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Edge Deletion and Invariance in Graphs using D-distance

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Abstract In this article we study the radius invarience and diameter invariance after edge deletion in a graph using D-distance. We give examples to show that $r^D(G) \lesseqgtr r^D(G')$ and $d^D(G) \lesseqgtr d^D(G')$. Next we study the invariance property in complete graphs, cyclic graphs and wheel graphs. The complete bipartite graph $K_{m,m-1}$ is also studied. Through out the article we use D- distance introduced by the first two authors. We end the article with some open problems.

1 Introduction

Let G be a graph and G' be the graph obtained from G by deleting an edge (sometimes we may call it as 'derived graph' of G). In this article we would like to study the radius and diameter D-radius and D-diameter of G' in comparison to those of G. In this context the following cases arise :

(i) $r^D(G) = r^D(G')$, $d^D(G) = d^D(G')$

(ii) $r^D(G) > r^D(G')$, $d^D(G) > d^D(G')$

(iii) $r^D(G) > r^D(G')$, $d^D(G) < d^D(G')$

(iv) $r^D(G) < r^D(G')$, $d^D(G) > d^D(G')$

(v) $r^D(G) < r^D(G')$, $d^D(G) < d^D(G')$

The examples discussed in the next section will show that all the above cases are possible.

Thus the study of invariance of D-radius and D-diameter becomes very interesting. Invariance problem were studied earlier, for example see [1, 3].

The D distance between vertices of a graph was introduced by Reddy Babu and Varma in [4]. Using this distance in a natural way we can define D-eccentricity of vertex ($e^D(v)$) and D radius, D-diameter of a graph ($r^D(G)$ and $d^D(G)$, resply.).

A graph is called *D-radius invariant* (*D-diameter invariant*, resply.) if $r^D(G) = r^D(G')$ ($dia^D(G) = dia^D(G')$, resply.).

In the present article, we study the D-radius invariant and D-diameter invariant properties of complete graph, cyclic graph, wheel graph and bipartite graph, $K_{m,m-1}$.

Some more classes of graphs will be studied in a separate article [5].

Through out this article unless otherwise specified G' denotes a graph obtained from G by deleting an edge.

We start with examples as mentioned above.

2 Examples

We begin this section with an example of a graph, which is both radius and diameter invariant.

Example 2.1. Consider the $(5, 7)$ graph G as shown in figure 1. Let G' be the derived graph, $G' = G \setminus \{v_1v_5\}$. Then the eccentricities of vertices in G and G' are as shown table 1.

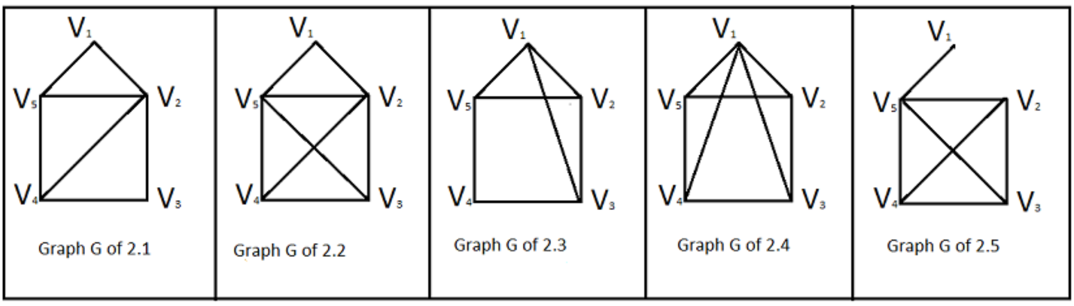

Figure 1. Various Graphs G

Eccentricity	v_1	v_2	v_3	v_4	v_5
G	10	8	10	10	10
G'	10	8	9	10	9

Table 1. Eccentricities in G and G'

From this table we can see that $r^D(G) = r^D(G')$ and $dia^D(G) = dia^D(G')$. Observe that in G' for some vertices the eccentricities have been changed.

In the next example we see that both radius and diameter of the derived graph decreases.

Example 2.2. Consider the $(5, 8)$ graph G as shown in figure 1. Let G' be the derived graph $G' = G \setminus \{v_1v_2\}$. Then the eccentricities of vertices in G and G' are as shown table 2.

Eccentricity	v_1	v_2	v_3	v_4	v_5
G	11	9	11	11	9
G'	10	10	10	10	8

Table 2. Eccentricities in G and G'

From this table we can see that $r^D(G) = 9$, $r^D(G') = 8$ and $dia^D(G) = 11$, $dia^D(G') = 10$. Hence $r^D(G) > r^D(G')$ and $dia^D(G) > dia^D(G')$. Observe that in G' all vertices of the eccentricities have been changed.

In the next example we see that radius decreases and diameter increases in the derived graph.

Example 2.3. Consider the $(5, 7)$ graph G as shown in figure 1, Let G' be the derived graph.$G' = G \setminus \{v_3v_4\}$. Then the eccentricities of vertices in G and G' are as shown table 3.

From this table we can see that $r^D(G) = 10$, $r^D(G') = 9$ and $dia^D(G) = 10$, $dia^D(G') = 12$. Hence $r^D(G) > r^D(G')$ and $dia^D(G) < dia^D(G')$. Observe that in G' for some vertices the eccentricities have not been changed.

In the next example we see that radius increases and diameter decreases in the derived graph.

Example 2.4. Consider the $(5, 8)$ graph G as shown in figure 1. Let G' be the derived graph.$G' = G \setminus \{v_1v_3\}$. Then the eccentricities of vertices in G and G' are as shown table 4.

From this table we can see that $r^D(G) = 8$, $r^D(G') = 10$ and $dia^D(G) = 11$, $dia^D(G') = 10$. Hence $r^D(G) < r^D(G')$ and $dia^D(G) > dia^D(G')$. Observe that in G' the eccentricities of all vertices have changed.

Similarly, in the next example we see that both radius and diameter of the derived graph increases.

Eccentricity	v_1	v_2	v_3	v_4	v_5
G	10	10	10	10	10
G'	9	9	12	12	10

Table 3. Eccentricities in G and G'

Eccentricity	v_1	v_2	v_3	v_4	v_5
G	8	11	11	11	11
G'	10	10	10	10	10

Table 4. Eccentricities in G and G'

Example 2.5. Consider the $(5, 7)$ graph G as shown in figure 1. Let G' be the derived graph,$G' = G \setminus \{v_3v_4\}$. Then the eccentricities of vertices in G and G' are as shown table 5.

Eccentricity	v_1	v_2	v_3	v_4	v_5
G	10	10	10	10	8
G'	9	9	12	12	10

Table 5. Eccentricities in G and G'

From this table we can see that $r^D(G) = 8$, $r^D(G') = 9$ and $dia^D(G) = 9$, $dia^D(G') = 12$. Hence $r^D(G) < r^D(G')$ and $dia^D(G) < dia^D(G')$. Observe that in G' for some vertices the eccentricities have been changed.

3 The Complete Graph K_n

In this section we study the invariance properties of the complete graph on n vertices. We begin with the following theorem.

Theorem 3.1. *Consider the complete graph K_n, ($n \geq 3$) and $G = K_n \setminus \{an\ edge\}$. Then we have $r^D(G) = 2n - 1$ and $dia^D(G) = 3(n - 1)$.*

Proof. Suppose G is the derived graph of the complete graph K_n, i.e., $G = K_n \setminus \{e\}$. Without loss of generality, assume that $e = v_1v_2$. We know that K_n is $(n - 1)$ regular graph.

In G, degree of v_1 = degree of v_2 = $n - 2$ and degree of all other vertices is $n - 1$. Then the shortest path between v_1, v_2 is of length 2 and hence the D-distance $d^D(v_1, v_2) = 2 + (n - 2) + (n - 2) + n - 1 = 3(n - 1)$. Then further $d^D(v_i, v_1) = 2n - 2$ for $i \neq 1$ and $d^D(v_i, v_j) = 2n - 1$ for $\{i, j\} \neq \{1, 2\}$. Then $e^D(v_i) = e^D(v_j) = 2n - 1$ and $e^D(v_1) = e^D(v_2) = 3n - 3$ for $\{i, j\} \neq \{1, 2\}$. Thus $r^D(G) = \min\{e^D(v)\} = 2n - 1$, $dia^D(G) = \max\{e^D(v)\} = 3n - 3$. □

Corollary 3.2. *We have*

$$r^D(K'_n) - r^D(K_n) = 0; \quad dia^D(K'_n) - dia^D(K_n) = n - 2$$

Proof. The corollary follows from the above theorem and the theorem 3.7 of [4]. □

If we delete two edges from complete graph, we have the following

Theorem 3.3. *Let G be as above with $n \geq 6$. Let G' be the graph derived graph of G. Then we have*

$$r^D(G') = r^D(G); \quad dia^D(G') = dia^D(G) - 1$$

Proof. Without loss of generality assume that $G = K_n \setminus \{v_1, v_2\}$ and $G' = G \setminus \{e = v_iv_j\}$. Then two cases arise, namely,

(i) e has a common vertex with v_1v_2,

(ii) e does not have any common vertex with $e = v_1v_2$.

Case(i): Assume that $e = v_1v_j$. Then degree of $v_1 = n-3$, the degree of $v_2 = n-2$, the degree of $v_j = n-2$ and the degree of all other vertices is $n-1$. Thus the degree sequence is $\{n-3, n-2, n-2, n-1, \cdots, n-1, n-1, n-1\}$. Then the shortest path between v_1 and v_2 is of length is 2 and hence we have $d^D(v_1, v_2) = (n-3)+(n-1)+(n-2)+2 = 3n-4$,
$d^D(v_1, v_j) = (n-3)+(n-1)+(n-2)+2 = 3n-4$,
$d^D(v_2, v_j) = (n-2)+(n-2)+1 = 2n-3$,
$d^D(v_1, v_l) = (n-3)+(n-1)+1 = 2n-3$ for $l \neq j$,
$d^D(v_2, v_l) = (n-2)+(n-1)+1 = 2n-2$ for $l \neq j$,
$d^D(v_l, v_m) = (n-1)+(n-1)+1 = 2n-1$ for $\{l, m\} \neq \{1, 2, j\}$.

Therefore $e^D(v_1) = e^D(v_2) = e^D(v_j) = 3n-4$, $e^D(v_l) = e^D(v_m) = 2n-1$ for $\{l, m\} \neq \{1, 2, j\}$. Thus $r^D(G') = \min\{e(v)\} = 2n-1$, $dia^D(G') = \max\{e(v)\} = 3n-4$ as claimed.

Case(ii): In this case e has no common vertex with v_1v_2. We have $deg(v_1) = deg(v_2) = deg(v_i) = deg(v_j) = n-2$ and the degree of all other vertices is $n-1$. Thus $\{n-2, n-2, n-2, n-2, \cdots, n-1\}$ is the degree sequence of G'. Then the D distances between these vertices are
$d^D(v_1, v_2) = 3n-4$,
$d^D(v_1, v_l) = 2n-2$ for $l \neq \{i, j\}$,
$d^D(v_1, v_i) = 2n-3$,
$d^D(v_2, v_l) = 2n-2$ for $l \neq \{i, j\}$,
$d^D(v_2, v_i) = d^D(v_2, v_j) = 2n-3$,
$d^D(v_i, v_j) = 3n-4$,
$d^D(v_l, v_m) = (n-1)+(n-1)+1 = 2n-1$ for $\{l, m\} \neq \{1, 2, i, j\}$.

Therefore the eccentricities are $e^D(v_1) = e^D(v_2) = e^D(v_i) = e^D(v_j) = 3n-4$, $e^D(v_l) = e^D(v_m) = 2n-1$ for $\{l, m\} \neq \{1, 2, i, j\}$. Thus $r^D(G') = \min\{e(v)\} = 2n-1$, $dia^D(G') = \max\{e(v)\} = 3n-4$ as claimed.

From above theorem 3.1 we have $r^D(G) = 2n-1$ and $dia^D(G) = 3n-3$. Therefore we get $r^D(G) = r^D(G')$ and $dia^D(G') = dia^D(G) - 1$. □

Theorem 3.4. *In the above situation, we have*

(i) if $n = 4$, then $r^D(G') = r^D(G) \pm 1$, $dia^D(G') = dia^D(G) - 1$.

(ii) if $n = 5$, then $r^D(G') = r^D(G)$ or $r^D(G) - 1$, $dia^D(G') = dia^D(G) - 1$.

4 The Cyclic Graph C_n

In the context of cyclic graphs we have the following

Theorem 4.1. $G = C_n \setminus \{an\ edge\}$ *Then we have*

$$r^D(G) = \begin{cases} 3m+1 & if\ n = 2m \\ 3m-2 & if\ n = 2m-1 \end{cases}$$

and

$$dia^D(G) = 3(n-1).$$

Proof. It is obvious that $G = C_n \setminus \{e\}$ is nothing but the path graph, P_n, on n vertices. Then the theorem follows from the proposition 3.9 of [4]. □

5 The Wheel Graph $W_{n,1}$

In this section we deal with the derived graph of wheel graph $W_{n,1}$. Let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertex set of $W_{n,1}$. We assume that v_0 is adjacent to all other vertices. Thus $deg(v_0) = n$ and degrees of all other vertices is 3. By abuse of notation, sometimes we may call v_0 as *c*entral vertex.

We need to consider the two cases $n \geq 10$ and $n < 10$ separately. Then we have

Theorem 5.1. *Let G be a derived graph of $W_{n,1}$ $(n \geq 10)$ by deleting an edge which is not adjacent to the central vertex. Then G is radius and diameter invariant.*

Proof. Let $G = W_{n,1} \setminus \{e_i\}$ where $e_i = v_i v_{i+1}$ $i \neq 0$.

In the derived graph G the degree of v_i = the degree of $v_{i+1} = 2$, the degree of $v_0 = n$ and the degree of all other vertices is 3. The shortest path from v_0 to v_i of length 1. Thus $d^D(v_0, v_i) = d^D(v_0, v_{i+1}) = n + 3$, $d^D(v_0, v_j) = n + 4$ for $j \neq i, i + 1$ and $d^D(v_j, v_k)$ can be any of $\{7, 10, 11, 14, 15, 19, \cdots, (n + 6), (n + 7), (n + 8)\}$. Thus $e^D(v_0) = n + 4$, $e^D(v_i) = e^D(v_{i+1}) = n + 3$ and $e^D(v_k) = n + 8$ for $k \neq 0, i, i + 1$. Hence $r^D(G) = n + 4$ and $dia^D(G) = n + 8$.

By proposition 3.10 of [4] we have $r^D(W_{n,1}) = n+4$ and $d^D(W_{n,1}) = n+8$. Thus it follows that G is radius and diameter invariant as required. □

Theorem 5.2. *Let G be a derived graph of wheel graph $W_{n,1}$, $(n \geq 10)$ by deleting an edge which is adjacent to the central vertex. Then the radius and diameter both are increased by 2. In otherwords, we have*

$$r^D(G) = r^D(W_{n,1}) + 2; \quad dia^D(G) = dia^D(W_{n,1}) + 2.$$

Proof. Let the vertex set of the wheel graph $W_{n,1}$ be $\{v_0, v_1, v_2, \cdots, v_n\}$ where v_0 is adjecnt to all other vertices.

Let $G = W_{n,1} \setminus \{e_i\}$ where $e_i = v_0 v_i$. In G the degree of $v_0 = (n - 1)$, the degree of $v_i = 2$ and the degree of all other vertices is 3. The shortest path between v_0 and v_i of length is 2. Thus $d^D(v_0, v_i) = n + 6$, $d^D(v_0, v_j) = n + 3$ for $j \neq i$ and $d^D(v_j, v_k)$ can be any of $\{6, 7, 10, 11, 14, 15, 18, 19, \cdots, n + 7, n + 10\}$ for $\{j, k\} \neq \{0, i\}$.

Thus $e^D(v_0) = n + 6$, $e^D(v_k) = n + 10$, $1 \leq k \leq n$. Therefore $r^D(G) = n + 6$ and $dia^D(G) = n + 10$. □

Next we consider the remaining cases, namely, $n \leq 9$. In this context we have the following

Theorem 5.3. *Let G be derived graph of the wheel graph $W_{n,1}$, $(n < 10)$ by deleting an edge. In this case the D-radius D-radius and D-diameter D-diameter of G are as follows:*

In the following G_1 denotes a graph obtained from $W_{n,1}$ by deleting an edge which is not *adjacent to v_0 and G_2 denotes a graph obtained from $W_{n,1}$ by deleting an edge which is adjacent to v_0.*

Sl. No.	$W_{n,1}$	$r^D(G_1)$	$dia^D(G_1)$	$r^D(G_2)$	$dia^D(G_2)$
1	$W_{3,1}$	7	9	7	9
2	$W_{4,1}$	10	10	8	0
3	$W_{5,1}$	11	11	9	12
4	$W_{6,1}$	12	14	10	14
5	$W_{7,1}$	13	14	11	15
6	$W_{8,1}$	14	18	12	16
7	$W_{9,1}$	15	18	13	17

Table 6. D-radius and D-diameter of $W_{n,1}$ minus an edge

Proof. We will prove this theorem in two parts. In first part we delete an edge which is not adjacent to the central vertex and in second part, an edge which is adjacent to the central vertex.

Part I:

G is obtained from $W_{n,1}$ by deleting an edge which is not adjacent to the central vertex. We will prove the theorem for each n separately.

$\mathbf{n = 3:}$

We have $G_1 = W_{3,1} \setminus \{e\}$, where $e = v_i v_j$, $1 \leq i, j \leq 3$. In G_1 the degree of sequence is $\{2, 2, 3, 3\}$. The distances between vertices are given by $d^D(v_0, v_i)$ = 6 or 7, and $d^D(v_i, v_j)$ = 6 or 9. Hence $e^D(v_0) = 7$ and $e^D(v_i) = 9$. Therefore $r^D(G_1) = 7$ and $dia^D(G_1) = 9$.

n = 4 :
$G = W_{4,1} \setminus \{e\}$, where $e = v_iv_j$, $1 \leq i, j \leq 4$. In G the degree of sequence is $\{2, 2, 3, 3, 4\}$. Thus $d^D(v_0, v_i) = \{7, 8\}$ $d^D(v_i, v_j) = \{10, 6\}$. Hence $e^D(v_i) = 10$, $e^D(v_0) = 8$. Therefore $r^D(G) = 8$, $dia^D(G) = 10$.
n = 5 :
$G = W_{5,1} \setminus \{e\}$, where $e = v_iv_j$, $1 \leq i, j \leq 5$. In G the degree of sequence is $\{2, 2, 3, 3, 3, 5\}$. $d^D(v_0, v_i) = \{8, 9\}$ $d^D(v_i, v_j) = \{6, 10, 11, 12\}$. Hence $e^D(v_i) = 12$, $e^D(v_0) = 9$. Therefore $r^D(G) = 9$, $dia^D(G) = 12$.
n = 6 :
$G = W_{6,1} \setminus \{e\}$, where $e = v_iv_j$, $1 \leq i, j \leq 6$. In G the degree of sequence is $\{2, 2, 3, 3, 3, 6\}$. Thus $d^D(v_0, v_i) = \{9, 10\}$ $d^D(v_i, v_j) = \{6, 7, 10, 11, 12, 13\}$. Hence $e^D(v_i) = 14$, $e^D(v_0) = 10$.. Therefore $r^D(G) = 10$, $dia^D(G) = 14$.
n = 7 :
$G = W_{7,1} \setminus \{e\}$, where $e = v_iv_j$, $1 \leq i, j \leq 7$. In G the degree of sequence is $\{2, 2, 3, 3, 3, 3, 7\}$. $d^D(v_0, v_i) = \{10, 11\}$ $d^D(v_i, v_j) = \{6, 7, 11, 13, 14, 15\}$. Hence $e^D(v_i) = 15$, $e^D(v_0) = 11$. Therefore $r^D(G) = 11$, $dia^D(G) = 15$.
n = 8 :
$G = W_{8,1} \setminus \{e\}$, where $e = v_iv_j$, $1 \leq i, j \leq 8$. In G the degree of sequence is $\{2, 2, 3, 3, 3, 3, 3, 8\}$. $d^D(v_0, v_i) = \{11, 12\}$ $d^D(v_i, v_j) = \{6, 7, 10, 11, 14, 15, 16\}$. Hence $e^D(v_i) = 16$, $e^D(v_0) = 12$. Therefore $r^D(G) = 12$, $dia^D(G) = 16$.
n = 9 :
$G = W_{9,1} \setminus \{e\}$, where $e = v_iv_j$, $1 \leq i, j \leq 9$. In G the degree of sequence is $\{2, 2, 3, 3, 3, 3, 3, 9\}$. $d^D(v_0, v_i) = \{12, 13\}$ $d^D(v_i, v_j) = \{6, 7, 10, 11, 14, 15, 16, 17\}$. Hence $e^D(v_i) = 17$, $e^D(v_0) = 13$. Therefore $r^D(G) = 13$, $dia^D(G) = 17$.
Part II:
G is obtained from $W_{n,1}$ by deleting an edge which is not adjacent to the central vertex. We can prove this part similar to Part-I. □

From the above theorem we get the following

Corollary 5.4. *Let G be the derived graph of the wheel graph $W_{n,1}$ by deleting an edge which is not adjacent to central vertex v_0. Then*

(i) G is radius invariant for $n \geq 3$

(ii) G is diameter invariant for $n \geq 6$.

Corollary 5.5. *Let G be the derived graph of the wheel graph $W_{n,1}$ by deleting an edge which is adjacent to central vertex v_0. Then*

(i) G is radius invariant for $n = 3$ only.

(ii) G is diameter invariant for $n = 5, 6$ only.

6 The Complete Bipartite Graph $K_{m,m-1}$

In this section we deal with complete bipartite graph on $(m, m - 1)$, vertices We have,

Theorem 6.1. *Let G be the graph $K_{m,m-1} \setminus \{an\ edge\}$. Then we have $r^D(G) = 3m - 1$ and $dia^D(G) = 4m - 1$.*

Proof. Consider the bipartite graph $K_{m,m-1}$ where $\{v_1, v_2, \cdots, v_m, u_1, \cdots, u_{m-1}\}$ is the vertex set of $K_{m,m-1}$. Let G be the graph $K_{m,m-1} \setminus \{v_i, u_j\}$.

In G the degree of $v_i = m-2$, the degree of $u_j = m-1$ and the degree of remaing $v_i's = m-1$ and $u_k's = m$ then the shortest path between v_i,u_j is of length 3. Then the D distance between v_i, u_j is $d^D(v_i, u_j) = (m - 2) + (m - 1) + (m - 1) + m + 3 = 4m - 1$,
$d^D(v_i, v_l) = (m - 2) + (m - 1) + m + 2 = 3m - 1$,
$d^D(u_k, u_j) = (m - 1) + (m - 1) + m + 2 = 3m$ for $k \neq j$,
$d^D(u_j, v_l) = (m - 1) + (m - 1) + 1 = 2m - 1$ for $l \neq j$,
$d^D(u_k, u_m) = m + m + (m - 2) + 2 = 3m$,

$d^D(v_i, u_k) = (m - 2 + m + 1 = 2m - 1,$
Then $e^D(v_i) = 4m - 1,\ \ e^D(v_l) = 3m - 1,\ \ e^D(u_j) = 4m - 1,$ and $e^D(u_k) = 3m.$

Thus $r^D(G) = min\{e(v_i), e(u_j)\} = 3m - 1, \quad dia^D = max\{e(v_i), e(u_j)\} = 4m - 1$ as required. □

7 Open Problems

We end this article with some open problems.
In view of the above, we can ask

OP1 Classify all the graphs G for which $G' = G \setminus \{e\}$ is D-radius invariant.

OP2 Classify all the graphs G for which $G' = G \setminus \{e\}$ is D-diameter invariant.

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