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SOME PROPERTIES OF ENDOMORPHISM OF MODULES

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Abstract An R-module M is called weakly Hopfian (respectively generalized co-Hopfian) if for every small epimorphism (respectively essential monomorphism) f of M is isomorphism. If M is quasi projective co-Hopfian (respectively quasi injective Hopfian) then M is weakly Hopfian (respectively generalized co-Hopfian). We prove an analogue to Hilbert's basis Theorem for weakly Hopfian (respectively, generalized co-Hopfian) module, M, i.e. if $M[x]/(x^{n+1})$ is weakly Hopfian (respectively, generalized co-Hopfian) $R[x]/(x^{n+1})$-module then M is weakly Hopfian (respectively, generalized co-Hopfian) R-module. And also we characterize the automorphisms of an R-module M satisfies the property (L).

1 Introduction and preliminaries

The study of modules by properties of their endomorphisms is a classical research subject. In the beginning of 1980, A. Kaidi and M. Sanghartroduced the concept of modules which satisfy the properties (I), (S) and (F). In 1987, P. Schupp showed, in [9], that the extension property (E) in the category of groups characterizes the inner automorphisms. And in [8], Pettet gave an easier proof of Schupp's result and proved at the same time that the inner automorphisms of a group G are also characterized by the lifting property (L) in the category of groups. We say the a module M satisfies the property (I) (resp. (S)), if any injective (resp. surjective) endomorphism of M is an automorphism of M, and we say that M satisfies the property (F), if for any endomorphism f of M there exists an integer $n \geq 1$ such that $M = Im(f^n) \oplus Ker(f^n)$, we say that $\alpha \in Aut_R(M)$ satisfies the property (E) if for all R-module N and any monomorphism $\lambda : M \to N$, there exists $\widetilde{\alpha} \in Aut_R(N)$ such that $\widetilde{\alpha}\lambda = \lambda\alpha$. And we say that $\alpha \in Aut_R(M)$ satisfies the property (L) if for all module N and any epimorphism $\lambda : N \to M$, there exists $\widetilde{\alpha} \in Aut_R(N)$ such that $\lambda\widetilde{\alpha} = \alpha\lambda$. In 1986, V.A. Hiremath, introduced the concept of Hopfian modules for modules satisfying the property (S). A bit later, K. Varadarajan, introduced the concept of co-Hopfian modules for modules satisfying the property (I). A submodule K of an R-module M is said to be small in M, written $K \ll M$, if for every submodule $L \subseteq M$ with $K + L = M$ implies $L = M$. A submodule K of an R-module M is said to be essential in M, written $K \leq^e M$, if for every submodule $L \subseteq M$ with $K \cap L = 0$ implies $L = 0$. In 2001, Haghany and Vedadi, [5], and in 2002, Ghorbani and Haghany, [4], respectively, introduced and investigated the weakly co-Hopfian (respectively generalized Hopfian) modules (i.e., every injective endomorphism has an essential image) (respectively every surjective endomorphism has a small kernel). Such modules and their generalizations were introduced and studied by many authors (for more information about this and others related topics, see, for instance,[4], [5], [6], [11], [13]). An R-module M is called weakly Hopfian (respectively generalized co-Hopfian) if for every small epimorphism (respectively essential monomorphism) f of M is isomorphism. The rings considered in this paper are associative with unit. Unless otherwise mentioned, all the modules considered are left unitary modules. The paper is organized as follows:

In Section 1, we recalling some well-known facts about Hopfian and co-Hopfian modules and some definitions.

In Section 2, we prove that for a quasi projective module M and N a fully invariant small submodule of M, if M is weakly Hopfian then M/N is weakly Hopfian, (Theorem 2.3). As a consequence we obtain for a finitely generated quasi-projective module M, if M is weakly Hopfian then $M/Jac(M)$ is weakly Hopfian, (Corollary 2.4). And we show that if M is quasi projective co-Hopfian (respectively quasi injective Hopfian) then M is weakly Hopfian (respectively generalized co-Hopfian), (Proposition 2.5) (respectively (Proposition 2.6)).

Varadarajan [11] showed that the left R-module M is Hopfian if and only if the left $R[x]$-module $M[x]$ is Hopfian if and only if the left $R[x]/(x^{n+1})$-module $M[x]/(x^{n+1})$ is Hopfian, where n is a non-negative integer and x is a commuting indeterminate over R. However, for any R-module $M \neq 0$, the $R[x]$-module $M[x]$ is never co-Hopfian. In fact, the map "multiplication by x" is injective and non surjective. We are motivated to prove that, if $M[x]/(x^{n+1})$ is weakly Hopfian (respectively, generalized co-Hopfian) $R[x]/(x^{n+1})$-module then M is weakly Hopfian (respectively, generalized co-Hopfian) R-module, (Theorem 2.8 and Theorem 2.10).

In Section 3, we prove that every automorphism of projective module satisfies the property (L), (Proposition 3.3), and we prove that an Automorphism α of finitely generated p-primary module M over Dedekind domain not a field R satisfies the property (L) if and only if there exists $k \in R$ invertible such that $\alpha = k.id_M$, (Theorem 3.9).

Let R be a ring and M an R-module. We recall the following definitions and facts:

Definition 1.1. M is called Hopfian (respectively co-Hopfian) if every surjective (respectively injective) endomorphism of M is an automorphism.

The ring R is called left Hopfian (respectively left co-Hopfian) if the left R-module R is Hopfian (respectively co-Hopfian).

Remark 1.2. $\diamond$

- Every Noetherian R-module M (i.e., M has ACC on submodules), is Hopfian [7].
- Every Artinian R-module M (i.e., M has DCC on submodules), is co-Hopfian) [7].
- The additive group $\mathbb{Q}$ of rational numbers is a non-Noetherian non-Artinian $\mathbb{Z}$-module, which is Hopfian and co-Hopfian [7].

Definition 1.3. An R-module M is said to be Fitting module if for any endomorphism f of M, there exists a positive integer $n \geq 1$ such that: $M = Kerf^n \oplus Imf^n$.

Remark 1.4. $\diamond$

- Every Artinian and Noetherian R-module is Fitting [1].
- Every Fitting R-module is Hopfian and co-Hopfian [1].

Definition 1.5. A ring R is said Dedekind finite if, $\forall a, b \in R$, $ab = 1 \Rightarrow ba = 1$. An R-module M is said Dedekind finite if $\text{End}_R(M)$ is Dedekind finite.

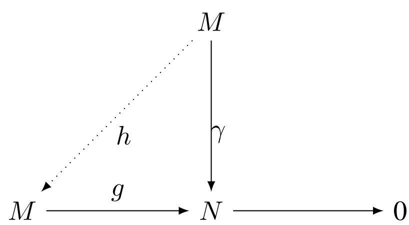
Remark 1.6. $\diamond$

- A ring R is left Hopfian if and only if R is Dedekind finite, if and only if R is right Hopfian [7].
- Every commutative ring is Hopfian [7].

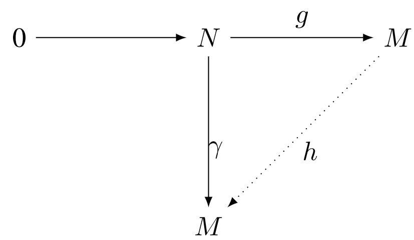
Definition 1.7. [13]. M is called weakly Hopfian (respectively generalized co-Hopfian) if every small surjective (respectively essential injective) endomorphism of M is an automorphism.

2 Notes on Generalizations of Hopfian and co-Hopfian modules

Definition 2.1. An R-module M is called quasi-projective (respectively quasi-injective) if for any surjective (respectively injective) homomorphism g of M onto N (respectively of N into M) and any homomorphism, γ of M (respectively N) to N (respectively to M), there exists an endomorphism h of M such that: $\gamma = gh$ (respectively $\gamma = hg$) (i.e., there exists $h : M \to M$ such that the diagram



respectively



commutes). Clearly, every projective module is quasi-projective and every injective module is quasi-injective.

Lemma 2.2. *(see [14]). Let M, N, and L be modules. Then two epimorphisms $f : M \to N$ and $g : N \to L$ are small if and only if gf is small.*

Theorem 2.3. *Let M be a quasi projective module and N be a fully invariant small submodule of M, if M is weakly Hopfian then M/N is weakly Hopfian.*

Proof. Assume that M is weakly Hopfian and let N be a fully invariant small submodule of M. If $f : M/N \to M/N$ is a small epimorphism, then by the small canonical epimorphism $\pi : M \to M/N$ and by lemma 2.2, we have $f\pi : M \to M/N$ is small epimorphism. Since M is quasi projective, there exists an endomorphism g of M such that $\pi g = f\pi$. This equality implies that g is small epimorphism, since M is weakly Hopfian, g is an automorphism. We have $f(x + N) = f\pi(x) = \pi g(x) = g(x) + N$, and $kerf = K/N$ where $N \subset K = \{x \in M; g(x) \in N\} = g^{-1}(N) \subset M$. Since N is a fully invariant small submodule of M thus $g^{-1}(N) \subset N$. Therefore $Kerf = g^{-1}(N)/N = 0$ and M/N is weakly Hopfian. □

Corollary 2.4. *Let M be a finitely generated quasi-projective module, if M is weakly Hopfian then $M/Jac(M)$ is weakly Hopfian.*

Proof. $Jac(M)$ is a fully invariant submodule of M, and since M is assumed finitely generated we have $Jac(M)$ is small in M. Thus is $M/Jac(M)$ is weakly Hopfian from Theorem 2.3. □

Proposition 2.5. *Let M be a quasi projective module, if M is co-Hopfian then it is weakly Hopfian.*

Proof. Let $f : M \to M$ be a small surjective endomorphism, since M is quasi projective, there exists $g : M \to M$, such that $fg = id_M$, then g is an injective endomorphism, since M is co-Hopfian, g is automorphism, which shows that f is an automorphism, then M is weakly Hopfian. □

Proposition 2.6. *Let M be a quasi injective module, if M is Hopfian then it is generalized co-Hopfian.*

Proof. Let $f : M \to M$ be an essential injective endomorphism, since M is quasi injective, there exists $g : M \to M$, such that $gf = id_M$, then g is a surjective endomorphism, since M is Hopfian, g is automorphism, which shows that f is an automorphism, then M is generalized co-Hopfian. □

Let M be an R-module. We will briefly recall the definitions of the modules $M[x]$ and $M[x]/(x^{n+1})$ from [10]. The elements of $M[x]$ are formal sums of the form $a_0 + a_1x + ... + a_kx^k$ with k an integer greater than or equal to 0 and $a_i \in M$. We denote this sum by $\sum_{i=1}^{k} a_ix^i$ (a_0x^0 is to be understood as the element $a_0 \in M$). Addition is defined by adding the corresponding coefficients. The $R[x]$-module structure is given by

$$(\sum_{i=0}^{k} \lambda_i x^i).(\sum_{j=0}^{z} a_j x^j) = \sum_{\mu=0}^{k+z} c_\mu x^\mu,$$

where $c_\mu = \sum_{i+j=\mu} \lambda_i a_j$, for any $\lambda_i \in R$, $a_j \in M$.

Any nonzero element β of $M[x]$ can be written uniquely as $(\sum_{i=k}^{l} m_i x^i)$ with $l \geq k \geq 0$, $m_i \in M$, $m_k \neq 0$ and $m_l \neq 0$. In this case, we refer to k as the order of β, l as the degree of β, m_k as the initial coefficient of β, and m_l as the leading coefficient of β.

Let n be any non-negative integer and

$$I_{n+1} = \{0\} \cup \{\beta; 0 \neq \beta \in R[x], \text{order of } \beta \geq n+1\}.$$

Then I_{n+1} is a two-sided ideal of $R[x]$. The quotient ring $R[x]/I_{n+1}$ will be called the truncated polynomial ring, truncated at degree $n+1$. Since R has an identity element, I_{n+1} is the ideal generated by x^{n+1}. Even when R does not have an identity element, we will "symbolically" denote the ring $R[x]/I_{n+1}$ by $R[x]/(x^{n+1})$. Any element of $R[x]/(x^{n+1})$ can be uniquely written as $(\sum_{i=0}^{n} \lambda_i x^i)$ with $\lambda_i \in R$.

Let

$$D_{n+1} = \{0\} \cup \{\beta; 0 \neq \beta \in M[x], \text{order of } \beta \geq n+1\}.$$

Then D_{n+1} is an $R[x]$-submodule of $M[x]$. Since $I_{n+1}M[x] \subset D_{n+1}$, we see that $R[x]/(x^{n+1})$ acts on $M[x]/D_{n+1}$. We denote the module $M[x]/D_{n+1}$ by $M[x]/(x^{n+1})$. The action of $R[x]/(x^{n+1})$ on $M[x]/(x^{n+1})$ is given by

$$(\sum_{i=0}^{n} \lambda_i x^i).(\sum_{j=0}^{n} a_j x^j) = \sum_{\mu=0}^{n} c_\mu x^\mu,$$

where $c_\mu = \sum_{i+j=\mu} \lambda_i a_j$, for any $\lambda_i \in R$, $a_j \in M$.

Any nonzero element β of $M[x]/D_{n+1}$ can be written uniquely as $(\sum_{i=k}^{n} m_i x^i)$ with $n \geq k \geq 0$, $m_i \in M$, $m_k \neq 0$. In this case, we refer to k as the order of β, m_k as the initial coefficient of β.

The $R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$-module $M[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$ is defined similarly.

Lemma 2.7. *[3, Lemma 2.1]. Let M be an R-module and $K \ll M$. Then $K[x]/(x^{n+1}) \ll M[x]/(x^{n+1})$ as $R[x]/(x^{n+1})$-modules, where $n \geq 0$.*

Theorem 2.8. *Let M be an R-module. If $M[x]/(x^{n+1})$ is weakly Hopfian $R[x]/(x^{n+1})$-module, then M is weakly Hopfian R-module.*

Proof. Let $f : M \to M$ be any small epimorphism in R-module, then $\alpha : M[x]/(x^{n+1}) \to M[x]/(x^{n+1})$ defined by $\alpha(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i) x^i$ is a surjective endomorphism in $R[x]/(x^{n+1})$-module, since $kerf$ is small in M then by Lemma 2.7, $ker\alpha = (kerf)[x]/(x^{n+1})$ is small in $M[x]/(x^{n+1})$. Since $M[x]/(x^{n+1})$ is weakly Hopfian $R[x]/(x^{n+1})$-module, α is automorphism in $M[x]/(x^{n+1})$. Then f is automorphism in M, and finally M is weakly Hopfian. □

Lemma 2.9. *[12, Lemma 1.7]. Let N be a submodule of an R-module M. Then N is essential in M as an R-module if and only if $N[x]/(x^{n+1})$ is essential in $M[x]/(x^{n+1})$ as an $R[x]/(x^{n+1})$-module.*

Theorem 2.10. *Let M be an R-module. If $M[x]/(x^{n+1})$ is generalized co-Hopfian $R[x]/(x^{n+1})$-module, then M is generalized co-Hopfian R-module.*

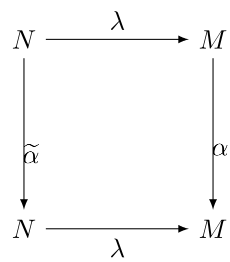
Proof. Let $f : M \to M$ be any essential injective endomorphism in R-module, then $\alpha : M[x]/(x^{n+1}) \to M[x]/(x^{n+1})$ defined by $\alpha(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i) x^i$ is a injective endomorphism in $R[x]/(x^{n+1})$-module, since Imf is essential in M then by Lemma 2.9, $Im\alpha = (Imf)[x]/(x^{n+1})$ is essential in $M[x]/(x^{n+1})$. Since $M[x]/(x^{n+1})$ is generalized co-Hopfian $R[x]/(x^{n+1})$-module, α is automorphism in $M[x]/(x^{n+1})$. Then f is automorphism in M, and finally M is generalized co-Hopfian. □

Theorem 2.11. *Let M be an R-module. If $M[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$ is weakly Hopfian (respectively, generalized co-Hopfian) $R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$-module, then M is weakly Hopfian (respectively, generalized co-Hopfian) R-module.*

Proof. Use induction and the
$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1})$-module isomorphism
$(M[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq M[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$
and ring isomorphism
$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$ □

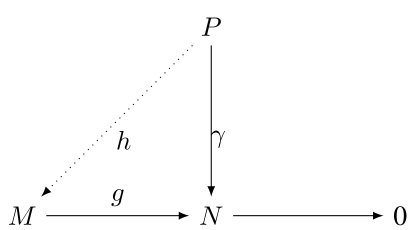
3 The lifting property

Definition 3.1. Let M be an R-module and $\alpha \in Aut_R(M)$, we say that the α has the lifting property if for all R-module N and any epimorphism $\lambda : N \to M$, there exists $\widetilde{\alpha} \in Aut_R(N)$ such that the following diagram is commutative:



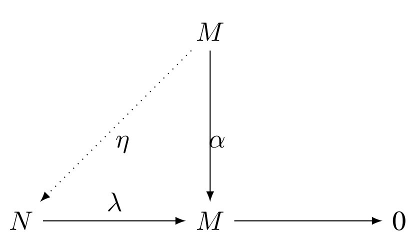
In other words: $\lambda\widetilde{\alpha} = \alpha\lambda$.

Definition 3.2. An R-module P is called projective if for any surjective homomorphism g of M onto N and any homomorphism, γ of P to N, there exists an homomorphism h of P to M such that: $\gamma = gh$, (i.e., there exists $h : P \to M$ such that the following diagram is commutative.

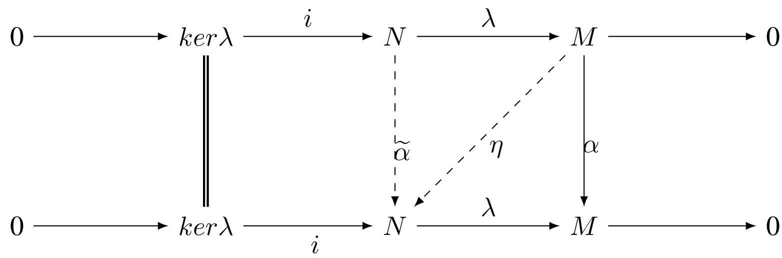


Proposition 3.3. *Let R be a ring and M be a projective R-module. If $\alpha \in Aut_R(M)$, then α has the lifting property.*

Proof. Let N be an R-module and λ be an epimorphism from N to M, since M is a projective module, there exists η from M to N such that the following daigram is commutative:



We take $\tilde{\alpha} = \eta\lambda$. Then the following diagram is commutative with exact rows:



Then by short five lemma, as α is an automorphism, then $\widetilde{\alpha}$ is an automorphism such that $\lambda\widetilde{\alpha} = \lambda\eta\lambda = \alpha\lambda$, hence α has the lifting property.

□

Definition 3.4. [2]. Let M be an R-module, an element m in M is said to be a torsion element if $m.r = 0$ for some $r \in R$. The set of all torsion elements in M is denoted by $T(M)$ and called the torsion submodule of M. We say that M is a torsion module if $M = T(M)$ and that M is torsion-free if $T(M) = 0$.

Lemma 3.5. *[2, Corollary 6.3.4] Let M be a finitely generated torsion-free module over a Dedekind domain R. Then M is projective.*

Corollary 3.6. *Let R be a Dedekind domain, and M be a finitely generated torsion-free R-module. If $\alpha \in Aut_R(M)$, then α has the lifting property.*

Proof. We can apply Proposition 3.3 and Lemma 3.5. □

Recall that, the annihilator of R-module M is defined to be the ideal $Ann(M)$ of R given by $Ann(M) = \{r \in R/m.r = 0 \text{ for all } m \in M\}$.

Definition 3.7. [2]. Let p be a nonzero prime ideal of R. An R-module M is called p-primary if $Ann(M) = p^n$ for some natural number n.

Lemma 3.8. *Let R be a Dedekind domain not a field, M be a finitely generated p-primary R-module and $\alpha \in Aut_R(M)$ has the lifting property, if $< m >$ is a direct summand of M then $\alpha(m) \in < m >$.*

Proof. We assume that M generated by elements $m_1, ..., m_s$ whose orders are respectively $p^{n_1}, ..., p^{n_s}$, if $n = max\{n_1, ..., n_s\}$, then $p^n M = 0$ and $p^{n-1}M \neq 0$. Let $< m >$ is a direct summand of M, there is a submodule M' of M such that

$$M = < m > \bigoplus M', \text{ where } m \in M.$$

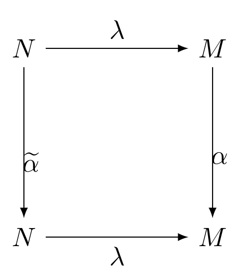
We prove that $\alpha(m) \in < m >$.

Consider $N = < m > \bigoplus M''$, and we define a homomorphism λ from N to M by:

$$\lambda(km + m'') = km + p^n m''.$$

for all $k \in R$ and $m'' \in M''$, is an epimorphism.

Using the fact That α checks the lifting property then there exists $\widetilde{\alpha} \in Aut(N)$ such that the following diagram is commutative:



i.e.,

$$\lambda\widetilde{\alpha} = \alpha\lambda$$

We can write $\widetilde{\alpha}(m) = km + m''$, then

$$\begin{aligned}\lambda\widetilde{\alpha}(m) &= \lambda(km + m'') \\ &= km + p^n m'' \\ &= km \\ &= \alpha\lambda(m) \\ &= \alpha(m).\end{aligned}$$

because $p^n m'' = 0$, thus $\alpha(m) \in < m >$.

□

Theorem 3.9. *Let R be a Dedekind domain not a field, M be a finitely generated p-primary R-module and $\alpha \in Aut_R(M)$. Then the following assertions are equivalent:*
i) α has the lifting property.
ii) there exists $k \in R$ invertible such that $\alpha = k.id_M$.

Proof. $i) \Rightarrow ii)$ By [2, Theorem 6.3.23], every finitely generated p-primary module over Dedekind Domain is a direct sum of cyclic modules of orders $p^{n_1}, ...p^{n_s}$ respectively.

Then $M = \bigoplus_{i=1}^{s} < m_i >$, then $\forall m \in M, \exists r_i \in R$ such that

$$m = \sum_{i=1}^{s} r_i m_i$$

then $\alpha(m) = \sum_{i=1}^{s} r_i\alpha(m_i)$, and by lemma 3.8, as $< m_i >$ is a direct summand of M, there exists $k_i \in R$ invertible such that

$$\alpha(m_i) = k_i m_i.$$

Let $1 \leq j < i \leq s$, We can write $M =< m_i > \bigoplus M_i$ with $m_j \in M_i$. It is easy to see that $< m_i + m_j > \bigoplus M_i = M$. So we have:

$$\begin{cases}\alpha(m_i) = k_i m_i \\ \alpha(m_j) = k_j m_j \\ \alpha(m_i + m_j) = k(m_i + m_j)\end{cases}$$

then $(k - k_i)m_i + (k - k_j)m_j = 0$, hence $(k - k_i)m_i = (k - k_j)m_j = 0$. Consequently p^{n_i} divides $k - k_i$ and p^{n_j} divides $k - k_j$. Hence p^{n_j} divides $k_j - k_i$ because $i > j$.

So there exists $t \in R$ such that $k_j = k_i + tp^{n_j}$

$$\begin{aligned}\alpha(m_j) &= k_j m_j \\ &= (k_i + tp^{n_j})m_j \\ &= k_i m_j + tp^{n_j} m_j \\ &= k_i m_j \\ &= km_j.\end{aligned}$$

Then

$$\begin{aligned}\alpha(m) &= \sum_{i=1}^{n} r_i\alpha(m_i) \\ &= \sum_{i=1}^{n} r_i k m_i \\ &= k\sum_{i=1}^{n} r_i m_i \\ &= km.\end{aligned}$$

Then there exists $k \in R$ invertible such that $\alpha = k.id_M$.

$ii) \Rightarrow i)$ Evident.

□

Corollary 3.10. *Let R be a Dedekind domain not a field, M be a finitely generated torsion R-module and $\alpha \in Aut_R(M)$. Then the following assertions are equivalent:*

i) α has the lifting property.

ii) there exists $k \in R$ invertible such that $\alpha/_{M_p} = k.id_{M_p}$.

Proof. $i) \Rightarrow ii)$ By [2, Theorem 6.3.23], every finitely generated torsion module over Dedekind domain is a direct sum of p-primary modules.

Then $M = \bigoplus M_p$, then $\alpha/_{M_p} \in Aut(M_p)$. Hence $\alpha/_{M_p}$ satisfies the lifting property (since α has the lifting property). Now by theorem 3.9, we have $\alpha/_{M_p} = k.id_{M_p}$ where $k \in R$ is an invertible.

$ii) \Rightarrow i)$ Evident.

□

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