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Primary Ideals of Lie Algebras

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Abstract In this paper, we introduce and advance the basic theory of primary ideals for Lie algebras and investigate their properties in details illustrated by several examples. We give some characterizations for ideals to be primary ideals. We also introduce the concept of strongly irreducible ideals. We study the interrelations among primary, prime, semi-prime, strongly irreducible, irreducible and maximal ideals in Lie algebra. We show that the maximal ideal is primary ideal. We also show that the concepts of strongly irreducible, primary and prime ideals are all equivalent for prime radical ideals.

1 Introduction

The concept of primary ideals has played an important role in the theory of rings, but has not been used in the study of Lie algebras. It is the purpose of the present paper to introduce the concept of primary ideals into Lie algebras and investigate their properties in details illustrated by several examples. We give some characterizations for ideals to be primary ideals. We also introduce the concept of strongly irreducible ideals. We study the interrelations among primary, prime, semi-prime, strongly irreducible, irreducible and maximal ideals. We show that the maximal ideal is primary. We also show that the concepts of strongly irreducible, primary and prime ideals are all equivalent for prime radical ideals.

Let F be a field of an arbitrary characteristic. Let L be always a Lie algebra over F, which is not necessarily finite dimensional.

The definitions and results given in this section are based on the work of Kawamoto in [1] and Aldosray in [2].

An ideal P of L is said to be prime if $[A,\ B] \subseteq P$ with A, B ideals of L implies $A \subseteq P$ or $B \subseteq P$.

The prime radical of an ideal A of L is the intersection of all the prime ideals of L containing A and is denoted by $r(A)$. We write r_L for $r(0)$, the intersection of all the prime ideals of L, and call it the prime radical of L. We say that an ideal A of L is prime radical ideal if $r(A) = A$.

An ideal Q of L is said to be semi-prime if $A^2 \subseteq Q$ with A an ideal of L implies $A \subseteq Q$.

It is clear that every prime ideal of L is prime radical ideal of L. Counterexample that the converse is not true was given by Kawamoto in [1].

2 Primary ideals

In this section, we introduce and study a new generalizations of prime ideals in Lie algebra.

Theorem 2.1. *The intersection of prime ideals of L is a semi-prime ideal of L.*

Proof. First, note that the intersection of ideals in L is an ideal of L, [3]. Now, let $\{P_\alpha : \alpha \in \Lambda\}$ be a collection of prime ideals of L, suppose that $A^2 \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha$ with A is an ideal of L, then $A^2 \subseteq P_\alpha$ for every $\alpha \in \Lambda$. Since each P_α is a prime ideal of L, then $A \subseteq P_\alpha$ for every $\alpha \in \Lambda$. Thus $A \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha$. Therefore, $\bigcap_{\alpha\in\Lambda} P_\alpha$ is a semi-prime ideal of L. □

Corollary 2.2. *The prime radical of any ideal in L is a semi-prime ideal of L.*

Corollary 2.3. *Let Q be an ideal of L. If $A^2 \subseteq Q$ with A an ideal of L, then $A \subseteq r(Q)$.*

Now, we introduce the concept of primary ideals in L as follows.

Definition 2.4. An ideal P of L is said to be **primary** if $[A, B] \subseteq P$ with A, B ideals of L implies $A \subseteq r(P)$ or $B \subseteq r(P)$.

Theorem 2.5. *If $r(P)$ is a prime ideal of L, then P is a primary ideal of L.*

Proof. Let A and B be two ideals of L with $[A, B] \subseteq P$. Since $P \subseteq r(P)$, then $[A, B] \subseteq r(P)$, but $r(P)$ is prime, thus $A \subseteq r(P)$ or $B \subseteq r(P)$. Therefore P is primary. □

By definition of prime radical and Theorem 2.5, we get the following corollary:

Corollary 2.6. *Every prime ideal of L is primary.*

The converse of Corollary 2.6 need not be true. In the next example, we show that there exists a primary ideal which is not prime.

Example 2.7. Let $L = gl_n \oplus S$, where gl_n is the set of all $n \times n$ matrices over the field F and S is any simple Lie algebra. The ideals of L are $0, S, sl_n, sl_n \oplus S, Z, Z \oplus S, gl_n$ and L, where sl_n is the set of all $n \times n$ matrices of trace zero and $Z = Z(gl_n) = \{A \in gl_n : [A, B] = AB - BA = 0$ for all $B \in gl_n\}$ is the centre of gl_n. However, $gl_n, Z \oplus S$ and L are the prime ideals only. Now sl_n is not prime ideal because $[Z, gl_n] = 0 \subseteq sl_n$ but neither $Z \subseteq sl_n$ nor $gl_n \subseteq sl_n$. Since $gl_n = r(sl_n) = L \cap gl_n$ is prime ideal, then by Theorem 2.5, sl_n is primary.

Theorem 2.8. *Let P be a primary ideal of L. If P is a prime radical ideal of L, then P is a prime ideal of L.*

Proof. Let A and B be two ideals of L with $[A, B] \subseteq P$, then we get $A \subseteq r(P) = P$ or $B \subseteq r(P) = P$. Thus, P is prime. □

We show that there are no implications between primary and semi-prime.

Example 2.9. Let L be the Lie algebra in Example 2.7, we have sl_n is primary ideal. We show that sl_n is not semi-prime. Now $Z^2 = 0 \subseteq sl_n$ but $Z \nsubseteq sl_n$. Therefore, sl_n can't be semi-prime.

Definition 2.10. ([4]) A Lie algebra L is said to be **simple** if it is non-abelian and contains no non-zero proper ideals.

Example 2.11. ([1]) Let $L = S_1 \oplus S_2 \oplus S_3$, where S_1, S_2 and S_3 be finite-dimensional simple Lie algebras. Then the ideals containing S_1 properly are $S_1 \oplus S_2, S_1 \oplus S_3$ and L. Hence S_1 is semi-prime. Since $[S_1 \oplus S_2, S_1 \oplus S_3] \subseteq S_1$, thus, S_1 is not prime. We prove that S_1 is a semi-prime ideal but not primary ideal. Now $r(S_1) = (S_1 \oplus S_2) \cap (S_1 \oplus S_3) = S_1$. Suppose that S_1 is primary, then it must be prime by Theorem 2.8, which is contradiction. Therefore, S_1 is not primary ideal.

Definition 2.12. ([5]) A Lie algebra L is said to be **semi-simple** if it is a direct sum of simple Lie algebras.

It is clear that every simple Lie algebra is semi-simple.

Lemma 2.13. *If A is an ideal of a semi-simple Lie algevra L, then $A^2 = A$.*

Proof. We can write $L = \bigoplus_{i \in I} S_i$ for some index set I, where S_i is simple ideal.

Now, we have

1. $[S_i, S_j] \subseteq S_i \cap S_j = 0$ for $i \neq j$.

2. Since $S_i^2 \subseteq S_i$, then $S_i^2 = 0$ or $S_i^2 = S_i$, because S_i is simple. Hence S_i is not abelian, thus $S_i^2 \neq 0$. Therefore, $S_i^2 = S_i$.

Let A be an ideal of L, then $A = \bigoplus_{i \in J} S_i$, where $J \subseteq I$.

Consequently

$$A^2 = [A, A] = [\bigoplus_{i \in J} S_i, \bigoplus_{i \in J} S_i] \overset{bilinearity}{=} \bigoplus_{i \neq j} [S_i, S_j] \oplus \bigoplus_{i = j} [S_i, S_j] = 0 \oplus \bigoplus_{i \in J} S_i = A.$$

□

Lemma 2.14. *Every ideal of a semi-simple Lie algebra L is semi-prime ideal.*

Proof. Let Q be an ideal of L, then for any ideal A of L, we have $A = A^2$ because L is a semi-simple Lie algebra, hence $A^2 \subseteq Q$ implies $A \subseteq Q$. Thus Q is semi-prime ideal. □

Definition 2.15. A proper ideal I of L is said to be **strongly irreducible** if for each pair of ideals A and B of L, $A \cap B \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$.

Definition 2.16. ([1]) An ideal N of L is said to be **irreducible** if $N = A \cap B$ with A, B ideals of L implies $N = A$ or $N = B$.

It is clear that every strongly irreducible ideal of L is an irreducible ideal of L.

Lemma 2.17. *If A and B are two ideals of L, then $[A, B] \subseteq A \cap B$*

Proof. Suppose that $z \in [A, B]$, then $z = \sum_i c_i[x_i, y_i]$ where $x_i \in A$ and $y_i \in B$. Hence $c_i[x_i, y_i] \in A$ and $c_i[x_i, y_i] \in B$ for every i, because A and B are ideals of L. Therefore $z \in A \cap B$. □

Theorem 2.18. *Every prime ideal of L is strongly irreducible.*

Proof. Let P be a prime ideal of L and suppose that A and B are two ideals of L. If $A \cap B \subseteq P$, by Lemma 2.17, we have $[A, B] \subseteq A \cap B$, so we obtain $[A, B] \subseteq P$. Since P is prime, then either $A \subseteq P$ or $B \subseteq P$. Thus P is strongly irreducible. □

Theorem 2.19. *Every strongly irreducible ideal of a semi-simple Lie algebra L is prime.*

Proof. Let P be a strongly irreducible ideal of L, then by Lemma 2.14, P is semi-prime. Let A and B be two ideals of L satisfying $[A, B] \subseteq P$. If we put $N = (A + P) \cap (B + P)$, then $N^2 \subseteq P$, hence, $N \subseteq P$. Thus, $A + P \subseteq P$ or $B + P \subseteq P$, because P is strongly irreducible. Therefore, $A \subseteq P$ or $B \subseteq P$. □

Example 2.20. In Example 2.7, we proved that sl_n is primary. Note that sl_n is not irreducible ideal, because $sl_n = gl_n \cap (sl_n \oplus S)$, but neither $sl_n = gl_n$ nor $sl_n = sl_n \oplus S$, hence sl_n is not strongly irreducible. Moreover, $sl_n \oplus S$ is strongly irreducible but not semi-prime because $Z^2 = (0) \subseteq sl_n \oplus S$, however $Z \nsubseteq sl_n \oplus S$.

Example 2.21. In Example 2.11, S_1 is semi-prime. Note that S_1 is not irreducible, because S_1 can be written as intersection of ideals containing it properly, i.e. $S_1 = (S_1 \oplus S_2) \cap (S_1 \oplus S_3)$. Thus, S_1 is not strongly irreducible.

Lemma 2.22. *If M is an ideal of L with $r(M) = L$, then M is primary ideal.*

Proof. Let A and B be two ideals of L such that $[A, B] \subseteq M$ then $A \subseteq L = r(M)$. Moreover, $B \subseteq L = r(M)$, therefore M is primary ideal. □

As in rings [6], we define the maximality of ideals as follows: A proper ideal M of L is said to be **maximal** if it is not properly contained in any other proper ideal of L i.e. $M \subseteq M' \subseteq L$ implies that $M = M'$ or $M' = L$

Kawamoto in [1] proved that every maximal ideal of L is irreducible ideal of L but the converse need not be true. In the next example, we show that there exists a strongly irreducible ideal which is not maximal.

Example 2.23. Let L be a 2-dimensional non-abelian Lie algebra. That is, $L = (x, y)$ with $[x, y] = y$. Then the ideals of L are $(0), (y)$ and L we have (0) is a strongly irreducible but not maximal. (y) is maximal but neither prime nor semi-prime, because $L^2 = (y)$. By definition L is prime but not maximal.

As Kawamoto proved in [1], semi-prime ideal need not be maximal ideal.

Lemma 2.24. *If M is a maximal prime radical ideal of L, then M is a prime ideal of L.*

Proof. Suppose that M is not prime, then L is the only prime ideal containing M. Thus, we have $r(M) = L$, which is a contradiction. Therefore, M is prime. □

In the next theorem, we show that primary concept generalized maximal concept.

Theorem 2.25. *Every maximal ideal M of L is a primary ideal of L.*

Proof. Suppose M is maximal ideal of L, then two cases arise:
Case I: If $r(M) = M$, then by Lemma 2.24, M is prime. Hence M is primary.
Case II: If $r(M) = L$, then by Lemma 2.22, M is primary. □

The converse of Theorem 2.25 need not be true. In the next example, we show that there exists a primary ideal which is not maximal.

Example 2.26. In example 2.23, we have (0) is primary but not maximal.

Remark 2.27. ([7]) For any element x of L, the smallest ideal of L containing x is denoted by $\langle x^L \rangle$.

Theorem 2.28. *Let P be an ideal of L. Then the following conditions are equivalent:*
i) P is primary.
ii) If $[a, B] \subseteq P$ for $a \in L$ and an ideal B of L, then either $a \in r(P)$ or $B \subseteq r(P)$.
iii) If $\left[a, < b^L >\right] \subseteq P$ for $a, b \in L$, then either $a \in r(P)$ or $b \in r(P)$.

Proof. i) ⇒ iii). For each $a \in L$,

$$< a^L >= \sum_{i=0}^{\infty} V_i \qquad where \qquad V_0 = (a) \qquad and \qquad V_i = [\dots[(a), \underbrace{L], \dots, \ L}_{i}].$$

If $\left[a, < b^L >\right] \subseteq P$, we assert that $\left[V_i, < b^L >\right] \subseteq P$ for all $i \geq 0$. In fact, it is true for $i = 0$. Let $i \geq 1$ and assume that the assertion is true for $i - 1$. Then

$$\begin{aligned}\left[V_i, < b^L >\right] &= \left[[V_{i-1}, L], < b^L >\right] \\ &\subseteq \left[\left[V_{i-1}, < b^L >\right], L\right] + \left[V_{i-1}, \left[L, < b^L >\right]\right] \\ &\subseteq [P, L] + \left[V_{i-1}, < b^L >\right] \subseteq P\end{aligned}$$

Thus we have the assertion. It follows that

$$\left[< a^L >, < b^L >\right] \subseteq P.$$

Since P is primary, either $< a^L > \subseteq r(P)$ or $< b^L > \subseteq r(P)$
Thus $a \in r(P)$ or $b \in r(P)$.

iii) ⇒ ii). Let $a \in (L - r(P))$ and let B be an ideal of L such that $[a, B] \subseteq P$. For any $b \in B$, $\left[a, < b^L >\right] \subseteq P$. Since the ideal $< b^L >$ is contained in B. As $a \notin r(P)$ iii) implies $b \in r(P)$. Hence $B \subseteq r(P)$

ii) ⇒ i). Let A and B be two ideals of L such that $[A, B] \subseteq P$ and $A \nsubseteq r(P)$. Since $[a, B] \subseteq P$ for any $a \in (A - r(P))$, we have $B \subseteq r(P)$ by ii).
Therefore P is primary. □

As it is well known, L is said to satisfy the minimal (maximal) condition for ideals, if for each infinite, descending chain of ideals $A_1 \supseteq A_2 \supseteq \dots$ (resp. for each , ascending chain $A_1 \subseteq A_2 \subseteq \dots$) an index m exists such that $A_i = A_k$ if $m < i$, $m < k$. We say in short: $L \in \text{Min} - \triangleleft$ (resp. $L \in \text{Max} - \triangleleft$). [8]
If $L \in \text{Max} - \triangleleft$, then semi-prime ideals have particular characterization.

Lemma 2.29. ([1]) *If $L \in \text{Max} - \triangleleft$ and A be an ideal of L, then the following statements are equivalent:*
i) A is a semi-prime ideal of L.
ii) A is a prime radical ideal of L.
iii) A is a finite intersection of prime ideals of L.

Theorem 2.30. *Let $L \in \mathrm{Max} - \triangleleft$, M is semi-prime and primary ideal if and only if M is prime ideal.*

Proof. Let M be semi-prime ideal of L, then by using Lemma 2.29, we get $r(M) = M$. If M is primary, then M is prime ideal by Theorem 2.8. Conversely, it is clear that every prime ideal is semi-prime and primary ideal. □

Theorem 2.31. *Let $L \in \mathrm{Max} - \triangleleft$, M is semi-prime and strongly irreducible if and only if M is prime ideal.*

Proof. If M is semi-prime ideal of L, then by Lemma 2.29, $r(M) = M$. Now $\bigcap_{\alpha\in\Lambda}\{N_\alpha :$ N_α is prime ideal containing M$\} = r(M) = M$. Since M is strongly irreducible and $N_\beta \subseteq M$ for some $\beta \in \Lambda$, but $M \subseteq N_\beta$, hence $M = N_\beta$ which is prime ideal. Therefore, M is prime ideal. Conversely, it is clear that every prime ideal is semi-prime and strongly irreducible ideal. □

The following corollary is an immediate result of Theorem 2.30 and Theorem 2.31.

Corollary 2.32. *Let $L \in \mathrm{Max}-\triangleleft$, and M be semi-prime ideal of L. Then the following statements are equivalent:*
i) M is prime.
ii) M is strongly irreducible.
iii) M is primary.

3 Primary Lie algebra

Definition 3.1. ([9]) A Lie algebra L is said to be **prime Lie algebra** if $[A, B] \neq (0)$ for any nonzero ideals $A, B \subset L$. A Lie algebra L is said to be **semi-prime Lie algebra** if $[A, A] \neq (0)$ for any nonzero ideal $A \subset L$.

Remark 3.2. A prime (semi-prime) Lie algebra is one in which (0) is prime (semi-prime) ideal of L.

Theorem 3.3. *A Lie algebra with non-zero center can't be semi-prime Lie algebra.*

Proof. Let L be a Lie algebra and $(0) \neq Z$ is the center. Suppose that L is a semi-prime Lie algebra, then (0) is semi-prime ideal. Since $Z^2 \subseteq (0)$, hence $Z \subseteq (0)$, which is a contradiction to $(0) \neq Z$. Therefore, L can't be semi-prime Lie algebra. □

Definition 3.4. A Lie algebra L is said to be **primary Lie algebra** if (0) is primary ideal.

Example 3.5. Let L be as in Example 2.23, (0) is irreducible but neither prime nor semi-prime, for $(y)^2 = (0)$. Apparently (0) is not maximal. As shown in Example 2.23, (y) is maximal but neither prime nor semi-prime. By definition L is prime ideal of L, but not a prime Lie algebra. Since $r(0) = L$, thus (0) is a primary ideal of L. Therefore, L is a primary Lie algebra but neither a prime nor a semi-prime Lie algebra.

Next, we give an example of a semi-simple Lie algebra that is neither prime nor primary Lie algebra.

Example 3.6. Let L be as in Example 2.11. Since $[S_1, S_2] \subseteq (0)$ but neither $S_1 \nsubseteq (0)$ nor $S_2 \nsubseteq (0)$, hence (0) is not prime ideal of L. Moreover, (0) is not primary ideal of L, because (0) is a prime radical ideal which is not a prime ideal of L. Thus L is neither a prime nor a primary Lie algebra.

Theorem 3.7. *A semi-simple Lie algebra is a semi-prime Lie algebra.*

Proof. Let L be a semi-simple Lie algebra. Since every ideal of semi-simple Lie algebra is semi-prime by Lemma 2.14, then (0) is semi-prime. Hence, L is semi-prime Lie algebra. □

Theorem 3.8. *For a proper ideal P in a Lie algebra L, the following conditions are equivalent:*
(i) P is a prime ideal.
(ii) If I and J are two ideals of L properly containing P, then $[I, J] \nsubseteq P$
(iii) L/P is a prime Lie algebra.

Proof. (i) $\Longrightarrow$ (ii) : This is clear.

(ii) $\Longrightarrow$ (iii) : Given ideals I and J in L/P such that $[I, J] = P$ (Zero in L/P), then $I = I'/P$ and $J = J'/P$ for some ideals I' and J' of L. Since $[I, J] = P$, then $[I', J'] \subseteq P$. Thus By (ii), $I' \subseteq P$ or $J' \subseteq P$. Hence $I = P$ (Zero in L/P) or $J = P$ (Zero in L/P). Therefore, L/P is a prime Lie algebra.

(iii) $\Longrightarrow$ (i) : If I and J are two ideals of L satisfying $[I, J] \subseteq P$, then $(I + P)/P$ and $(J + P)/P$ are ideals of L/P whose product is P (Zero in L/P). Then either $(I + P)/P = P$ or$(J + P)/P = P$ whence either $I \subseteq P$ or $J \subseteq P$. Therefore, P is a prime ideal. □

Theorem 3.9. *For a proper ideal P in a Lie algebra L, the following conditions are equivalent:*
(i) P is a primary ideal.
(ii) If I and J are two ideals of L properly containing $r(P)$, then $[I, J] \nsubseteq P$
(iii) L/P is a primary Lie algebra.

Proof. (i) $\Longrightarrow$ (ii) : This is clear.

(ii) $\Longrightarrow$ (iii) : Given ideals I and J in L/P such that $[I, J] = P$ (Zero in L/P), then $I = I'/P$ and $J = J'/P$ for some ideals I' and J' of L. Since $[I, J] = P$, then $[I', J'] \subseteq P$. Hence, by (ii), $I' \subseteq r(P)$ or $J' \subseteq r(P)$. Thus, $I \subseteq r(P)$ or $J \subseteq r(P)$. Therefore, L/P is a primary Lie algebra.

(iii) $\Longrightarrow$ (i) : If I and J are two ideals of L satisfying $[I, J] \subseteq P$, then $(I + P)/P$ and $(J+P)/P$ are ideals of L/P whose product is P (Zero in L/P). Thus either $(I+P)/P \subseteq r(P)$ or$(J + P)/P \subseteq r(P)$ whence either $I + P \subseteq r(P)$ or $J + P \subseteq r(P)$. Thus, we get $I \subseteq r(P)$ or $J \subseteq r(P)$. Therefore, P is a primary ideal. □

4 Homomorphic Image

In the beginning of this section we set some notions from [10] and we prove some useful lemmas.

Suppose that L_1 and L_2 are two Lie algebras. A linear map $f : L_1 \to L_2$ is called a Lie algebra homomorphism if $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L_1$.

Lemma 4.1. *Let $f : L_1 \to L_2$ be a surjective Lie homomorphism, let A and B be two ideals of L_1 , then* $\Big[f^{-1}(A), f^{-1}(B)\Big] \subseteq f^{-1}([A, B])$.

Proof. Since $[A, B] \subseteq [A, B]$, then $[f(f^{-1}(A), f(f^{-1}(B)] \subseteq f(f^{-1}([A, B]))$. Hence, we have $f([f^{-1}(A), f^{-1}(B)]) \subseteq [A, B]$. Thus, $f^{-1}(f([f^{-1}(A), f^{-1}(B)])) \subseteq f^{-1}([A, B])$. Therefore, $[f^{-1}(A), f^{-1}(B)] \subseteq f^{-1}(f([f^{-1}(A), f^{-1}(B)])) \subseteq f^{-1}([A, B])$. □

Lemma 4.2. *Let $f : L_1 \to L_2$ be a surjective Lie homomorphism, let P be an ideal of L_1. If* $ker(f) \subseteq P$*, then* $f^{-1}f(P) = P$

Proof. In general, we have $P \subseteq f^{-1}f(P)$.

Let ker$(f) \subseteq P$. If $x \in f^{-1}f(P)$ then $f(x) = f(p)$ for some $p \in P$, hence $(x - p) \in ker(f)$. Now $x = (x - p) + p$, with $p \in P$ and $x - p$ $\in$ker$(f) \subseteq P$. Thus, $x \in P$. Therefore, $f^{-1}f(P) \subseteq P$. □

Kawamoto in [1], pointed out that the ideal contains the kernel of a surjective homomorphism is a prime ideal if and only if it's image under this surjective homomorphism is a prime ideal.

Theorem 4.3. *Let L_1 and L_2 be two Lie algebras and let $f : L_1 \to L_2$ be a surjective Lie homomorphism. An ideal P of L_1 containing* Ker$f$ *is primary if and only if $f(P)$ is primary ideal of L_2.*

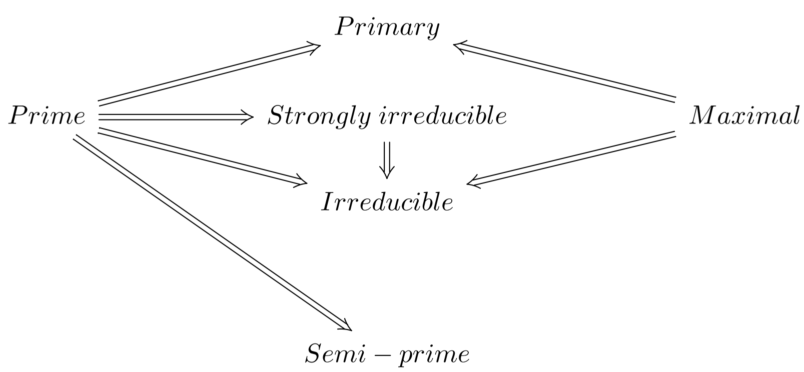
Proof. Let A and B be two ideals of L_2 with $[A, B] \subseteq f(P)$, then $f^{-1}([A, B]) \subseteq f^{-1}f(P)$. Since P contains Ker$f$, by Lemmas(4.1 and 4.2), we have, $\Big[f^{-1}(A), f^{-1}(B)\Big] \subseteq P$. Thus $f^{-1}(A) \subseteq r(P)$ *or* $f^{-1}(B) \subseteq r(P)$ because P is primary ideal of L_1. Since f is surjective, then $A \subseteq f(r(P))$ or $B \subseteq f(r(P))$. Thus, $A \subseteq r(f(P))$ *or* $B \subseteq r(f(P))$, hence $f(P)$ is a primary ideal of L_2.

Conversely, let A and B be two ideals of L_1 with $[A, B] \subseteq P$. Then $f([A, B]) \subseteq f(P)$, hence $[f(A), f(B)] \subseteq f(P)$. Now $f(P)$ is a primary ideal of L_2, so $f(A) \subseteq r(f(P))$ *or* $f(B) \subseteq r(f(P))$. Since P contains Ker$f$, therefore $A \subseteq r(P)$ *or* $B \subseteq r(P)$, by Lemma 4.2. □

5 Conclusions

1) Every prime ideal of L is a primary.
2) Every prime ideal of L is a strongly irreducible.
3) Every strongly irreducible ideal of L is an irreducible.
4) Every prime ideal of L is a semi-prime.
5) Every strongly irreducible ideal of a semi-simple Lie algebra L is a prime.
6) Every maximal ideal of L is a primary.
7) Every maximal ideal of L is an irreducible.
8) The concepts of strongly irreducible, primary and prime ideals are all equivalent for prime radical ideals.
9) A Lie algebra with non-zero center can't be semi-prime Lie algebra.
10) A semi-simple Lie algebra is a semi-prime Lie algebra.
11) P is a primary ideal if and only if L/P is a primary Lie algebra.
12) The ideal contains the kernel of a surjective homomorphism is a primary ideal if and only if it's image under this surjective homomorphism is a primary ideal.

The following diagram summarizes the interrelations between ideals which considered in this paper.



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