m with $D = rad(G) = 3$. Then*

$$RCW(G) \leq \frac{n(n-2)}{2} - \frac{2m}{3}.$$

Equality holds when $G \cong C_6$.

Proof. Since G be a graph with $D = rad(G) = 3$, we have $e(u) = 3$, for every vertex u in G. Define the sets $A_i(u) = \{v \in V \mid \frac{1}{1+D-d_G(u,v)} = \frac{1}{4-i}\}$ for $i = 1, 2, 3$.
Clearly, $\mid \bigcup_{i=1}^{3} A_i(u) \mid = n$.

$$\mid A_2(u) \mid + \mid A_3(u) \mid = n - 1 - d_G(u). \tag{3.1}$$

Since $|A_1(u)| = d_G(u)$ and $|A_2(u)| \geq 2$, for otherwise, there is a vertex $w \in A_2(u)$ such that $e(w) \leq 2$, a contradiction. Thus,

$$\begin{aligned} RCD(u|G) &= \sum_{v \in V} \frac{1}{1+D-d_G(u,v)} \\ &= \frac{|A_1(u)|}{D} + \frac{|A_2(u)|}{D-1} + \frac{|A_3(u)|}{D-2} \\ &= \frac{1}{3}|A_1(u)| + \frac{1}{2}|A_2(u)| + |A_3(u)| \\ &= \frac{d_G(u)}{3} + \frac{1}{2}\left[|A_2(u)| + |A_3(u)|\right] + \frac{1}{2}|A_3(u)|. \end{aligned} \tag{3.2}$$

Now, from (3.1), we have the following

$$\begin{aligned} |A_3(u)| &= n-1-d_G(u)-|A_2(u)| \\ &\leq n-1-d_G(u)-2 \qquad \text{since } |A_2(u)|\geq 2 \\ &= n-3-d_G(u). \end{aligned}$$

Using (3.1) and above argument in (3.2), we get

$$RCD(u|G) \leq \frac{3n-6-d_G(u)}{3}. \tag{3.3}$$

Next, using (3.3) in (1.5), we get the required result. □

Theorem 3.2. *Let G be a connected graph of order n, size m and $D = rad(G) = \alpha \geq 3$. Then*

$$RCW(G) \leq \frac{1}{D-1}\left[\frac{n(n-1)}{2} - \frac{m}{D} + n\left(\sum_{i=3}^{\alpha-1}\frac{(i-2)}{(D-(i-1))} + \frac{(\alpha-2)}{2(D-(\alpha-1))}\right)\right]. \tag{3.4}$$

Equality holds if and only if $G \cong C_{2\alpha}$.

Proof. Let u be any vertex in G, then define the set $A_i(u) = \{v \in V \mid \frac{1}{1+D-d_G(u,v)} = \frac{1}{D-(i-1)}\}$, for $i = 1, 2, \ldots, \alpha$. By (1.4), we have

$$\begin{aligned} RCD(u|G) &= \frac{|A_1(u)|}{D} + \frac{|A_2(u)|}{D-1} + \frac{|A_3(u)|}{D-2} + \ldots + \frac{|A_\alpha(u)|}{D-(\alpha-1)} \\ &= \frac{d_G(u)}{D} + \frac{n-1-d_G(u)}{D-1} + \sum_{i=3}^{\alpha-1}\frac{(i-2)|A_{\alpha-1}(u)|}{(D-(i-1))(D-1)} \\ &+ \frac{(\alpha-2)|A_\alpha(u)|}{2(D-(\alpha-1))(D-1)}. \end{aligned} \tag{3.5}$$

Since, $|A_\alpha(u)| \geq 1$ and $|A_i(u)| \geq 2$, for $i = 1, 2, \ldots, \alpha-1$, then (3.5) becomes

$$\begin{aligned} RCD(u|G) &\leq \frac{n-1}{D-1} - \frac{d_G(u)}{D(D-1)} \\ &+ 2\sum_{i=3}^{\alpha-1}\frac{(i-2)}{(D-(i-1))(D-1)} + \frac{(\alpha-2)}{(D-(\alpha-1))(D-1)} \end{aligned} \tag{3.6}$$

Using (3.6) in (1.5), we get the required result and easily we can see that, the equality is holds for the graph $G \cong C_{2\alpha}$.

Next, For the equality in (3.4), we have to prove that $G \cong C_{2\alpha}$. Suppose $G \ncong C_{2\alpha}$, then $|A_{1i}(u)| \geq 3$, for $i = 2, 3, \ldots, \alpha-1$. This implies (3.6) becomes

$$RCD(u|G) \leq \frac{n-1}{D-1} - \frac{d_G(u)}{D(D-1)} + 3\sum_{i=3}^{\alpha-1}\frac{(i-2)}{(D-(i-1))(D-1)} + \frac{(\alpha-2)}{(D-(\alpha-1))(D-1)}.$$

Hence from (1.5), we get

$$RCW(G) \leq \frac{1}{D-1}\left[\frac{n(n-1)}{2} - \frac{m}{D} + \frac{n}{2}\left(3\sum_{i=3}^{\alpha-1}\frac{(i-2)}{(D-(i-1))} + \frac{(\alpha-2)}{(D-(\alpha-1))}\right)\right].$$

This is a contradiction, so that $G \cong C_{2\alpha}$. This completes the proof. □

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