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NON-TRIVIAL TRIGONOMETRIC SUMS ARISING FROM SOME OF RAMANUJAN THETA FUNCTION IDENTITIES

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Abstract In this article, we deduce some interesting non-trivial trigonometric identities from Ramanujan theta function identities.

1 Introduction

Ramanujan [9, P. 309], recorded the following beautiful relationship between his theta functions and trigonometric functions:

Theorem 1. [3, P. 140]: Let $f(a,b)$ be defined in (1.10) below. Let m, n, p, r and k be positive numbers such that $m+n=p+r=k$, Then as q tends to 1-,

$$\frac{f(-q^m,-q^n)}{f(-q^p,-q^r)} \sim \frac{\sin(\frac{m\pi}{k})}{\sin(\frac{p\pi}{k})}.$$

This relation can be effectively employed to evaluate various generalized as well as particular trigonometric sums.

Evaluation of generalized as well as particular trigonometric sums has been one of the interesting and important topics, studied by various mathematicians. It finds its applications in various branchs of mathematics. For the detailed history of trigonometric sums, one may refer [5], where B. C. Berndt and B. P. Yeap along with giving the history of trigonometric sums, have generalized the trigonometric sums in different dimensions by establishing some reciprocity theorems using contour integration techinique. After this, along with A. Zaharesue, Berndt [6], generalizes several non-trivial trigonometric sums of Z. G. Liu [8], which were particular to level 7. In all these papers, authour mainly use the theory of elliptic functions and the contour integration. M. Beck, Matthias and Halloran, Mary [1], also deduced several of the trigonometric sums found in [5], [6], through the method of discrete Fourier analysis. Recently K. N. Harshitha, K. R. Vasuki and M. V. Yathirajsharma [7], used the theory of theta functions to evaluate generalize trigonometric functions. This motivated us to write this paper.

The purpose of this paper is to obtain following non-trivial trigonometric sums, which easily follow from Theorem 1 and Ramanujan's theta functions identities.

Theorem 2. : The following identities hold :

$$\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})} - \frac{\sin(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})} + \frac{\sin(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})} = 1, \tag{1.1}$$

$$\frac{\sin(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\sin(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}+\frac{\sin(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}=2, \tag{1.2}$$

$$\frac{\sin(\frac{4\pi}{13})}{\sin(\frac{2\pi}{13})}-\frac{\sin(\frac{6\pi}{13})}{\sin(\frac{3\pi}{13})}-\frac{\sin(\frac{2\pi}{13})}{\sin(\frac{\pi}{13})}+\frac{\sin(\frac{5\pi}{13})}{\sin(\frac{4\pi}{13})}-\frac{\sin(\frac{3\pi}{13})}{\sin(\frac{5\pi}{13})}+\frac{\sin(\frac{\pi}{13})}{\sin(\frac{6\pi}{13})}=-1, \tag{1.3}$$

$$\frac{\sin(\frac{4\pi}{13})}{\sin(\frac{2\pi}{13})}\frac{\sin(\frac{6\pi}{13})}{\sin(\frac{3\pi}{13})}-\frac{\sin(\frac{2\pi}{13})}{\sin(\frac{\pi}{13})}\frac{\sin(\frac{3\pi}{13})}{\sin(\frac{5\pi}{13})}-\frac{\sin(\frac{5\pi}{13})}{\sin(\frac{4\pi}{13})}\frac{\sin(\frac{\pi}{13})}{\sin(\frac{6\pi}{13})}=1, \tag{1.4}$$

$$\frac{\sin(\frac{2\pi}{13})}{\sin(\frac{4\pi}{13})}\frac{\sin(\frac{3\pi}{13})}{\sin(\frac{6\pi}{13})}-\frac{\sin(\frac{\pi}{13})}{\sin(\frac{2\pi}{13})}\frac{\sin(\frac{5\pi}{13})}{\sin(\frac{3\pi}{13})}-\frac{\sin(\frac{4\pi}{13})}{\sin(\frac{5\pi}{13})}\frac{\sin(\frac{6\pi}{13})}{\sin(\frac{\pi}{13})}=-4, \tag{1.5}$$

$$\frac{\sin(\frac{6\pi}{13})}{\sin(\frac{3\pi}{13})}\frac{\sin(\frac{2\pi}{13})}{\sin(\frac{\pi}{13})}\frac{\sin(\frac{5\pi}{13})}{\sin(\frac{4\pi}{13})}-\frac{\sin(\frac{4\pi}{13})}{\sin(\frac{2\pi}{13})}\frac{\sin(\frac{3\pi}{13})}{\sin(\frac{5\pi}{13})}\frac{\sin(\frac{\pi}{13})}{\sin(\frac{6\pi}{13})}=3, \tag{1.6}$$

$$\frac{\sin(\frac{2\pi}{15})}{\sin(\frac{\pi}{15})}-\frac{\sin(\frac{7\pi}{15})}{\sin(\frac{4\pi}{15})}-\frac{\sin(\frac{3\pi}{15})}{\sin(\frac{6\pi}{15})}=0, \tag{1.7}$$

$$\frac{\sin(\frac{6\pi}{17})}{\sin(\frac{3\pi}{17})}-\frac{\sin(\frac{4\pi}{17})}{\sin(\frac{2\pi}{17})}-\frac{\sin(\frac{8\pi}{17})}{\sin(\frac{4\pi}{17})}+\frac{\sin(\frac{2\pi}{17})}{\sin(\frac{\pi}{17})}+\frac{\sin(\frac{7\pi}{17})}{\sin(\frac{5\pi}{17})}-\frac{\sin(\frac{5\pi}{17})}{\sin(\frac{6\pi}{17})}+\frac{\sin(\frac{3\pi}{17})}{\sin(\frac{7\pi}{17})}-\frac{\sin(\frac{\pi}{17})}{\sin(\frac{8\pi}{17})}=1, \tag{1.8}$$

and

$$\frac{\sin(\frac{6\pi}{17})}{\sin(\frac{3\pi}{17})}\frac{\sin(\frac{7\pi}{17})}{\sin(\frac{5\pi}{17})}+\frac{\sin(\frac{4\pi}{17})}{\sin(\frac{2\pi}{17})}\frac{\sin(\frac{\pi}{17})}{\sin(\frac{8\pi}{17})}-\frac{\sin(\frac{8\pi}{17})}{\sin(\frac{4\pi}{17})}\frac{\sin(\frac{2\pi}{17})}{\sin(\frac{\pi}{17})}-\frac{\sin(\frac{5\pi}{17})}{\sin(\frac{6\pi}{17})}\frac{\sin(\frac{3\pi}{17})}{\sin(\frac{7\pi}{17})}=-1. \tag{1.9}$$

Identities (1.1) and (1.2) can be found in Liu [8, Corollary 7], identities (1.3)- (1.9) seems to be new.
In Section 2, we note down the basic definitions and the Ramanujan theta function identities from which the above trigonometric sums follows. In Section 3, we prove Theorem 2.

2 Preliminaries

For complex numbers a and q with $|q| < 1$, define as usual

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Ramanujan's theta function $f(a,b)$ is defined as

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab,ab)_\infty, \qquad |ab|<1.$$

and

$$f(-q)=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}=(q;q)_\infty.$$

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In the scattered places of his second notebook, Ramanujan has recorded the following identities :

$$\frac{f(-q^2,-q^5)}{q^{\frac{2}{7}}f(-q,-q^6)}-\frac{f(-q^3,-q^4)}{q^{\frac{1}{7}}f(-q^2,-q^5)}+q^{\frac{3}{7}}\frac{f(-q,-q^6)}{f(-q^3,-q^4)}=\frac{f(-q^{\frac{1}{7}})}{q^{\frac{2}{7}}f(-q^7)}+1, \tag{2.1}$$

$$\frac{q^{\frac{2}{7}}f(-q,-q^6)}{f(-q^2,-q^5)}-\frac{q^{\frac{1}{7}}f(-q^2,-q^5)}{f(-q^3,-q^4)}+\frac{f(-q^3,-q^4)}{q^{\frac{3}{7}}f(-q,-q^6)}-2= \frac{1}{2}\Big(3\frac{f(-q^{\frac{1}{7}})}{q^{\frac{2}{7}}f(-q^7)}+\Big(4\Big(\frac{f(-q^{\frac{1}{7}})}{q^{\frac{2}{7}}f(-q^7)}\Big)^3+21\Big(\frac{f(-q^{\frac{1}{7}})}{q^{\frac{2}{7}}f(-q^7)}\Big)^2+28\Big(\frac{f(-q^{\frac{1}{7}})}{q^{\frac{2}{7}}f(-q^7)}\Big)\Big)^{\frac{1}{2}}\Big), \tag{2.2}$$

$$\frac{f(-q^{\frac{1}{13}})}{q^{\frac{7}{13}}f(-q^{13})}=\frac{f(-q^4,-q^9)}{q^{\frac{7}{13}}f(-q^2,-q^{11})}-\frac{f(-q^6,-q^7)}{q^{\frac{6}{13}}f(-q^3,-q^{10})}-\frac{f(-q^2,-q^{11})}{q^{\frac{5}{13}}f(-q,-q^{12})}+\frac{f(-q^5,-q^8)}{q^{\frac{2}{13}}f(-q^4,-q^9)} +1-q^{\frac{5}{13}}\frac{f(-q^3,-q^{10})}{f(-q^5,-q^8)}+q^{\frac{15}{13}}\frac{f(-q,-q^{12})}{f(-q^6,-q^7)}, \tag{2.3}$$

$$1+\frac{f^2(-q)}{qf^2(-q^{13})}=\frac{f(-q^4,-q^9)}{q^{\frac{7}{13}}f(-q^2,-q^{11})}\frac{f(-q^6,-q^7)}{q^{\frac{6}{13}}f(-q^3,-q^{10})}-\frac{f(-q^2,-q^{11})}{q^{\frac{5}{13}}f(-q,-q^{12})}q^{\frac{5}{13}}\frac{f(-q^3,-q^{10})}{f(-q^5,-q^8)} -\frac{f(-q^5,-q^8)}{q^{\frac{2}{13}}f(-q^4,-q^9)}q^{\frac{15}{13}}\frac{f(-q,-q^{12})}{f(-q^6,-q^7)}, \tag{2.4}$$

$$-4-\frac{f^2(-q)}{qf^2(-q^{13})}=\frac{q^{\frac{7}{13}}f(-q^2,-q^{11})}{f(-q^4,-q^9)}\frac{q^{\frac{6}{13}}f(-q^3,-q^{10})}{f(-q^6,-q^7)}-\frac{q^{\frac{5}{13}}f(-q,-q^{12})}{f(-q^2,-q^{11})}\frac{f(-q^5,-q^8)}{q^{\frac{5}{13}}f(-q^3,-q^{10})} -\frac{q^{\frac{2}{13}}f(-q^4,-q^9)}{f(-q^5,-q^8)}\frac{f(-q^6,-q^7)}{q^{\frac{15}{13}}f(-q,-q^{12})}, \tag{2.5}$$

$$3+\frac{f^2(-q)}{qf^2(-q^{13})}=\frac{f(-q^6,-q^7)}{q^{\frac{6}{13}}f(-q^3,-q^{10})}\frac{f(-q^2,-q^{11})}{q^{\frac{5}{13}}f(-q,-q^{12})}\frac{f(-q^5,-q^8)}{q^{\frac{2}{13}}f(-q^4,-q^9)} -\frac{f(-q^4,-q^9)}{q^{\frac{7}{13}}f(-q^2,-q^{11})}q^{\frac{5}{13}}\frac{f(-q^3,-q^{10})}{f(-q^5,-q^8)}q^{\frac{15}{13}}\frac{f(-q,-q^{12})}{f(-q^6,-q^7)}, \tag{2.6}$$

$$\frac{f(-q^2,-q^{13})}{f(-q,-q^{14})}-\frac{f(-q^7,-q^8)}{f(-q^4,-q^{11})}-\frac{f(-q^3,-q^{12})}{f(-q^6,-q^9)}=0, \tag{2.7}$$

$$\frac{f(-q^{\frac{1}{17}})}{q^{\frac{12}{17}}f(-q^{17})}=\frac{f(-q^6,-q^{11})}{q^{\frac{12}{17}}f(-q^3,-q^{14})}-\frac{f(-q^4,-q^{13})}{q^{\frac{11}{17}}f(-q^2,-q^{15})}-\frac{f(-q^8,-q^9)}{q^{\frac{10}{17}}f(-q^4,-q^{13})}+\frac{f(-q^2,-q^{15})}{q^{\frac{7}{17}}f(-q,-q^{16})} +\frac{f(-q^7,-q^{10})}{q^{\frac{5}{17}}f(-q^5,-q^{12})}-1-q^{\frac{3}{17}}\frac{f(-q^5,-q^{12})}{f(-q^6,-q^{11})}+q^{\frac{14}{17}}\frac{f(-q^3,-q^{14})}{f(-q^7,-q^{10})}+q^{\frac{28}{17}}\frac{f(-q,-q^{16})}{f(-q^8,-q^9)}, \tag{2.8}$$

and

$$\frac{f(-q^6,-q^{11})}{q^{\frac{12}{17}}f(-q^3,-q^{14})}\frac{f(-q^7,-q^{10})}{q^{\frac{5}{17}}f(-q^5,-q^{12})}+\frac{f(-q^4,-q^{13})}{q^{\frac{11}{17}}f(-q^2,-q^{15})}q^{\frac{28}{17}}\frac{f(-q,-q^{16})}{f(-q^8,-q^9)}-\frac{f(-q^8,-q^9)}{q^{\frac{10}{17}}f(-q^4,-q^{13})}\frac{f(-q^2,-q^{15})}{q^{\frac{7}{17}}f(-q,-q^{16})}$$
$$-q^{\frac{3}{17}}\frac{f(-q^5,-q^{12})}{f(-q^6,-q^{11})}q^{\frac{14}{17}}\frac{f(-q^3,-q^{14})}{f(-q^7,-q^{10})}=-1. \quad (2.9)$$

Identities (2.1)-(2.2) can be found in [9, p. 300], identities (2.3)-(2.6) can be found in [9, p. 244], identity (2.7) can be found in [9, p. 326], identities (2.8)-(2.9) can be found in [9, p. 247]. Proofs of the above identities can be found in [2].

3 Main results

To prove our main results, we require the following limits

$$\lim_{q\to 1^-}\frac{f(-q)}{q^2f(-q^{49})}=\lim_{q\to 1^-}\frac{(q,q)_\infty}{q^2(q^{49},q^{49})_\infty}=\lim_{q\to 1^-}\prod_{\substack{m=1\\ m\neq k\times 49\\ k\in N}}^{\infty}(1-q^m)=0, \quad (3.1)$$

$$\lim_{q\to 1^-}\frac{f(-q)}{q^7f(-q^{169})}=\lim_{q\to 1^-}\frac{(q,q)_\infty}{q^7(q^{169},q^{169})_\infty}=\lim_{q\to 1^-}\prod_{\substack{m=1\\ m\neq k\times 169\\ k\in N}}^{\infty}(1-q^m)=0, \quad (3.2)$$

$$\lim_{q\to 1^-}\frac{f^2(-q)}{qf^2(-q^{13})}=\lim_{q\to 1^-}\frac{(q,q)_\infty^2}{q(q^{13},q^{13})_\infty^2}=\lim_{q\to 1^-}\prod_{\substack{m=1\\ m\neq k\times 13\\ k\in N}}^{\infty}(1-q^m)=0, \quad (3.3)$$

and

$$\lim_{q\to 1^-}\frac{f(-q)}{q^{12}f(-q^{289})}=\lim_{q\to 1^-}\frac{(q,q)_\infty}{q^{12}(q^{289},q^{289})_\infty}=\lim_{q\to 1^-}\prod_{\substack{m=1\\ m\neq k\times 289\\ k\in N}}^{\infty}(1-q^m)=0. \quad (3.4)$$

Proof of (2.1)- (2.9): Using (3.1)- (3.4) and Theorem 1, taking $\lim_{q\to 1^-}$ on both sides of each of the identities (2.1)-(2.9), we correspondingly obtain the trigonometric identities in Theorem 2.

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