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Atanassov's Intuitionistic Anti Fuzzy Interior Ideals of Semigroups

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Abstract In this paper the concepts of intuitionistic anti fuzzy interior ideal in semigroups is introduced and some important characterizations have been obtained.

1 Introduction

After the introduction of the classical notion of fuzzy sets by Zadeh[9] in 1965, many scientists used fuzzy sets in different fields of science. The use of fuzzy sets in algebraic structures was done by Rosenfeld[8] in 1971. He defined fuzzy subgroups and discussed their important properties. The idea of fuzzy ideals, fuzzy bi-ideals in semigroup was given by Kuroki[6]. With the passage of time researchers introduced many extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar fuzzy sets etc. In 1986, K. T. Atanassov[1] introduced the notion of intuitionistic fuzzy sets, which is the generalization of fuzzy sets. Fuzzy set gives the degree of membership of an element in a given set, but intuitionistic fuzzy set gives both degree of membership and degree of non-membership. The degree of membership and the degree of non-membership are the real numbers between 0 and 1, having sum not greater than 1. For more details on intuitionistic fuzzy sets, we refer [1, 2].

In [5] K. H. Kim and Y. B. Jun introduced the concept of intuitionistic fuzzy ideals of semigroups. In [4] K. H. Kim and J. G. Lee gave the notion of intuitionistic fuzzy bi-ideals of semigroups. In [3] Y. B. Jun introduced intuitionistic fuzzy bi-ideals of ordered semigroups and studied natural equivalence relation on the set of all intuitionistic fuzzy bi-ideals of an ordered semigroup. In [7], T. Nagaiah, S.K. Majumder, P. N. Swamy and T. Srinivas introduced the concept of intuitionistic anti fuzzy bi-ideals in semigroups. In this paper the concept of intuitionistic anti fuzzy interior ideals in a semigroup has been introduced. Here a regular semigroup has been characterized in terms of intuitionistic anti fuzzy interior ideal.

2 Preliminaries

Let S be a semigroup. Now recall the following from[1, 5].

• By a subsemigroup of S we mean a non-empty subset A of S such that $A^2 \subseteq A$.

• A subsemigroup A of S is called an interior ideal of S if $SAS \subseteq A$.

• A non-empty subset A of a semigroup S is called a left(right) ideal of S if $SA \subseteq A$(resp. $AS \subseteq A$).

• A non-empty subset A of a semigroup S is called a two sided ideal(ideal) of S if it is both a left ideal and a right ideal of S.

• Let X be a non-empty set. A fuzzy set of X is a function $\Psi : X \to [0,1]$ and the complement of Ψ, denoted by $\overline{\Psi}$ is a fuzzy set in X given by $\overline{\Psi}(x) = 1 - \Psi(x)$ for all $x \in X$.

• An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form

$$A = \{(x, \Psi_A(x), \Omega_A(x)) : x \in X\},$$

where the functions $\Psi_A : X \to [0,1]$ and $\Omega_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership, respectively and

$$0 \leq \Psi_A(x) + \Omega_A(x) \leq 1$$

for all $x \in X$.

• An intuitionistic fuzzy set $A = \{(x, \Psi_A(x), \Omega_A(x)) : x \in X\}$ in X can be identified as an order pair (Ψ_A, Ω_A) in $I^X X I^X$. For the sake of simplicity, we use the symbol $A = (\Psi_A, \Omega_A)$ for IFS $A = \{(x, \Psi_A(x), \Omega_A(x)) : x \in X\}$.

• Let X be a non-empty set and let $A = (\Psi_A, \Omega_A)$ and $B = (\Psi_B, \Omega_B)$ be any two IFSs in X. Then the following operations and relations are valid [1].

(1) $A \subseteq B$ iff $\Psi_A \leq \Psi_B$ and $\Omega_A \geq \Omega_B$
(2) $A = B$ iff $A \subset B$ and $B \subset A$
(3) $\overline{A} = (\overline{\Psi}_A, \overline{\Omega}_A) = (\Omega_A, \Psi_A)$
(4) $A \cap B = (\Psi_A \wedge \Psi_B, \Omega_A \vee \Omega_B)$
(5) $A \cup B = (\Psi_A \vee \Psi_B, \Omega_A \wedge \Omega_B)$
(6) $\Box A = (\Psi_A, 1 - \Psi_A), \Diamond A = (1 - \Omega_A, \Omega_A)$.

• An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy subsemigroup of S if

(1) $\Psi_A(xy) \geq \min\{\Psi_A(x), \Psi_A(y)\} \forall x, y \in S$,
(2) $\Omega_A(xy) \leq \max\{\Omega_A(x), \Omega_A(y)\} \forall x, y \in S$.

• An intuitionistic fuzzy subsemigroup $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy interior ideal of S if $\Psi_A(xay) \geq \Psi_A(a)$ and $\Omega_A(xay) \leq \Omega_A(a)$ for all $x, a, y \in S$.

• An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy left ideal of S if $\Psi_A(xy) \geq \Psi_A(y)$ and $\Omega_A(xy) \leq \Omega_A(y)$ for all $x, y \in S$.

• An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy right ideal of S if $\Psi_A(xy) \geq \Psi_A(x)$ and $\Omega_A(xy) \leq \Omega_A(x)$ for all $x, y \in S$.

• An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic fuzzy two-sided ideal or an intuitionistic fuzzy ideal of S if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S.

3 Intuitionistic Anti Fuzzy Interior Ideal

In this section we introduce the concepts of intuitionistic anti fuzzy interior ideal of a semigroup and investigate some of its important properties.

Definition 3.1. An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy subsemigroup of S if

(1) $\Psi_A(xy) \leq \max\{\Psi_A(x), \Psi_A(y)\} \forall x, y \in S$,
(2) $\Omega_A(xy) \geq \min\{\Omega_A(x), \Omega_A(y)\} \forall x, y \in S$.

Definition 3.2. An intuitionistic anti fuzzy subsemigroup $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy interior ideal of S if

(1) $\Psi_A(xay) \leq \Psi_A(a) \forall x, y, a \in S$,
(2) $\Omega_A(xay) \geq \Omega_A(a) \forall x, y, a \in S$.

Definition 3.3. An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy left ideal of S if

(1) $\Psi_A(xy) \leq \Psi_A(y) \forall x, y \in S$,
(2) $\Omega_A(xy) \geq \Omega_A(y) \forall x, y \in S$.

Definition 3.4. An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy right ideal of S if

(1) $\Psi_A(xy) \leq \Psi_A(x) \forall x, y \in S$,
(2) $\Omega_A(xy) \geq \Omega_A(x) \forall x, y \in S$.

Definition 3.5. An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy two-sided ideal or an intuitionistic anti fuzzy ideal of S if it is both an intuitionistic anti fuzzy left and an intuitionistic anti fuzzy right ideal of S.

Example 3.6. Let $S = \{a, b, c\}$ be a semigroup with the following cayley table.

$\cdot$	a	b	c
a	b	c	a
b	c	b	a
c	a	b	c

Let us define an IFS $A = (\Psi_A, \Omega_A)$ in S by $\Psi_A(a) = 0.4, \Psi_A(b) = \Psi_A(c) = 0.3$ and $\Omega_A(a) = 0.5, \Omega_A(b) = \Omega_A(c) = 0.7$.
The routine calculation shows that $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S which is not an intuitionistic anti fuzzy ideal of S since $\Psi_A(bc) \nleq \Psi_A(b)$.

4 Main Results

Lemma 4.1. *If an IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is an intuitionistic anti fuzzy interior ideal of S, then so is $\Box A := (\Psi_A, \overline{\Psi}_A)$.*

Proof. Suppose an IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is an intuitionistic anti fuzzy interior ideal of S. Then by definition it is clear that Ψ_A is an intuitionistic anti fuzzy interior ideal of S. It is sufficient to show that $\overline{\Psi}_A$ is an intuitionistic anti fuzzy interior ideal of S.

For any $x, y \in S$, we have

$$\begin{aligned} \overline{\Psi}_A(xy) &= 1 - \Psi_A(xy) \\ &\geq 1 - \max\{\Psi_A(x), \Psi_A(y)\} \\ &= \min\{1 - \Psi_A(x), 1 - \Psi_A(y)\} \\ &= \min\{\overline{\Psi}_A(x), \overline{\Psi}_A(y)\} \end{aligned}$$

Hence $\overline{\Psi}_A$ is an intuitionistic anti fuzzy subsemigroup of S.
Again for any $x, y, a \in S$, we have

$$\begin{aligned} \overline{\Psi}_A(xay) &= 1 - \Psi_A(xay) \\ &\geq 1 - \Psi_A(a) \\ &= \overline{\Psi}_A(a). \end{aligned}$$

Hence $\Box A$ is an intuitionistic anti fuzzy interior ideal of S. This completes the proof. □

Similarly we can prove the following lemma.

Lemma 4.2. *If $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy interior ideal of a semigroup S then so is $\Diamond A = (\overline{\Omega}_A, \Omega_A)$.*

Combining Lemmas 4.1 and 4.2 we obtain the following theorem.

Theorem 4.3. *$A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy interior ideal of a semigroup S if and only if $\Box A$ and $\Diamond A$ are intuitionistic anti fuzzy interior ideals of S.*

Theorem 4.4. *An IFS $A = (\Psi_A, \Omega_A)$ of a semigroup S is an intuitionistic anti fuzzy interior ideal of S if and only if $\overline{A}$ is an intuitionistic fuzzy interior ideal of S.*

Proof. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy interior ideal of S. To show that $\overline{A} = (\overline{\Psi}_A, \overline{\Omega}_A) = (\Omega_A, \Psi_A)$ is an intuitionistic fuzzy interior ideal. For any $x, y \in S$, we have

$$\begin{aligned} \overline{\Psi}_A(xy) &= 1 - \Psi_A(xy) \\ &\geq 1 - \max\{\Psi_A(x), \Psi_A(y)\} \\ &= \min\{1 - \Psi_A(x), 1 - \Psi_A(y)\} \\ &= \min\{\overline{\Psi}_A(x), \overline{\Psi}_A(y)\} \end{aligned}$$

and

$$\begin{aligned}\overline{\Omega}_A(xy) &= 1-\Omega_A(xy)\\ &\leq 1-\min\{\Omega_A(x),\Omega_A(y)\}\\ &= \max\{1-\Omega_A(x),1-\Omega_A(y)\}\\ &= \max\{\overline{\Omega}_A(x),\overline{\Omega}_A(y)\}.\end{aligned}$$

Hence $\overline{A}=(\overline{\Psi}_A,\overline{\Omega}_A)$ is an intuitionistic fuzzy subsemigroup of S. Let $x,y,a\in S$. Then

$$\begin{aligned}\overline{\Psi}_A(xay) &= 1-\Psi_A(xay)\\ &\geq 1-\Psi_A(a)\\ &= \overline{\Psi}_A(a)\end{aligned}$$

and

$$\begin{aligned}\overline{\Omega}_A(xay) &= 1-\Omega_A(xay)\\ &\leq 1-\Omega_A(a)\\ &= \overline{\Omega}_A(a).\end{aligned}$$

Hence $\overline{A}$ is an intuitionistic fuzzy interior ideal of S.

Conversely suppose that $\overline{A}$ is an intuitionistic fuzzy interior ideal of S. To show that $A=(\Psi_A,\Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S. Let $x,y\in S$. Then

$$\begin{aligned}\Psi_A(xy) &= 1-\overline{\Psi}_A(xy)\\ &\leq 1-\min\{\overline{\Psi}_A(x),\overline{\Psi}_A(y)\}\\ &= \max\{1-\overline{\Psi}_A(x),1-\overline{\Psi}_A(y)\}\\ &= \max\{\Psi_A(x),\Psi_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\Omega_A(xy) &= 1-\overline{\Omega}_A(xy)\\ &\geq 1-\max\{\overline{\Omega}_A(x),\overline{\Omega}_A(y)\}\\ &= \min\{1-\overline{\Omega}_A(x),1-\overline{\Omega}_A(y)\}\\ &= \min\{\Omega_A(x),\Omega_A(y)\}.\end{aligned}$$

Hence A is an intuitionistic anti fuzzy subsemigroup of S. Let $x,y,a\in S$. Then

$$\begin{aligned}\Psi_A(xay) &= 1-\overline{\Psi}_A(xay)\\ &\leq 1-\overline{\Psi}_A(a)\\ &= \Psi_A(a)\end{aligned}$$

and

$$\begin{aligned}\Omega_A(xay) &= 1-\overline{\Omega}_A(xay)\\ &\geq 1-\overline{\Omega}_A(a)\\ &= \Omega_A(a).\end{aligned}$$

Hence A is an intuitionistic anti fuzzy interior ideal of S. This completes the proof. □

Theorem 4.5. *If an IFS $A=(\Psi_A,\Omega_A)$ in S is an intuitionistic anti fuzzy interior ideal of a semigroup S, then so is $\Box A:=(\Psi_A,\overline{\Psi}_A)$ where $\overline{\Psi}_A=1-\Psi_A$.*

Proof. Since A is an intuitionistic anti fuzzy interior ideal of S then $\Psi_A(xy)\leq\max\{\Psi_A(x),\Psi_A(y)\}$ and $\Psi_A(xay)\leq\Psi_A(a)\forall x,y,a\in S$. Now

$$\begin{aligned}\overline{\Psi}_A(xy) &= 1-\Psi_A(xy)\\ &\geq 1-\max\{\Psi_A(x),\Psi_A(y)\}\\ &= \min\{1-\Psi_A(x),1-\Psi_A(y)\}\\ &= \min\{\overline{\Psi}_A(x),\overline{\Psi}_A(y)\}\end{aligned}$$

and $\overline{\Psi}_A(xay)=1-\Psi_A(xay)\geq 1-\Psi_A(a)=\overline{\Psi}_A(a)$.

Hence A is an intuitionistic anti fuzzy interior ideal of S. This completes the proof. □

Theorem 4.6. *An IFS $A=(\Psi_A,\Omega_A)$ in S is an intuitionistic anti fuzzy interior ideal of a semigroup S if and only if the fuzzy sets Ψ_A and $\overline{\Omega}_A$ are anti fuzzy interior ideals of S.*

Proof. Let $A=(\Psi_A,\Omega_A)$ be an intuitionistic anti fuzzy interior ideal of S. Then Ψ_A is an anti fuzzy interior ideal of S. Let $x,y,a\in S$. Then

$$\begin{aligned}\overline{\Omega}_A(xy) &= 1-\Omega_A(xy)\\ &\leq 1-\min\{\Omega_A(x),\Omega_A(y)\}\\ &= \max\{1-\Omega_A(x),1-\Omega_A(y)\}\\ &= \max\{\overline{\Omega}_A(x),\overline{\Omega}_A(y)\}.\end{aligned}$$

also $\overline{\Omega}_A(xay)=1-\Omega_A(xay)\leq 1-\Omega_A(a)=\overline{\Omega}_A(a)$.

Hence $\overline{\Omega}_A$ is an anti fuzzy interior ideal of S.

Conversely, suppose that Ψ_A and $\overline{\Omega}_A$ are anti fuzzy interior ideals of S. Let $x,y,a\in S$. Then we have to show that $A=(\Psi_A,\Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S. Since Ψ_A is an anti fuzzy interior ideal then $\Psi_A(xy)\leq\max\{\Psi_A(x),\Psi_A(y)\}$ and $\Psi_A(xay)\leq\Psi_A(a)$. Now

$$\begin{aligned}1-\Omega_A(xy) &= \overline{\Omega}_A(xy)\\ &\leq \max\{\overline{\Omega}_A(x),\overline{\Omega}_A(y)\}\\ &= \max\{1-\Omega_A(x),1-\Omega_A(y)\}\\ &= 1-\min\{\Omega_A(x),\Omega_A(y)\}\\ \Rightarrow 1-\Omega_A(xy) &\leq 1-\min\{\Omega_A(x),\Omega_A(y)\}\\ \Rightarrow \Omega_A(xy) &\geq \min\{\Omega_A(x),\Omega_A(y)\}\end{aligned}$$

Also $1-\Omega_A(xay)=\overline{\Omega}_A(xay)\leq\overline{\Omega}_A(a)=1-\Omega_A(a)$. This implies $\Omega_A(xay)\geq\Omega_A(a)$. Hence $A=(\Psi_A,\Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S. This completes the proof. □

Theorem 4.7. *Let $\{A_i : i\in I\}$ be a collection of intuitionistic anti fuzzy interior ideals of a semigroup S then their union $\bigcup_{i\in I}A_i=(\bigvee_{i\in I}\Psi_{A_i},\bigwedge_{i\in I}\Omega_{A_i})$ is an intuitionistic anti fuzzy interior ideal of S, where*

$\bigvee_{i\in I}\Psi_{A_i}(x)=\sup\{\Psi_{A_i}(x):i\in I,x\in S\}$ *and* $\bigwedge_{i\in I}\Omega_{A_i}(x)=\inf\{\Omega_{A_i}(x):i\in I,x\in S\}$.

Proof. As $\bigcup_{i\in I}A_i=(\bigvee_{i\in I}\Psi_{A_i},\bigwedge_{i\in I}\Omega_{A_i})$, where $\bigvee_{i\in I}\Psi_{A_i}(x)=\sup\{\Psi_{A_i}(x):i\in I,x\in S\}$ and $\bigwedge_{i\in I}\Omega_{A_i}(x)=\inf\{\Omega_{A_i}(x):i\in I,x\in S\}$. For $x,y\in S$, we have

$$\begin{aligned}(\bigvee_{i\in I}\Psi_{A_i})(xy) &= \sup\{\Psi_{A_i}(xy):i\in I,xy\in S\}\\ &\leq \sup\{\max\{\Psi_{A_i}(x),\Psi_{A_i}(y)\}:i\in I,x,y\in S\}\\ &\leq \max\{\sup\{\Psi_{A_i}(x):i\in I,x\in S\},\sup\{\Psi_{A_i}(y):i\in I,y\in S\}\}\\ &= \max\{\bigvee_{i\in I}\Psi_{A_i}(x),\bigvee_{i\in I}\Psi_{A_i}(y)\}\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in I}\Omega_{A_i})(xy) &= \inf\{\Omega_{A_i}(xy) : i\in I, xy\in S\}\\ &\geq \inf\{\min\{\Omega_{A_i}(x),\Omega_{A_i}(y)\} : i\in I, x,y\in S\}\\ &\geq \min\{\inf\{\Omega_{A_i}(x) : i\in I, x\in S\}, \inf\{\Omega_{A_i}(y) : i\in I, y\in S\}\}\\ &= \min\{\bigwedge_{i\in I}\Omega_{A_i}(x), \bigwedge_{i\in I}\Omega_{A_i}(y)\}\end{aligned}$$

Hence $\bigcup_{i\in I} A_i$ is an intuitionistic anti fuzzy subsemigroup of S. Also for any $x,y,a\in S$

$$\begin{aligned}(\bigvee_{i\in I}\Psi_{A_i})(xay) &= \sup\{\Psi_{A_i}(xay) : i\in I, xay\in S\}\\ &\leq \sup\{\Psi_{A_i}(a) : i\in I, a\in S\}\\ &= \bigvee_{i\in I}\Psi_{A_i}(a)\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in I}\Omega_{A_i})(xay) &= \inf\{\Omega_{A_i}(xay) : i\in I, xay\in S\}\\ &\geq \inf\{\Omega_{A_i}(a) : i\in I, a\in S\}\\ &= \bigwedge_{i\in I}\Omega_{A_i}(a).\end{aligned}$$

Hence $\bigcup_{i\in I} A_i$ is an intuitionistic anti fuzzy interior ideal of S. Similarly we can prove the other cases also. This completes the proof. □

Definition 4.8. For any $t\in[0,1]$ and a fuzzy subset Ψ of a semigroup S, the set $U(\Psi;t)=\{x\in S:\Psi(x)\geq t\}$(resp. $L(\Psi;t)=\{x\in S:\Psi(x)\leq t\}$) is called an upper(resp. lower) t-level cut of Ψ.

Theorem 4.9. *If $A=(\Psi_A,\Omega_A)$ is an intuitionistic anti fuzzy interior ideal of a semigroup S, then the upper and lower level cuts $L(\Psi_A;t)$ and $U(\Omega_A;t)$ are interior ideals of S, for every $t\in Im(\Psi_A)\cap Im(\Omega_A)$.*

Proof. Let $t\in Im(\Psi_A)\cap Im(\Omega_A)$. Let $A=(\Psi_A,\Omega_A)$ be an intuitionistic anti fuzzy interior ideal of S. Then $A=(\Psi_A,\Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S. Let $x,y\in L(\Psi_A;t)$. Then $\Psi_A(x)\leq t$ and $\Psi_A(y)\leq t$ whence $\max\{\Psi_A(x),\Psi_A(y)\}\leq t$. Now $\Psi_A(xy)\leq \max\{\Psi_A(x),\Psi_A(y)\}$. Hence $\Psi_A(xy)\leq t, i.e., xy\in L(\Psi_A;t)$. Consequently, $L(\Psi_A;t)$ is a subsemigroup of S.

Now let $x,z\in S$ and $y\in L(\Psi_A;t)$. Then $\Psi_A(y)\leq t$. Now, since $A=(\Psi_A,\Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S, hence $\Psi_A(xyz)\leq\Psi_A(y)\leq t$. Consequently, $xyz\in L(\Psi_A;t)$. Hence $L(\Psi_A;t)$ is an interior ideal of S. Similarly we can prove other case also. □

Theorem 4.10. *If $A=(\Psi_A,\Omega_A)$ is an intuitionistic fuzzy subset of a semigroup S such that the non-empty sets $L(\Psi_A;t)$ and $U(\Omega_A;t)$ are interior ideals of S, for $t\in[0,1]$, then $A=(\Psi_A,\Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S.*

Proof. For $t\in[0,1]$, let us assume that the non-empty sets $L(\Psi_A;t)$ and $U(\Omega_A;t)$ are interior ideals of S. Now, we shall show that $A=(\Psi_A,\Omega_A)$ satisfies the conditions of Definition 3.1 and

Definition 3.2. Let $x, y \in S$. Let $t_0 = \max\{\Psi_A(x), \Psi_A(y)\}$ and $t_1 = \min\{\Omega_A(x), \Omega_A(y)\}$. Then $x, y \in L(\Psi_A; t_0)$ and $x, y \in U(\Omega_A; t_1)$. So $xy \in L(\Psi_A; t_0)$ and $xy \in U(\Omega_A; t_1)$ which implies that $\Psi_A(xy) \leq t_0 = \max\{\Psi_A(x), \Psi_A(y)\}$ and $\Omega_A(xy) \geq t_1 = \min\{\Omega_A(x), \Omega_A(y)\}$. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S.

Again let $x, a, y \in S$. Let $t_2 = \Psi_A(a)$ and $t_3 = \Omega_A(a)$. Then $a \in L(\Psi_A; t_2)$ and $a \in U(\Omega_A; t_3)$. So $xay \in L(\Psi_A; t_2)$ and $xay \in U(\Omega_A; t_3)$ which implies that $\Psi_A(xay) \leq t_2 = \Psi_A(a)$ and $\Omega_A(xay) \geq t_3 = \Omega_A(a)$. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S. □

Theorem 4.11. *Let I be a non-empty subset of a semigroup S. If two fuzzy subsets Ψ and Ω are defined on S by*

$$\Psi(x) := \begin{cases} \alpha_0 & \text{if } x \in I \\ \alpha_1 & \text{if } x \in S - I \end{cases}$$

and

$$\Omega(x) := \begin{cases} \beta_0 & \text{if } x \in I \\ \beta_1 & \text{if } x \in S - I \end{cases}$$

where $0 \leq \alpha_0 < \alpha_1, 0 \leq \beta_1 < \beta_0$ and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$. Then $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy interior ideal of S and $U(\Omega; \alpha_0) = I = L(\Psi; \beta_0)$.

Proof. Let I be an interior ideal of S. Then I is a subsemigroup of S. Let $x, y \in S$. Then following four cases may arise:

(1) : If $x \in I$ and $y \in I$, (2) : If $x \notin I$ and $y \in I$, (3) : If $x \in I$ and $y \notin I$, (4) : If $x \notin I$ and $y \notin I$.

In Case 4, $\Psi(x) = \Psi(y) = \alpha_1$ and $\Omega(x) = \Omega(y) = \beta_1$. Then $\max\{\Psi(x), \Psi(y)\} = \alpha_1$ and $\min\{\Omega(x), \Omega(y)\} = \beta_1$. Now $\Psi(xy) = \alpha_0$ and α_1 or $\Omega(xy) = \beta_0$ and β_1 according as $xy \in I$ or $xy \notin I$. Again $\alpha_0 < \alpha_1$ and $\beta_1 < \beta_0$. Hence we see that $\Psi(xy) \leq \max\{\Psi(x), \Psi(y)\}$ and $\Omega(xy) \geq \min\{\Omega(x), \Omega(y)\}$. Hence $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy subsemigroup of S. For other cases, by using a similar argument we can deduce that $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy subsemigroup of S.

Now, let $x, y, z \in S$. Then following two cases may arise:

(1) : If $y \in I$, (2) : If $y \notin I$.

In Case 2, $\Psi(y) = \alpha_1$ and $\Omega(y) = \beta_1$. Now $\Psi(xyz) = \alpha_0$ or α_1 and $\Omega(xyz) = \beta_0$ or β_1 according as $xyz \in I$ or $xyz \notin I$. Again $\alpha_0 < \alpha_1$ and $\beta_1 < \beta_0$. Hence we see that $\Psi(xyz) \leq \Psi(y)$ and $\Omega(xyz) \geq \Omega(y)$. Hence $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy interior ideal of S. For other case, by using a similar argument we can deduce that $A = (\Psi, \Omega)$ is an intuitionistic anti fuzzy interior ideal of S.

In order to prove the converse, we first observe that by definition of Ψ and Ω, $U(\Omega; \alpha_0) = I = L(\Psi; \beta_0)$. Then the proof follows from Theorem 4.10. □

Definition 4.12. A mapping Φ from a semigroup S to another semigroup T is called a homomorphism if $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in S$.

Theorem 4.13. *Let $f : S \to T$ be an homomorphism of semigroups and $B = (\Psi_B, \Omega_B)$ is an intuitionistic anti fuzzy interior ideal of T. Then the pre-image $f^{-1}(B)$ of B under f is an intuitionistic anti fuzzy interior ideal of S, where $f^{-1}(B)(x) := (f^{-1}(\Psi_B)(x), f^{-1}(\Omega_B)(x)) = (\Psi_B(f(x)), \Omega_B(f(x)))$.*

Proof. Let $B = (\Psi_B, \Omega_B)$ be an intuitionistic anti fuzzy interior ideal of T and let $x, y \in S$. Then

$$\begin{aligned} f^{-1}(\Psi_B(xy)) &= \Psi_B(f(xy)) \\ &= \Psi_B(f(x)f(y)) \\ &\leq \max\{\Psi_B(f(x)), \Psi_B(f(y))\} \\ &= \max\{f^{-1}(\Psi_B(x)), f^{-1}(\Psi_B(y))\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\Omega_B(xy)) &= \Omega_B(f(xy)) \\ &= \Omega_B(f(x)f(y)) \\ &\geq \min\{\Omega_B(f(x)), \Omega_B(f(y))\} \\ &= \min\{f^{-1}(\Omega_B(x)), f^{-1}(\Omega_B(y))\}. \end{aligned}$$

Hence $f^{-1}(B) = (f^{-1}(\Psi_B), f^{-1}(\Omega_B))$ is an intuitionistic anti fuzzy subsemigroup of S. Now for any $a, x, y \in S$ we have

$$\begin{aligned} f^{-1}(\Psi_B(xay)) &= \Psi_B(f(xay)) \\ &= \Psi_B(f(x)f(a)f(y)) \\ &\leq \Psi_B(f(a)) \\ &= f^{-1}(\Psi_B(a)) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\Omega_B(xay)) &= \Omega_B(f(xay)) \\ &= \Omega_B(f(x)f(a)f(y)) \\ &\geq \Omega_B(f(a)) \\ &= f^{-1}(\Omega_B(a)). \end{aligned}$$

Therefore $f^{-1}(B) = (f^{-1}(\Psi_B), f^{-1}(\Omega_B))$ is an intuitionistic anti fuzzy interior ideal of S. □

Proposition 4.14. *Let θ be an endomorphism and $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy interior ideal of a semigroup S. Then $A[\theta]$ is also an intuitionistic anti fuzzy interior ideal of S, where $A[\theta](x) := (\Psi_A[\theta](x), \Omega_A[\theta](x)) = (\Psi_A(\theta(x)), \Omega_A(\theta(x))) \forall x \in S$.*

Proof. Let $x, y \in S$. Then

$$\begin{aligned} \Psi_A[\theta](xy) &= \Psi_A(\theta(xy)) = \Psi_A(\theta(x)\theta(y)) \\ &\leq \max\{\Psi_A(\theta(x)), \Psi_A(\theta(y))\} \\ &= \max\{\Psi_A[\theta](x), \Psi_A[\theta](y)\} \end{aligned}$$

and

$$\begin{aligned} \Omega_A[\theta](xy) &= \Omega_A(\theta(xy)) = \Omega_A(\theta(x)\theta(y)) \\ &\geq \min\{\Omega_A(\theta(x)), \Omega_A(\theta(y))\} \\ &= \min\{\Omega_A[\theta](x), \Omega_A[\theta](y)\}. \end{aligned}$$

Hence $A[\theta]$ is an intuitionistic anti fuzzy subsemigroup of S. Similarly we can prove other cases also. This completes the proof. □

Proposition 4.15. *Let S be a semigroup and $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy ideal of S. Then A is an intuitionistic anti fuzzy interior ideal of S.*

Proof. Let $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy ideal of S and $x, y \in S$. Then $\Psi_A(xy) \leq \Psi_A(x)$ and $\Psi_A(xy) \leq \Psi_A(y)$, which implies that $\Psi_A(xy) \leq \min\{\Psi_A(x), \Psi_A(y)\}$. Again $\Omega_A(xy) \geq \Omega_A(x)$ and $\Omega_A(xy) \geq \Omega_A(y)$, which implies $\Omega_A(xy) \geq \max\{\Omega_A(x), \Omega_A(y)\}$. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy subsemigroup of S.

Now, let $x, a, y \in S$. Then $\Psi_A(xay) = \Psi_A(x(ay)) \leq \Psi_A(ay) \leq \Psi_A(a)$ and $\Omega_A(xay) = \Omega_A(x(ay)) \geq \Omega_A(ay) \geq \Omega_A(a)$.

Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy interior ideal of S. □

Definition 4.16. A semigroup S is said to be regular if for each element a of S, there exists an element $x \in S$ such that $a = axa$.

Proposition 4.17. *Let S be a regular semigroup and $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy interior ideal of S. Then A is an intuitionistic anti fuzzy ideal of S.*

Proof. Let $x, y \in S$. Since S is regular, for any $x \in S$ there exist $a \in S$ such that $x = xax$. Then $\Psi_A(xy) = \Psi_A(xaxy) \le \Psi_A(x)$ and $\Omega_A(xy) = \Omega_A(xaxy) \ge \Omega_A(x)$. So $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy right ideal of S.

Similarly, we can prove that $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy left ideal of S. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy ideal of S. □

Remark 4.18. From the above two propositions it is clear that in regular semigroups the concept of intuitionistic anti fuzzy ideals and intuitionistic anti fuzzy interior ideals coincide.

Definition 4.19. An interior ideal M of a semigroup S is called a characteristic interior ideal of S if $f(M) = M$ for all $f \in Aut(S)$.

Definition 4.20. An intuitionistic anti fuzzy interior ideal $A = (\Psi_A, \Omega_A)$ of a semigroup S is called an intuitionistic anti fuzzy characteristic interior ideal of S if $\Psi_A(f(x)) = \Psi_A(x)$ and $\Omega_A(f(x)) = \Omega_A(x) \forall x \in S$ and $f \in Aut(S)$.

Theorem 4.21. *If $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy characteristic interior ideal of a semigroup S then the lower and upper level cuts $L(\Psi_A; t)$ and $U(\Omega_A; t)$ are characteristic interior ideals of S for every $t \in Im(\Psi_A) \cap Im(\Omega_A)$.*

Proof. Let $t \in Im(\Psi_A) \cap Im(\Omega_A)$. Let $f \in Aut(S)$ and $x \in U(\Omega_A; t)$. Then $\Omega_A(x) \ge t$ and so $\Omega_A(f(x)) \ge t$. Hence $f(x) \in U(\Omega_A; t)$. This implies that $f(U(\Omega_A; t)) \subset U(\Omega_A; t)$. Again, let $x \in U(\Omega_A; t)$ and $y \in S$ such that $f(y) = x$. Then, $\Omega_A(y) = \Omega_A(f(y)) = \Omega_A(x) \ge t$, so $y \in U(\Omega_A; t)$. Consequently, $f(y) \in f(U(\Omega_A; t))$, whence $x \in f(U(\Omega_A; t))$. Hence $U(\Omega_A; t) \subset f(U(\Omega_A; t)))$. So we have $U(\Omega_A; t) = f(U(\Omega_A; t)))$. Hence $U(\Omega_A; t)$ is a characteristic interior ideal of S. By using similar argument we can show that $L(\Psi_A; t)$ is a characteristic interior ideal of S. □

Theorem 4.22. *If $A = (\Psi_A, \Omega_A)$ is an intuitionistic fuzzy subset of a semigroup S such that the non- empty sets $U(\Omega_A; t)$ and $L(\Psi_A; t)$ are characteristic interior ideals of S for all $t \in [0, 1]$. Then $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy characteristic interior ideal of S.*

Proof. Let $U(\Omega_A; t)$ and $L(\Psi_A; t)$ are characteristic interior ideals of S for all $t \in [0, 1]$. Let $f \in Aut(S), x \in S$ and $\Omega_A(x) = t_0$ and $\Psi_A(x) = t_1$. Then $x \in U(\Omega_A; t_0)$ and $x \in L(\Psi_A; t_1)$. Since, by hypothesis, $U(\Omega_A; t_0) = f(U(\Omega_A; t_0))$ and $L(\Psi_A; t_1) = f(L(\Psi_A; t_1))$, so we see that $f(x) \in U(\Omega_A; t_0)$ and $f(x) \in L(\Psi_A; t_1)$. Hence $\Omega_A(f(x)) \ge t_0$ and $\Psi_A(f(x)) \le t_1$. Let $t_2 = \Omega_A(f(x))$ and $t_3 = \Psi_A(f(x))$. Then $t_2 \ge t_0$ and $t_3 \le t_1$ and $f(x) \in U(\Omega_A; t_2) = f(U(\Omega_A; t_2)), f(x) \in L(\Psi_A; t_3) = f(L(\Psi_A; t_3))$. Since f is one-one, we have $x \in U(\Omega_A; t_2)$ and $x \in L(\Psi_A; t_3)$. This implies that $\Omega_A(x) \ge t_2$ and $\Psi_A(x) \le t_3$. Hence $t_0 \ge t_2$ and $t_1 \le t_3$. Thus we obtain $\Omega_A(f(x)) = \Omega_A(x)$ and $\Psi_A(f(x)) = \Psi_A(x)$. Hence $A = (\Psi_A, \Omega_A)$ is an intuitionistic anti fuzzy characteristic interior ideal of S. □

Proposition 4.23. *Let $\alpha \ge 0$ be a real number and $A = (\Psi_A, \Omega_A)$ be an intuitionistic anti fuzzy interior ideal of a semigroup S. Then so is $A^\alpha = (\Psi_A^\alpha, \Omega_A^\alpha)$, where $\Psi_A^\alpha(x) = (\Psi_A(x))^\alpha$ and $\Omega_A^\alpha(x) = (\Omega_A(x))^\alpha$ for all $x \in S$.*

Proof. Let $x, y \in S$. Without any loss of generality, suppose $\Psi_A(x) \ge \Psi_A(y)$ and $\Omega_A(x) \le \Omega_A(y)$. Then $\Psi_A^\alpha(x) \ge \Psi_A^\alpha(y)$ and $\Omega_A^\alpha(x) \le \Omega_A^\alpha(y)$. Now $\Psi_A(xy) \le \max\{\Psi_A(x), \Psi_A(y)\} = \Psi_A(x)$ and $\Omega_A(xy) \ge \min\{\Omega_A(x), \Omega_A(y)\} = \Omega_A(x)$. Then

$$\Psi_A^\alpha(xy) = (\Psi_A(xy))^\alpha \le (\Psi_A(x))^\alpha = \Psi_A^\alpha(x) = \max\{\Psi_A^\alpha(x), \Psi_A^\alpha(y)\}$$

and

$$\Omega_A^\alpha(xy) = (\Omega_A(xy))^\alpha \ge (\Omega_A(x))^\alpha = \Omega_A^\alpha(x) = \min\{\Omega_A^\alpha(x), \Omega_A^\alpha(y)\}.$$

Consequently, $A^\alpha = (\Psi_A^\alpha, \Omega_A^\alpha)$ is an intuitionistic anti fuzzy subsemigroup of S.

Again let $x, a, y \in S$. Then

$$\Psi_A(xay) \le \Psi_A(a) \text{ and } \Omega_A(xay) \ge \Omega_A(a).$$

Now

$$\Psi_A^\alpha(xay) = (\Psi_A(xay))^\alpha \le (\Psi_A(a))^\alpha = \Psi_A^\alpha(a)$$

and

$$\Omega_A^{\alpha}(xay) = (\Omega_A(xay))^{\alpha} \geq (\Omega_A(a))^{\alpha} = \Omega_A^{\alpha}(a).$$

Hence $A^{\alpha} = (\Psi_A^{\alpha}, \Omega_A^{\alpha})$ is an intuitionistic anti fuzzy interior ideal of S. □

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