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A NOTE ON RESIDUALLY SMALL RINGS AND MODULES

Rahul Kumar and Atul Gaur

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Abstract In this note, how the residual smallness of $R(+)M$ is related to the residual smallness of the ring R and the R-module M is discussed.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity; all ring extensions and ring homomorphisms are unital. Recently, residual smallness of rings and modules is studied by Oman and Salminen in [4, 5]. Let R be a ring. An infinite R-module M is said to be a residually small R-module if for each nonzero $m \in M$, there exists an R-submodule N of M not containing m such that $|M/N| < |M|$ (as usual, $|Y|$ denotes the cardinality of a set Y). An infinite ring R is said to be a residually small ring if R is a residually small R-module. In this note, we discuss the residual smallness of idealization of a module. Recall from [2, p.2] that if R is a ring and M is an R-module, then the idealization $R(+)M$ is a ring with additive structure is that of the abelian group $R \oplus M$, and its multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$, for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Note that R can be treated as a subring of $R(+)M$ via the canonical injective ring homomorphism $r \mapsto (r, 0)$. For further study on idealization, see [1]. In this note, we discuss the residual smallness of $R(+)M$ when $|M| < |R|$, $|R| < |M|$, and $|R| = |M|$.

2 Results

We start with the case $|M| < |R|$.

Theorem 2.1. *Let R be a ring and M be an R-module such that $|M| < |R|$. If $R(+)M$ is a residually small ring, then R is a residually small ring. The converse holds, when M is finitely generated.*

Proof. By [4, Proposition 3.8], R is a residually small ring. Conversely, assume that R is residually small and $M = Rx_1 + Rx_2 + \cdots + Rx_n$ for some $x_1, x_2, \ldots, x_n \in M$. It follows that $M \cong R^n/N$ where N is the kernel of the map $\phi : R^n \to M$ defined as $\phi(r_1, r_2, \ldots, r_n) = r_1x_1 + r_2x_2 + \cdots + r_nx_n$. Since $|M| < |R|$, $N \neq \{0\}$. Now, take a nonzero $(r, m) \in R(+)M$. If $r \neq 0$, then there exists an ideal J of R not containing r such that $|R/J| < |R|$. Now, by [1, Theorem 3.1], $J(+)M$ is an ideal of $R(+)M$ and $\frac{R(+)M}{J(+)M} \cong R/J$. It follows that $|\frac{R(+)M}{J(+)M}| = |R/J| < |R| = |R(+)M|$. Also, $(r, m) \notin J(+)M$. Thus, we are done. Now, assume that $r = 0$ and so m is nonzero. Note that R can be treated as an R-submodule of R^n via the map $r \mapsto (r, r, \ldots, r)$. Set $I = N \cap R$. Note that I is an ideal of R and $|R/I| \leq |R^n/N| = |M|$. Clearly, IM is the zero submodule of M. Therefore, by [1, Theorem 3.1], $I(+)\{0\}$ is an ideal of $R(+)M$ and $\frac{R(+)M}{I(+)\{0\}} \cong (R/I)(+)M$. It follows that

$$\Big|\frac{R(+)M}{I(+)\{0\}}\Big| = |(R/I)(+)M| = |M| < |R| = |R(+)M|$$

Also, $(0, m) \notin I(+)\{0\}$. Thus, $R(+)M$ is a residually small ring. □

From the last theorem, one may naturally think if the residual smallness of M is also a necessary condition for $R(+)M$ to be a residually small ring. The answer is no, as is evident from the next example.

Example 2.2. Let $R = \mathbb{Q} \times \mathbb{Q}[[X]]$ and $M = \mathbb{Q} \times \{0\}$. Then M is not residually small as M is a simple R-module. Now, we assert that $R(+)M$ is a residually small ring. First note that R is a residually small ring, by [4, Example 2.1(iv)] and [4, Theorem 3.4]. Also, $|M| < |R|$. Thus, the assertion follows by Theorem 2.1.

Now, if we consider $|R| < |M|$, then the residual smallness of M is a necessary and sufficient condition for $R(+)M$ to be a residually small ring. This is precisely our next result.

Theorem 2.3. *Let R be a ring and M be an R-module such that $|R| < |M|$. Then $R(+)M$ is a residually small ring if and only if M is a residually small R-module.*

Proof. Let $R(+)M$ be a residually small ring. Then it is easy to see that $R(+)M$ is a residually small R-module and M is an R-submodule of $R(+)M$. Thus, by [5, Proposition 1], M is a residually small R-module. Conversely, assume that M is residually small. Take any nonzero $(r, m) \in R(+)M$. If m is nonzero, then there exists an R-submodule N of M not containing m such that $|M/N| < |M|$. It follows that $(r, m) \notin \{0\}(+)N$. Now, by [1, Theorem 3.1], $\{0\}(+)N$ is an ideal of $R(+)M$ and $\frac{R(+)M}{\{0\}(+)N} \cong R(+)M/N$. It follows that

$$\left|\frac{R(+)M}{\{0\}(+)N}\right| = |R(+)M/N| < |M| = |R(+)M|$$

Thus, we are done. Now, assume that $m = 0$. Then clearly $r \neq 0$ and hence $(r, m) \notin \{0\}(+)M$. Again by [1, Theorem 3.1], $\{0\}(+)M$ is an ideal of $R(+)M$ and $\frac{R(+)M}{\{0\}(+)M} \cong R$. Moreover, $\left|\frac{R(+)M}{\{0\}(+)M}\right| = |R| < |M| = |R(+)M|$. Therefore, $R(+)M$ is a residually small ring. □

Our last theorem discusses the case $|R| = |M|$. Recall from [3] that an ideal I of a ring R is said to be large if $|R/I| < |R|$.

Theorem 2.4. *Let R be a ring and M be an R-module such that $|R| = |M|$. If $R(+)M$ is a residually small ring, then R is a residually small ring and M is a residually small R-module. Converse holds when every nonzero submodule N of M contained IM for some large ideal I of R.*

Proof. Let $R(+)M$ be a residually small ring such that $|R| = |M|$. Then by [4, Proposition 3.8], R is a residually small ring. It is easy to see that $R(+)M$ is a residually small R-module and M is an R-submodule of $R(+)M$. Thus, by [5, Proposition 1], M is a residually small R-module. Conversely, assume that R is a residually ring and M is a residually small R-module. Let (r, m) be a nonzero element in $R(+)M$. If $r \neq 0$, then there exists an ideal I of R not containing r such that $|R/I| < |R|$. It follows that $(r, m) \notin I(+)M$. Now, by [1, Theorem 3.1], $I(+)M$ is an ideal of $R(+)M$ and $\frac{R(+)M}{I(+)M} \cong R/I$. Therefore,

$$\left|\frac{R(+)M}{I(+)M}\right| = |R/I| < |R| = |R(+)M|$$

and hence we are done. Now, we may assume that $r = 0$ and so $m \neq 0$. Then there exists an R-submodule N of M not containing m such that $|M/N| < |M|$. Since N is nonzero, $IM \subseteq N$ for some ideal large ideal I of R. It follows that $|R/I| < |R|$. Moreover, by [1, Theorem 3.1], $I(+)N$ is an ideal of $R(+)M$ and $\frac{R(+)M}{I(+)N} \cong R/I(+)M/N$. Since $|R| = |M|$,

$$\left|\frac{R(+)M}{I(+)N}\right| = |R/I(+)M/N| < |R| = |R(+)M|$$

Note that $(0, m) \notin I(+)N$ and hence the result holds. □

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Author information

Rahul Kumar and Atul Gaur, Department of Mathematics, University of Delhi, Delhi 110007, India.
E-mail: rahulkmr977@gmail.com, gaursatul@gmail.com

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