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PRESENTATIONS OF GENERAL LINEAR GROUPS WITH JORDAN REGULAR GENERATORS

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Abstract In this article, we obtain presentations of the general linear groups $GL(2,\mathbb{Z}_{16})$, $GL(2,\mathbb{Z}_{18})$, $GL(2,\mathbb{Z}_{20})$, $GL(2,\mathbb{Z}_{24})$, $GL(2,\mathbb{Z}_{28})$, $GL(2,\mathbb{Z}_{30})$, $GL(2,\mathbb{Z}_{32})$, $GL(2,\mathbb{Z}_{36})$, $GL(2,\mathbb{Z}_{38})$, $GL(2,\mathbb{Z}_{40})$ and $GL(2,\mathbb{Z}_{42})$ with Jordan regular units as generators.

1 Introduction

Let R be an arbitrary ring. Lie regular elements and units in R were introduced and studied by Sharma, Yadav and Kanwar in [5]. They defined that an element $a \in R$ is called a Lie regular element if $a = eu - ue$ for some idempotent $e \in R$ and some unit $u \in R$. A Lie regular element which is also a unit is called a Lie regular unit. They used Lie regular units to describe the general linear groups and obtained presentations of $GL(2,\mathbb{Z}_4)$, $GL(2,\mathbb{Z}_6)$, $GL(2,\mathbb{Z}_8)$, $GL(2,\mathbb{Z}_9)$, $GL(2,\mathbb{Z}_{10})$, $GL(2,\mathbb{Z}_{14})$, $GL(2,\mathbb{Z}_{15})$, $GL(2,\mathbb{Z}_{22})$, $GL(2,\mathbb{Z}_{25})$, $GL(2,\mathbb{Z}_{26})$, $GL(2,\mathbb{Z}_{27})$ and $GL(2,\mathbb{Z}_{34})$ having Lie regular units as generators in [5, 6]. We call an element $a \in R$, a Jordan regular element if $a = eu + ue$ for some idempotent $e \in R$ and some unit $u \in R$. A Jordan regular element which is also a unit is called a Jordan regular unit. Jordan regular elements and Jordan regular units were introduced by the authors in [3]. For a commutative ring R with unity, the authors have studied Jordan regular units in $M(2,R)$ in [3]. It is proved that if 2 is a unit in R, then every unit in $M(2,R)$ is a Jordan regular unit, but if 2 is not a unit in R, then it is not necessary that every unit in $M(2,R)$ is a Jordan regular unit. Jordan regular units in $GL(2,\mathbb{Z}_{2^n})$ and $GL(2,\mathbb{Z}_{2n})$ have been obtained in [3]. Further we have proved that for $n \geq 2$, the general linear group $GL(2,F_{2^n})$ can be generated by Jordan regular units, see [4]. Here F_q denotes a finite field containing q elements. In the same paper, presentations of $GL(2,F_4)$, $GL(2,F_8)$, $GL(2,F_{16})$ and $GL(2,F_{32})$ have been obtained having Jordan regular units as generators.

In this article, we use Jordan regular units to obtain presentations of the general linear groups $GL(2,\mathbb{Z}_{16})$, $GL(2,\mathbb{Z}_{18})$, $GL(2,\mathbb{Z}_{20})$, $GL(2,\mathbb{Z}_{24})$, $GL(2,\mathbb{Z}_{28})$, $GL(2,\mathbb{Z}_{30})$, $GL(2,\mathbb{Z}_{32})$, $GL(2,\mathbb{Z}_{36})$, $GL(2,\mathbb{Z}_{38})$, $GL(2,\mathbb{Z}_{40})$ and $GL(2,\mathbb{Z}_{42})$.

We have used GAP(Groups Algorithms-programming) software for all the algebraic computations throughout this paper.

2 Presentations of General Linear Groups over $\mathbb{Z}_n$

In this section, we find presentations of general linear groups over $\mathbb{Z}_n$ for even n having Jordan regular units as generators. Let ϕ denote the Euler's totient function.

Proposition 2.1. *[6, Proposition 2.1] For any prime p, $|GL(2,\mathbb{Z}_{p^n})| = p^{2n-1}(p+1)(\phi(p^n))^2$.*

Corollary 2.2. *[6, Corollary 2.2] For any $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are distinct primes, $|GL(2,\mathbb{Z}_n)| = \prod_{i=1}^{k} p_i^{2\alpha_i-1}(p_i+1)(\phi(p_i^{\alpha_i}))^2$.*

Corollary 2.3. *[6, Corollary 2.3] For any two distinct primes p and q, $|GL(2,\mathbb{Z}_{pq})| = pq(p+1)(q+1)(p-1)^2(q-1)^2$.*

Corollary 2.4. *[6, Corollary 2.3]* $|SL(2,\mathbb{Z}_n)| = \frac{|GL(2,\mathbb{Z}_n)|}{\phi(n)}$.

Proposition 2.5. *If L denotes the group of lower triangular matrices in $SL(2,\mathbb{Z}_n)$, then $|L| = n\phi(n)$.*

Proof. Any element in L is of the form $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$, where $a,b,c \in \mathbb{Z}_n$ and $ab = 1$. So $a \in \mathcal{U}(\mathbb{Z}_n)$ and $|\mathcal{U}(\mathbb{Z}_n)| = \phi(n)$. Also for each value of a, there are n choice for c, hence $|L| = n\phi(n)$. □

Lemma 2.6. *The elements $a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 1 \\ 12 & 9 \end{pmatrix}$ and $c = \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}$ are Jordan regular units in $M(2,\mathbb{Z}_{16})$.*

Proof. The proof is clear once we observe that $a = e_1u_1 + u_1e_1$, $b = e_2u_2 + u_2e_2$ and $c = e_3u_3 + u_3e_3$, where $e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 1 & 0 \\ 11 & 0 \end{pmatrix}$ are idempotents in $M(2,\mathbb{Z}_{16})$ and $u_1 = \begin{pmatrix} 1 & 15 \\ 1 & 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 & 3 \\ 5 & 0 \end{pmatrix}$ are units in $M(2,\mathbb{Z}_{16})$. □

Theorem 2.7. *Presentation of $GL(2,\mathbb{Z}_{16})$ is*
$GL(2,Z_{16}) = \langle\, a,\ b,\ c \mid\ a^{16},\ b^{16},\ c^8,\ b^8c^4,\ c^5bab^4cb^{15}ab^{15}ca^{15}c^7b^{15}a^{15},\ b^3c(b^{15}a)^2b^{15}(c^7a)^2c,\ c^5bcbac^4a^7,\ c^5bcbab^{12}ab^2a^2b^2,\ a^7bc^6b^8c^5a^{15}cb^{15},\ (c^7a^{15}b^{15}cba)^3,\ ab^{15}cacb^{15}ac^6b^{10}a^7, a^{14}b^8a^2b^8, a^{12}b^{12}a^4b^4, a^{14}b^2a^4b^{10}a^{14}b^4, a^{12}c^6a^4c^2 \rangle$,
where $a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 1 \\ 12 & 9 \end{pmatrix}$ and $c = \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}$ are Jordan regular units.

Proof. Let G be a group having the above presentation. Since $a,b,c \in GL(2,Z_{16})$, so $G \leq GL(2,\mathbb{Z}_{16})$. Let $x = bc^7aca^{15}b^{15} = \begin{pmatrix} 4 & 5 \\ 7 & 1 \end{pmatrix}$ and $y = ab^{15}aba^3 = \begin{pmatrix} 13 & 11 \\ 4 & 1 \end{pmatrix}$. Then $x,y \in G$, $o(x) = 12$ and $o(y) = 16$. Then $H = \langle x,y \rangle \leq SL(2,\mathbb{Z}_{16})$. Let L be the group consisting of all lower triangular matrices in $SL(2,\mathbb{Z}_{16})$. By Proposition 2.5, $|L| = 128$. Let $u = y^{12}xy^3x^{11} = \begin{pmatrix} 1 & 0 \\ 15 & 1 \end{pmatrix}$ and let $H_1 = \langle u \rangle$. Then $|H_1| = 16$. Let $v = y^4xy^{15}x^2 = \begin{pmatrix} 11 & 0 \\ 1 & 3 \end{pmatrix}$ and $w = yx^{11}yx^5 = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$. Then $o(v) = 16$, $o(w) = 2$ and $vw = wv$. Thus $H_2 = \langle v,w \rangle$ is an abelian subgroup of H of order 32 and $H_1 \cap H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \right\}$. So $|H_1H_2| = 128 = |L|$. Since $H_1H_2 \subseteq L$, so $H_1H_2 = L \leq H \leq SL(2,\mathbb{Z}_{16})$.

Let $k = x^{11}y^5x^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $K = \langle k \rangle$. Then $|K| = 16$. As $L \cap K = 1$, so $|LK| = 2048$. Both $K, L \leq H$, hence $KL \subseteq H \leq SL(2,\mathbb{Z}_{16})$. By Corollary 2.4, $|SL(2,\mathbb{Z}_{16})| = 3072$. Thus $|H| > \frac{1}{2}|SL(2,\mathbb{Z}_{16})|$ and we conclude that $H = SL(2,\mathbb{Z}_{16})$.

Let $r = b^{-1}ac^{-1}b^{-2}ac^{-1}a = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ and $s = cac^2a^{-1}b^{-1} = \begin{pmatrix} 11 & 0 \\ 0 & 5 \end{pmatrix}$. Then $o(r) = o(s) = 4$ and if $P = \langle r,s \rangle$, then $|P| = 16$. Elements of P are listed below:
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 \\ 0 & 13 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}$, $\begin{pmatrix} 7 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}$, $\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$, $\begin{pmatrix} 11 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 11 & 0 \\ 0 & 13 \end{pmatrix}$, $\begin{pmatrix} 13 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 13 & 0 \\ 0 & 9 \end{pmatrix}$, $\begin{pmatrix} 15 & 0 \\ 0 & 5 \end{pmatrix}$ and $\begin{pmatrix} 15 & 0 \\ 0 & 13 \end{pmatrix}$.

Now $P \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \right\}$. Therefore $|PH| = 24576$. Since $PH \subseteq G \leq GL(2, \mathbb{Z}_{16})$ and by Proposition 2.1, $|GL(2, \mathbb{Z}_{16})| = 24576 = |PH|$, so $G = GL(2, \mathbb{Z}_{16})$. □

Lemma 2.8. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 17 & 17 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{18})$.

Proof. The proof is clear once we observe that $a = eu + ue$, $b = e'b + be'$ and $c = e'c + ce'$, where $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2, \mathbb{Z}_{18})$ and $u = \begin{pmatrix} 10 & 7 \\ 17 & 1 \end{pmatrix}$ is a unit in $M(2, \mathbb{Z}_{18})$. □

Theorem 2.9. *Presentation of* $GL(2, \mathbb{Z}_{18})$ *is*
$GL(2, Z_{18}) = \langle\, a,\ b,\ c \mid a^2,\ b^2,\ c^{12},\ ac^2ac^{10},\ bc^2bc^{10},\ (ca)^2(bc)^2(ac)^2a(cb)^3cacbcab,\ (bc)^2b(ac)^2bac^7abcba(ca)^2ab(ac)^2b(cb)^2,\ (ac)^{11}ac^7 \rangle$,
where $a = \begin{pmatrix} 1 & 0 \\ 17 & 17 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a, b, c \in GL(2, \mathbb{Z}_{18})$, so $G \leq GL(2, \mathbb{Z}_{18})$. Let $x = (cb)^2cab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = (cb)^2c = \begin{pmatrix} 0 & 17 \\ 1 & 0 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 18$ and $o(y) = 4$. Let $H = \langle x, y \rangle$. Clearly $H \leq SL(2, \mathbb{Z}_{18})$. Let L be the group consisting of all lower triangular matrices in $SL(2, \mathbb{Z}_{18})$. Then by Proposition 2.5, $|L| = 108$. Let $u = xyx = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and let $H_1 = \langle u \rangle$. Then $|H_1| = 18$. Let $v = xyx^{16}yx^4yx^3 = \begin{pmatrix} 5 & 0 \\ 9 & 11 \end{pmatrix}$ and let $H_2 = \langle v \rangle$. Then $|H_2| = 6$. Both H_1 and H_2 are subgroups of L and $|H_1H_2| = 108 = |L|$ as $H_1 \cap H_2 = 1$. Hence $H_1H_2 = L \leq H \leq SL(2, \mathbb{Z}_{18})$.

Let $m = x^9 = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}$ and $n = x^5yx^3yx^2y = \begin{pmatrix} 5 & 4 \\ 5 & 15 \end{pmatrix}$. Then $o(m) = 2$ and $o(n) = 18$. Let $K = \langle m, n \rangle$. Then $|K| = 54$. Now K contains following elements and their inverses:
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 17 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 17 \\ 1 & 11 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 9 & 10 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 17 & 9 \end{pmatrix},$
$\begin{pmatrix} 3 & 2 \\ 7 & 17 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 16 & 17 \end{pmatrix}, \begin{pmatrix} 3 & 11 \\ 7 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 11 \\ 16 & 17 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 15 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 15 & 16 \end{pmatrix},$
$\begin{pmatrix} 5 & 4 \\ 5 & 15 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 14 & 15 \end{pmatrix}, \begin{pmatrix} 5 & 13 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 5 & 13 \\ 14 & 15 \end{pmatrix}, \begin{pmatrix} 6 & 5 \\ 13 & 14 \end{pmatrix}, \begin{pmatrix} 7 & 6 \\ 3 & 13 \end{pmatrix},$
$\begin{pmatrix} 7 & 6 \\ 12 & 13 \end{pmatrix}, \begin{pmatrix} 7 & 15 \\ 12 & 13 \end{pmatrix}, \begin{pmatrix} 8 & 7 \\ 11 & 12 \end{pmatrix}, \begin{pmatrix} 9 & 8 \\ 1 & 11 \end{pmatrix}, \begin{pmatrix} 9 & 8 \\ 10 & 11 \end{pmatrix}, \begin{pmatrix} 9 & 17 \\ 10 & 11 \end{pmatrix},$
$\begin{pmatrix} 10 & 9 \\ 9 & 10 \end{pmatrix}, \begin{pmatrix} 12 & 11 \\ 7 & 17 \end{pmatrix}, \begin{pmatrix} 13 & 3 \\ 15 & 16 \end{pmatrix}$ and $\begin{pmatrix} 14 & 13 \\ 5 & 15 \end{pmatrix}$.
Also $L \cap K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} \right\}$. Hence $|LK| = 2916$. As $K, L \leq H$, so $KL \subseteq H \leq SL(2, \mathbb{Z}_{18})$ and by Corollary 2.4, $|SL(2, \mathbb{Z}_{18})| = 3888$. Thus $|H| > \frac{1}{2}|SL(2, \mathbb{Z}_{18})|$ and so $H = SL(2, \mathbb{Z}_{18})$.

Let $p = bc = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ and let $P = \langle p \rangle$. Clearly $p \notin H$. Now $|P| = 6$ and $|PH| = 23328$ as $P \cap H = 1$. Also $PH \subseteq G \leq GL(2, \mathbb{Z}_{18})$. By Corollary 2.2, $|GL(2, \mathbb{Z}_{18})| = 23328 = |PH|$. Thus $G = GL(2, \mathbb{Z}_{18})$. □

Lemma 2.10. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 19 & 19 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{20})$.

Proof. The proof is clear once we observe that $a = e_1u_1 + u_1e_1$, $b = e_2u_2 + u_2e_2$ and $c = e_3u_3 + u_3e_3$, where $e_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ are idempotents in $M(2, \mathbb{Z}_{20})$ and $u_1 = \begin{pmatrix} 11 & 8 \\ 19 & 1 \end{pmatrix}$, $u_2 = b$, $u_3 = \begin{pmatrix} 1 & 1 \\ 19 & 0 \end{pmatrix}$ are units in $M(2, \mathbb{Z}_{20})$. □

Theorem 2.11. *Presentation of* $GL(2, \mathbb{Z}_{20})$ *is*
$GL(2, Z_{20}) = \langle\, a,\, b,\, c \mid a^2, b^8, c^{20}, ab^2ab^6, cb^2c^{19}b^6, ac^{19}bacbc^2abc^5bab^7cac^2abc^4, b^6a(c^{19}b)^2cac^7, (ac^{17}ab^7ac^3a)^8,\ b^3c^3ab^5c^{19}ac^{18},\ (ac^{17})^2b^5(ac^3)^2ab^2c^2ac^{18}b,\ c^2bc^{19}b(ca)^2cbc^2b\rangle$,
where $a = \begin{pmatrix} 1 & 0 \\ 19 & 19 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a, b, c \in GL(2, \mathbb{Z}_{20})$, so $G \leq GL(2, \mathbb{Z}_{20})$. Let $x = c$, $y = cac^{-1}bcb^2ab = \begin{pmatrix} 3 & 0 \\ 9 & 7 \end{pmatrix}$ and $z = cbacbc^{-1}bcab = \begin{pmatrix} 16 & 5 \\ 15 & 11 \end{pmatrix}$. Then $x, y, z \in G$, $o(x) = 20$, $o(y) = 4$ and $o(z) = 3$. Let $H = \langle x, y, z\rangle$. Clearly $H \leq SL(2, \mathbb{Z}_{20})$. Let L be the group consisting of all lower triangular matrices in $SL(2, \mathbb{Z}_{20})$. By Proposition 2.5, $|L| = 160$. Let $u = x^{-1}y^{-1}x^3z^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and let $H_1 = \langle u\rangle$. Then $|H_1| = 20$. Let $v = y$ and $w = z^{-1}xz^{-1}x^9 = \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$. Then $o(w) = 2$ and $wv = vw$. Let $H_2 = \langle v, w\rangle$. Now H_2 is an abelian subgroup of H of order 8. Also $H_1 \cap H_2 = 1$. Hence $|H_1H_2| = 160$. Since $H_1H_2 \subseteq L$, so $H_1H_2 = L \leq H \leq SL(2, \mathbb{Z}_{20})$.

Let $K = \langle x\rangle$, then $|K| = 20$, $L \cap K = 1$ and $|LK| = 3200$. Both $K, L \leq H$, so $KL \subseteq H \leq SL(2, \mathbb{Z}_{20})$. Thus $|H| \geq 3200$. Since by Corollary 2.4, $|SL(2, \mathbb{Z}_{20})| = 5760$ and $|H| > \frac{1}{2}|SL(2, \mathbb{Z}_{20})|$, so $H = SL(2, \mathbb{Z}_{20})$.

Let $r = c^2ac^{-1}ab = \begin{pmatrix} 1 & 0 \\ 0 & 17 \end{pmatrix}$ and $s = cac^{-1}b^{-1} = \begin{pmatrix} 13 & 0 \\ 0 & 19 \end{pmatrix}$. Then $o(r) = o(s) = 4$. If $P = \langle r, s\rangle$, then $|P| = 16$. Elements of P are listed below:
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 11 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 19 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 11 \end{pmatrix}$ and $\begin{pmatrix} 17 & 0 \\ 0 & 19 \end{pmatrix}$.

Now $P \cap H = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}\right\}$ and so $|PH| = 46080$. Also $PH \subseteq G \leq GL(2, \mathbb{Z}_{20})$ and by Corollary 2.2, $|GL(2, \mathbb{Z}_{20})| = 46080 = |PH|$. Thus $G = GL(2, \mathbb{Z}_{20})$. □

Lemma 2.12. *The elements* $a = \begin{pmatrix} 1 & 5 \\ 12 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 & 2 \\ 17 & 17 \end{pmatrix}$, $c = \begin{pmatrix} 19 & 3 \\ 14 & 11 \end{pmatrix}$ *and* $d = \begin{pmatrix} 3 & 1 \\ 16 & 7 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{24})$.

Proof. The proof is clear once we observe that $a = e_1u_1 + u_1e_1$, $b = e_2u_2 + u_2e_2$, $c = e_3u_3 + u_3e_3$ and $d = e_4u_4 + u_4e_4$, where $e_1 = \begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 9 & 8 \\ 9 & 16 \end{pmatrix}$,

$e_4 = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$ are idempotents in $M(2,\mathbb{Z}_{24})$ and $u_1 = \begin{pmatrix} 1 & 5 \\ 7 & 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 & 5 \\ 17 & 7 \end{pmatrix}$, $u_3 = \begin{pmatrix} 16 & 3 \\ 17 & 5 \end{pmatrix}$, $u_4 = \begin{pmatrix} 1 & 1 \\ 7 & 2 \end{pmatrix}$ are units in $M(2,\mathbb{Z}_{24})$. □

Theorem 2.13. *Presentation of $GL(2,\mathbb{Z}_{24})$ is*
$GL(2,Z_{24}) = \langle a,\ b,\ c,\ d \mid a^{24},\ b^8,\ c^8, d^8,\ (bc)^{12}, (bd)^{12}, (cd)^6, ab^4a^{23}b^4, cb^4c^7b^4,\ db^4d^7b^4, c^4a^{12}, a^{18}b^6a^6b^2,\ a^{16}b^4a^8b^4,\ a^{12}b^6a^{12}b^2,\ a^{22}c^6a^2c^2,\ a^{12}c^7a^4cb^7a^8b,\ a^{19}c^7a^6c^7a^{23}c^2,\ c^7a^8cb^7a^{16}b, a^6c^5ac^2a^{17}c,\ a^6c^5a^3c^2a^{15}c,\ a^{12}c^5a^4c^3b^7a^8b,\ c^5a^8c^3b^7a^{16}b,\ c^6a^{10}c^2a^{14},\ a^{18}c^5a^{15}c^2a^{15}c, a^3d^7a^3dc^7a^{18}c,\ a^{12}d^7a^4db^7a^8b,\ a^3d^5a^3d^3c^7a^{18}c,\ b^6c^6b^2c^2,\ c^6b^2c^2b^6,\ b^6d^6b^2d^2,\ c^6d^7c^2b^6db^2, c^5d^7c^3dc^7a^{13}ca^{11},\ c^6d^6c^2d^2,\ c^6d^5c^2d^3c^7a^{18}ca^6,\ (ad^7)^2(cd)^2a^{22}c^7b^6c^7,\ (cd)^2cb^2ca^{22}, ba^{23}bc^7baca^{23}b^7c^7b^7cb^7cac^7,\ c^7a(db)^2b^3a^{13}dc^7bd,\ bda^{23}b^7a^{23}bd^7bdc^7dca^{22}(b^7c^7)^2d^2, b^7ada^{23}b(ab^7)^2a^{23}bdbab^7c^7d^7bcd\rangle$,
where $a = \begin{pmatrix} 1 & 5 \\ 12 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 & 2 \\ 17 & 17 \end{pmatrix}$, $c = \begin{pmatrix} 19 & 3 \\ 14 & 11 \end{pmatrix}$ *and* $d = \begin{pmatrix} 3 & 1 \\ 16 & 7 \end{pmatrix}$ *are Jordan regular units.*

Proof. : Let G be a group having the above presentation. Since $a,b,c,d \in GL(2,\mathbb{Z}_{24})$, so $G \leq GL(2,\mathbb{Z}_{24})$. Let $x = b^{-1}d^{-1}b^{-1}c^{-1}bc^{-1}a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = c^{-1}bc^{-1}d^{-1}a = \begin{pmatrix} 12 & 1 \\ 23 & 0 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 24$ and $o(y) = 4$. Let $H = \langle x, y\rangle$. Clearly $H \leq SL(2,\mathbb{Z}_{24})$. Let L be the group of all lower triangular matrices in $SL(2,\mathbb{Z}_{24})$. By Proposition 2.5, $|L| = 192$. Let $h = y^{-1}x^{-1}y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and let $H_1 = \langle h\rangle$. Then $|H_1| = 24$. Let $u = y^{-1}xyx^{-1}y^2x = \begin{pmatrix} 11 & 0 \\ 1 & 11 \end{pmatrix}$ and $v = yx^{-2}yx^3y^{-1}x^2yx^3 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$. Now $o(u) = 24$, $o(v) = 2$ and $uv = vu$. Hence $H_2 = \langle u, v\rangle$ is a group of order 48. Let $w = yx^{-1}y^{-1} = \begin{pmatrix} 13 & 0 \\ 1 & 13 \end{pmatrix}$ and let $H_3 = \langle w\rangle$. Then $|H_3| = 24$. Also $H_2 \cap H_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 8 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 10 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \right\}$. Therefore $|H_2H_3| = 96$. Now if $T = \langle u, v, w\rangle$, then $H_2H_3 \subseteq T \leq L$. Thus $|H_2H_3| \leq |T| \leq |L|$, which implies that either $|T| = 96$ or $|T| = 192$. But $k = y^{-1}x^{-3}y = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \notin T$, so $|T| = 96$ and $H_2H_3 = T$. Now $H_1 \cap T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 8 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 10 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \right\}$. Hence $|H_1T| = 192 = |L|$ and $H_1T = L \leq H \leq SL(2,\mathbb{Z}_{24})$.

Let D be the derived subgroup of $SL(2,\mathbb{Z}_{24})$. Then $D = \langle l, m\rangle$, where $l = xy^{-1}xyx^{-2} = \begin{pmatrix} 0 & 1 \\ 23 & 3 \end{pmatrix}$, $m = x^2y^{-1}x = \begin{pmatrix} 2 & 1 \\ 1 & 13 \end{pmatrix}$ and $|D| = 768$. Since $l, m \in H$, so $D \leq H$. Now $L, D \subseteq H$ and $L \cap D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 12 & 5 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 6 & 7 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 18 & 7 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 6 & 11 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 18 & 11 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 12 & 13 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 12 & 17 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 6 & 19 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 18 & 19 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ 6 & 23 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ 18 & 23 \end{pmatrix} \right\}$. Thus $|LD| = 9216$.

Now $LD \subseteq H \leq SL(2,\mathbb{Z}_{24})$ and by Corollary 2.4, $|SL(2,\mathbb{Z}_{24})| = 9216$. So $LD = H = SL(2,\mathbb{Z}_{24})$.

Let $n = cb^{-1}a^{-1}b^{-1}cb = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$, $p = abc^{-1}b^{-1}c^{-1}b = \begin{pmatrix} 17 & 0 \\ 0 & 13 \end{pmatrix}$, $q = dcbda^{-1}ba^2 = \begin{pmatrix} 13 & 0 \\ 0 & 23 \end{pmatrix}$, $r = c^{-1}a^{-1}b^{-1}d^2b = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$, $s = bda^2c^{-1}b = \begin{pmatrix} 19 & 0 \\ 0 & 1 \end{pmatrix}$ and $t = b^{-1}a^{-1}c^{-1}bca = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$. Then $o(n) = o(p) = o(q) = o(r) = o(s) = o(t) = 2$. Let $P = \langle n,p,q,r,s,t\rangle$. Then $P \leq G$ and $|P| = 64$. P consists of the diagonal matrices having diagonal elements in $\mathcal{U}(\mathbb{Z}_{24})$. Also $P \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 0 & 19 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ 0 & 23 \end{pmatrix} \right\}$ and so $|PH| = 73728$. Now $PH \subseteq G \leq GL(2,\mathbb{Z}_{24})$ and by Corollary 2.2, $|GL(2,\mathbb{Z}_{24})| = 73728 = |PH|$. Thus $G = GL(2,\mathbb{Z}_{24})$. □

Lemma 2.14. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 27 & 27 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 27 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2,\mathbb{Z}_{28})$.

Proof. The proof is clear once we observe that $a = eu + ue$, $b = e'b + be'$, $c = e'c + ce'$ and $d = e'd + de'$, where $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2,\mathbb{Z}_{28})$ and $u = \begin{pmatrix} 15 & 13 \\ 27 & 0 \end{pmatrix}$ is a unit in $M(2,\mathbb{Z}_{28})$. □

Theorem 2.15. *Presentation of* $GL(2,\mathbb{Z}_{28})$ *is*
$GL(2,Z_{28}) = \langle a,b,c,d \mid a^2, b^2, c^{12}, d^4,\ ac^2ac^{10},\ bc^2bc^{10},\ (ad^2)^2,\ (bd^2)^2,\ c^2dc^{10}d^3,\ cd^2c^{11}d^2,$
$(cd)^4(ad)^7ab(ac^{11})^2d^3(c^{11}a)^2b, (ac^{11})^2bc^3(ac)^3b^3cac, cdbc^{11}bd, cdadba(cb)^2adacdbadacbc^5acdcab,$
$c^7(db)^2a(cb)^2(dc)^2ac, dc^{11}d(ca)^2bcbadbd((ca)^2b)^2\rangle,$
where $a = \begin{pmatrix} 1 & 0 \\ 27 & 27 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 27 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a,b,c,d \in GL(2,\mathbb{Z}_{28})$, so $G \leq GL(2,\mathbb{Z}_{28})$. Let $x = dab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = dbac^{-1}a(dba)^2ca = \begin{pmatrix} 14 & 13 \\ 15 & 14 \end{pmatrix}$. Then both $x, y \in G$, $o(x) = 28$ and $o(y) = 4$. Let $H = \langle x,y\rangle$. Clearly $H \leq SL(2,\mathbb{Z}_{28})$. Let L be the group consisting of all lower triangular matrices in $SL(2,\mathbb{Z}_{28})$. By Proposition 2.5, $|L| = 336$. Let $u = yx^{-2}yx^4yx^{-3}yx^4 = \begin{pmatrix} 27 & 0 \\ 1 & 27 \end{pmatrix}$ and $v = (yx)^3 = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}$. Now $o(u) = 28$, $o(v) = 2$ and $uv = vu$. If $H_1 = \langle u,v\rangle$, then $|H_1| = 56$. Let $w = x^{-5}y^{-1}xyx^{-1}yx^2 = \begin{pmatrix} 3 & 0 \\ 2 & 19 \end{pmatrix}$ and let $H_2 = \langle w\rangle$. Then $|H_2| = 6$. Also $H_1 \cap H_2 = 1$, so $|H_1H_2| = 336$. Now $H_1, H_2 \leq L$, hence $H_1H_2 = L \leq H \leq SL(2,\mathbb{Z}_{28})$.

Let $K = \langle x\rangle$, then $|K| = 28$ and $L \cap K = 1$. Thus $|LK| = 9408$. Now $KL \subseteq H \leq SL(2,\mathbb{Z}_{28})$. By Corollary 2.4, $|SL(2,\mathbb{Z}_{28})| = 16128$. As $|H| > \frac{1}{2}|SL(2,\mathbb{Z}_{28})|$, so $H = SL(2,\mathbb{Z}_{28})$.

Let $p = cd^{-1}cb = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $q = c^{-3}d = \begin{pmatrix} 17 & 0 \\ 0 & 19 \end{pmatrix}$, $r = cdc^3b = \begin{pmatrix} 13 & 0 \\ 0 & 23 \end{pmatrix}$

and $s = dc^6b = \begin{pmatrix} 15 & 0 \\ 0 & 13 \end{pmatrix}$. Then $o(p) = o(q) = o(r) = 6$, whereas $o(s) = 2$. Let $P = \langle p,\ q,\ r,\ s\rangle$. Then $|P| = 144$. P consists of the diagonal matrices having diagonal elements in $\mathcal{U}(\mathbb{Z}_{28})$. Also $P \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 19 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 25 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ 0 & 11 \end{pmatrix}, \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}, \begin{pmatrix} 27 & 0 \\ 0 & 27 \end{pmatrix} \right\}$. Thus $|PH| = 193536$. Now $PH \subseteq G \leq GL(2, \mathbb{Z}_{28})$ and by Corollary 2.2, $|GL(2, \mathbb{Z}_{28})| = 193536 = |PH|$. Thus $G = GL(2, \mathbb{Z}_{28})$. □

Lemma 2.16. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 29 & 29 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 7 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 29 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{30})$.

Proof. The proof is clear once we observe that $a = eu + ue$, $b = e'b + be'$, $c = e'c + ce'$ and $d = e'd + de'$, where $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2, \mathbb{Z}_{30})$ and $u = \begin{pmatrix} 16 & 13 \\ 29 & 1 \end{pmatrix}$ is a unit in $M(2, \mathbb{Z}_{30})$. □

Theorem 2.17. *Presentation of* $GL(2, \mathbb{Z}_{30})$ *is*
$GL(2, Z_{30}) = \langle a, b, c, d \mid a^2, b^2, c^8, d^4, (bc)^4, (bd)^2, (cd)^4, ac^2ac^6, bc^2bc^6, (ad^2)^2, (bd^2)^2, c^2dc^6d^3, cd^2c^7d^2, ad^3bcacbad(c^7a)^2b, adac^3(da)^2cacb(da)^2cadcac, db(ac)^2adcb(c(ad)^2)^2bcd(ad)^2c^5adac, c^5ad^3adbacadac(bdc^3a)^7, abcdacbcabadacab\rangle$,
where $a = \begin{pmatrix} 1 & 0 \\ 29 & 29 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 7 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 29 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a, b, c, d \in GL(2, \mathbb{Z}_{30})$, so $G \leq GL(2, \mathbb{Z}_{30})$. Let $x = dab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = ba = \begin{pmatrix} 29 & 29 \\ 1 & 0 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 30$ and $o(y) = 3$. Let $H = \langle x, y\rangle$. Clearly $H \leq SL(2, \mathbb{Z}_{30})$. Let L be the group consisting of all lower triangular matrices in $SL(2, \mathbb{Z}_{30})$. By Proposition 2.5, $|L| = 240$. Let $u = (x^{-1}y)^2x^3y^{-1}x^2 = \begin{pmatrix} 19 & 0 \\ 7 & 19 \end{pmatrix}$ and $v = yx^{-3}y^{-1}x^6yx^{-2}yx^7 = \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$. Then $o(u) = 30$, $o(v) = 2$ and $uv = vu$. Let $H_1 = \langle u, v\rangle$. Then $|H_1| = 60$. Let $w = y^{-1}xy^{-1}x^4yx^{-2}yx^2 = \begin{pmatrix} 13 & 0 \\ 7 & 7 \end{pmatrix}$ and let $H_2 = \langle w\rangle$. Then $|H_2| = 12$ and $H_1 \cap H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 20 & 1 \end{pmatrix} \right\}$, so $|H_1H_2| = 240$. Now $H_1, H_2 \leq L$, hence $H_1H_2 = L \leq H \leq SL(2, \mathbb{Z}_{30})$.

Let $r = yx^{-2}yx^{-5}y^{-1}x^2y^{-1}x = \begin{pmatrix} 1 & 11 \\ 15 & 16 \end{pmatrix}$ and $s = x^{-4}yxy = \begin{pmatrix} 29 & 5 \\ 0 & 29 \end{pmatrix}$. Then $o(r) = 15$, $o(s) = 6$ and $r^i \neq s^j$ for $0 \leq i \leq 14$, $0 \leq j \leq 5$. Also $rs^3 = sr^{11}$. Let $K = \langle r, s\rangle$. The canonical form of K is $\{r^is^j | 0 \leq i \leq 14, 0 \leq j \leq 5\}$ and $|K| = 90$. Also $L \cap K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 29 & 0 \\ 15 & 29 \end{pmatrix} \right\}$ and $|LK| = 10800$. Both $K, L \leq H$, so $|KL| \leq$

$|H| \leq |SL(2, \mathbb{Z}_{30})|$. By Corollary 2.4, $|SL(2, \mathbb{Z}_{30})| = 17280$. As $|H| > \frac{1}{2}|SL(2, \mathbb{Z}_{30})|$, therefore $H = SL(2, \mathbb{Z}_{30})$.

Let $p = d^{-1}c^2b = \begin{pmatrix} 7 & 0 \\ 0 & 23 \end{pmatrix}$, $q = d^{-1}c^{-1}db = \begin{pmatrix} 17 & 0 \\ 0 & 29 \end{pmatrix}$, $l = d^2c^4 = \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$ and $m = c^{-1}b = \begin{pmatrix} 1 & 0 \\ 0 & 13 \end{pmatrix}$. Then $o(p) = o(q) = o(m) = 4$, whereas $o(l) = 2$. Let $P = \langle p, q, l, m\rangle$. Then $|P| = 64$. P consists of the diagonal matrix having diagonal elements in $\mathcal{U}(\mathbb{Z}_{30})$. Also $P \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 0 & 19 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 29 & 0 \\ 0 & 29 \end{pmatrix} \right\}$ and $|PH| = 138240$. Now $PH \subseteq G \leq GL(2, \mathbb{Z}_{30})$ and by Corollary 2.2, $|GL(2, \mathbb{Z}_{30})| = 138240 = |PH|$. Thus $G = GL(2, \mathbb{Z}_{30})$. □

Lemma 2.18. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 31 & 31 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 31 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{32})$.

Proof. The proof is clear once we observe that $a = eu + ue$, $b = e'b + be'$, $c = e'c + ce'$ and $d = e'd + de'$, where $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2, \mathbb{Z}_{32})$ and $u = \begin{pmatrix} 17 & 15 \\ 31 & 0 \end{pmatrix}$ is a unit in $M(2, \mathbb{Z}_{32})$. □

Theorem 2.19. *Presentation of* $GL(2, \mathbb{Z}_{32})$ *is*
$GL(2, Z_{32}) = \langle a, b, c, d \mid a^2, b^2, c^{16}, d^4, (ab)^3, (bd)^2, ac^2ac^{14}, bc^2bc^{14}, (ad^2)^2, (bd^2)^2, cd^2c^{15}d^2, c^2dc^{14}d^3, b(ad)^3c^2bd^3a(c^{15}a)^2, c^3bda(dcba)^2dc(ac)^2(da)^2, cba(dadc)^5bab(da)^2cabcad^3ad, c(baca)^3dacd^3(ca)^2bcb, (cd)^5acd(cac)^3bcadcacbc^9ada\rangle$,
where $a = \begin{pmatrix} 1 & 0 \\ 31 & 31 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 31 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a, b, c, d \in GL(2, \mathbb{Z}_{32})$, so $G \leq GL(2, \mathbb{Z}_{32})$. Let $x = dab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = c^{-1}ac^{-1}dacabcada = \begin{pmatrix} 1 & 13 \\ 5 & 2 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 32$ and $o(y) = 24$. Let $H = \langle x, y\rangle$. Clearly $H \leq SL(2, \mathbb{Z}_{32})$. Let L be the group consisting of all lower triangular matrices in $SL(2, \mathbb{Z}_{32})$. By Proposition 2.5, $|L| = 512$. Let $u = x^{-4}yx^{-1}y^3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and let $H_1 = \langle u\rangle$. Then $|H_1| = 32$. Let $v = yxy^{-1}x^3yxy^{-1}x^2 = \begin{pmatrix} 3 & 0 \\ 1 & 11 \end{pmatrix}$ and $w = yx^{-1}y^{-1}(xy^{-1})^2x^{-1}yx^2 = \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}$. Then $o(v) = 32$, $o(w) = 2$ and $vw = wv$. Thus $H_2 = \langle v, w\rangle$ is an abelian subgroup of H of order 64. Also $H_1 \cap H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 16 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 24 & 1 \end{pmatrix} \right\}$. So $|H_1H_2| = 512$. Now $H_1, H_2 \leq L$ and hence $H_1H_2 = L \leq H \leq SL(2, \mathbb{Z}_{32})$.

Let $k = x^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and let $K = \langle k\rangle$. Then $|K| = 32$. Also $L \cap K = 1$, so $|LK| = 16384$. Now $LK \subseteq H \leq SL(2, \mathbb{Z}_{32})$. By Corollary 2.4, $|SL(2, \mathbb{Z}_{32})| = 24576$. Since $|H| > \frac{1}{2}|SL(2, \mathbb{Z}_{32})|$, so $H = SL(2, \mathbb{Z}_{32})$.

Let $p = d^{-1}c^3bc = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $q = c^{-1}d^{-1}c^{-4} = \begin{pmatrix} 7 & 0 \\ 0 & 19 \end{pmatrix}$, $r = d^{-1}c^5bc = \begin{pmatrix} 9 & 0 \\ 0 & 15 \end{pmatrix}$ and $s = c^{-1}d^{-1}c^{-1}(bc^{-1})^2b = \begin{pmatrix} 31 & 0 \\ 0 & 17 \end{pmatrix}$. Now $o(p) = o(q) = 8$, $o(r) = 4$ and $o(s) = 2$. Let $P = \langle p, q, r, s\rangle$. Then $|P| = 256$. P consists of the diagonal matrices having diagonal elements in $\mathcal{U}(\mathbb{Z}_{32})$. Also $P \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 11 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 25 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 0 & 27 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 0 & 29 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}, \begin{pmatrix} 27 & 0 \\ 0 & 19 \end{pmatrix}, \begin{pmatrix} 29 & 0 \\ 0 & 21 \end{pmatrix}, \begin{pmatrix} 31 & 0 \\ 0 & 31 \end{pmatrix} \right\}$ and so $|PH| = 393216$. Now $PH \subseteq G \leq GL(2, Z_{32})$ and by Proposition 2.1, $|GL(2, \mathbb{Z}_{32})| = 393216$, which implies that $G = GL(2, \mathbb{Z}_{32})$. □

Lemma 2.20. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 35 & 35 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 35 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{36})$.

Proof. The proof is clear once we observe that $a = eu + ue$, $b = e'b + be'$, $c = e'c + ce'$ and $d = e'd + de'$, where $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2, \mathbb{Z}_{36})$ and $u = \begin{pmatrix} 19 & 16 \\ 35 & 1 \end{pmatrix}$ is a unit in $M(2, \mathbb{Z}_{36})$. □

Theorem 2.21. *Presentation of* $GL(2, \mathbb{Z}_{36})$ *is*
$GL(2, Z_{36}) = \langle a,\ b,\ c,\ d \mid a^2, b^2, c^{12}, d^4,\ (ab)^3,\ (ac)^{24}, (bd)^2, ac^2ac^{10}, bc^2bc^{10},\ dc^2d^3c^{10},\ (ad^2)^2,$ $(bd^2)^2,\ (ad)^2abcbacb(da)^2b(cabca)^2,\ c^5dcac(baca)^5(abc)^3acbacabcabad,\ dcd(cab)^2cbcabca,$ $cd^2c^{11}d^2, c^3(bc)^2(ac)^2abcbcd(ca)^3d^3, abcdbc^{11}d^3a\rangle$,
where $a = \begin{pmatrix} 1 & 0 \\ 35 & 35 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 35 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a, b, c, d \in GL(2, \mathbb{Z}_{36})$, so $G \leq GL(2, \mathbb{Z}_{36})$. Let $x = dab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = d^{-1} = \begin{pmatrix} 0 & 1 \\ 35 & 0 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 36$ and $o(y) = 4$ and $H = \langle x, y\rangle \leq SL(2, \mathbb{Z}_{36})$. Let L be the group consisting of all lower triangular matrices in $SL(2, \mathbb{Z}_{36})$. By Proposition 2.5, $|L| = 432$. Let $u = xy^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $H_1 = \langle u\rangle$ has order 36. Let $v = x^7y^{-1}x^{-5} = \begin{pmatrix} 7 & 0 \\ 1 & 31 \end{pmatrix}$ and $w = (yx^{-4}yx^4)^2 = \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}$. Now $o(v) = 36$, $o(w) = 2$ and $vw = wv$. So $H_2 = \langle v, w\rangle$, has order 72. Also $H_1 \cap H_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 18 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 24 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 30 & 1 \end{pmatrix} \right\}$ and so $|H_1H_2| = 432$. Now $H_1, H_2 \leq L$, therefore $H_1H_2 = L \leq H \leq SL(2, \mathbb{Z}_{36})$.

Let $l = y^{-1}x^8 = \begin{pmatrix} 0 & 35 \\ 1 & 8 \end{pmatrix}$, $m = x^{-6}y^{-1}xyx^{-3}yxyx^{-2}y = \begin{pmatrix} 13 & 2 \\ 14 & 5 \end{pmatrix}$ and $K = \langle l, m\rangle$. Then $|K| = 192$. K consists of following matrices:

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 35 & 28 \end{pmatrix}, \pm\begin{pmatrix} 0 & 17 \\ 19 & 8 \end{pmatrix}, \pm\begin{pmatrix} 1 & 4 \\ 4 & 17 \end{pmatrix}, \pm\begin{pmatrix} 1 & 8 \\ 28 & 9 \end{pmatrix}, \pm\begin{pmatrix} 1 & 12 \\ 32 & 25 \end{pmatrix},$$

$$\pm\begin{pmatrix} 1 & 18 \\ 18 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 22 \\ 22 & 17 \end{pmatrix}, \pm\begin{pmatrix} 1 & 26 \\ 10 & 9 \end{pmatrix}, \pm\begin{pmatrix} 1 & 30 \\ 14 & 25 \end{pmatrix}, \pm\begin{pmatrix} 2 & 3 \\ 5 & 26 \end{pmatrix},$$

$$\pm\begin{pmatrix} 2 & 21 \\ 23 & 26 \end{pmatrix}, \pm\begin{pmatrix} 3 & 4 \\ 8 & 23 \end{pmatrix}, \pm\begin{pmatrix} 3 & 14 \\ 34 & 15 \end{pmatrix}, \pm\begin{pmatrix} 3 & 22 \\ 26 & 23 \end{pmatrix}, \pm\begin{pmatrix} 3 & 32 \\ 16 & 15 \end{pmatrix},$$

$$\pm\begin{pmatrix} 4 & 1 \\ 15 & 4 \end{pmatrix}, \pm\begin{pmatrix} 4 & 11 \\ 5 & 32 \end{pmatrix}, \pm\begin{pmatrix} 4 & 13 \\ 35 & 24 \end{pmatrix}, \pm\begin{pmatrix} 4 & 17 \\ 3 & 4 \end{pmatrix}, \pm\begin{pmatrix} 4 & 19 \\ 33 & 4 \end{pmatrix},$$

$$\pm\begin{pmatrix} 4 & 29 \\ 23 & 32 \end{pmatrix}, \pm\begin{pmatrix} 4 & 31 \\ 17 & 24 \end{pmatrix}, \pm\begin{pmatrix} 4 & 35 \\ 21 & 4 \end{pmatrix}, \pm\begin{pmatrix} 5 & 8 \\ 12 & 5 \end{pmatrix}, \pm\begin{pmatrix} 5 & 10 \\ 6 & 5 \end{pmatrix},$$

$$\pm\begin{pmatrix} 5 & 14 \\ 10 & 21 \end{pmatrix}, \pm\begin{pmatrix} 5 & 16 \\ 4 & 13 \end{pmatrix}, \pm\begin{pmatrix} 5 & 26 \\ 30 & 5 \end{pmatrix}, \pm\begin{pmatrix} 5 & 28 \\ 24 & 5 \end{pmatrix}, \pm\begin{pmatrix} 5 & 32 \\ 28 & 21 \end{pmatrix},$$

$$\pm\begin{pmatrix} 5 & 34 \\ 22 & 13 \end{pmatrix}, \pm\begin{pmatrix} 6 & 5 \\ 19 & 22 \end{pmatrix}, \pm\begin{pmatrix} 6 & 13 \\ 11 & 30 \end{pmatrix}, \pm\begin{pmatrix} 6 & 23 \\ 1 & 22 \end{pmatrix}, \pm\begin{pmatrix} 6 & 31 \\ 29 & 30 \end{pmatrix},$$

$$\pm\begin{pmatrix} 7 & 6 \\ 22 & 19 \end{pmatrix}, \pm\begin{pmatrix} 7 & 24 \\ 4 & 19 \end{pmatrix}, \pm\begin{pmatrix} 8 & 1 \\ 35 & 0 \end{pmatrix}, \pm\begin{pmatrix} 8 & 5 \\ 23 & 28 \end{pmatrix}, \pm\begin{pmatrix} 8 & 9 \\ 27 & 8 \end{pmatrix},$$

$$\pm\begin{pmatrix} 8 & 15 \\ 13 & 20 \end{pmatrix}, \pm\begin{pmatrix} 8 & 19 \\ 17 & 0 \end{pmatrix}, \pm\begin{pmatrix} 8 & 23 \\ 5 & 28 \end{pmatrix}, \pm\begin{pmatrix} 8 & 27 \\ 9 & 8 \end{pmatrix}, \pm\begin{pmatrix} 8 & 33 \\ 31 & 20 \end{pmatrix},$$

$$\pm\begin{pmatrix} 9 & 8 \\ 28 & 17 \end{pmatrix}, \pm\begin{pmatrix} 9 & 10 \\ 26 & 1 \end{pmatrix}, \pm\begin{pmatrix} 9 & 26 \\ 10 & 17 \end{pmatrix}, \pm\begin{pmatrix} 9 & 28 \\ 8 & 1 \end{pmatrix}, \pm\begin{pmatrix} 10 & 3 \\ 5 & 34 \end{pmatrix},$$

$$\pm\begin{pmatrix} 10 & 9 \\ 27 & 10 \end{pmatrix}, \pm\begin{pmatrix} 10 & 13 \\ 31 & 26 \end{pmatrix}, \pm\begin{pmatrix} 10 & 17 \\ 19 & 18 \end{pmatrix}, \pm\begin{pmatrix} 10 & 21 \\ 23 & 34 \end{pmatrix}, \pm\begin{pmatrix} 10 & 27 \\ 9 & 10 \end{pmatrix},$$

$$\pm\begin{pmatrix} 10 & 31 \\ 13 & 26 \end{pmatrix}, \pm\begin{pmatrix} 10 & 35 \\ 1 & 18 \end{pmatrix}, \pm\begin{pmatrix} 11 & 12 \\ 32 & 35 \end{pmatrix}, \pm\begin{pmatrix} 11 & 30 \\ 14 & 35 \end{pmatrix}, \pm\begin{pmatrix} 12 & 5 \\ 7 & 24 \end{pmatrix},$$

$$\pm\begin{pmatrix} 12 & 13 \\ 35 & 32 \end{pmatrix}, \pm\begin{pmatrix} 12 & 23 \\ 25 & 24 \end{pmatrix}, \pm\begin{pmatrix} 12 & 31 \\ 17 & 32 \end{pmatrix}, \pm\begin{pmatrix} 13 & 2 \\ 14 & 5 \end{pmatrix}, \pm\begin{pmatrix} 13 & 4 \\ 8 & 33 \end{pmatrix},$$

$$\pm\begin{pmatrix} 13 & 8 \\ 12 & 13 \end{pmatrix}, \pm\begin{pmatrix} 13 & 10 \\ 6 & 13 \end{pmatrix}, \pm\begin{pmatrix} 13 & 20 \\ 32 & 5 \end{pmatrix}, \pm\begin{pmatrix} 13 & 22 \\ 26 & 33 \end{pmatrix}, \pm\begin{pmatrix} 13 & 26 \\ 30 & 13 \end{pmatrix},$$

$$\pm\begin{pmatrix} 13 & 28 \\ 24 & 13 \end{pmatrix}, \pm\begin{pmatrix} 14 & 1 \\ 15 & 14 \end{pmatrix}, \pm\begin{pmatrix} 14 & 5 \\ 19 & 30 \end{pmatrix}, \pm\begin{pmatrix} 14 & 7 \\ 13 & 22 \end{pmatrix}, \pm\begin{pmatrix} 14 & 17 \\ 3 & 14 \end{pmatrix},$$

$$\pm\begin{pmatrix} 14 & 19 \\ 33 & 14 \end{pmatrix}, \pm\begin{pmatrix} 14 & 23 \\ 1 & 30 \end{pmatrix}, \pm\begin{pmatrix} 14 & 25 \\ 31 & 22 \end{pmatrix}, \pm\begin{pmatrix} 14 & 35 \\ 21 & 14 \end{pmatrix}, \pm\begin{pmatrix} 15 & 4 \\ 20 & 3 \end{pmatrix},$$

$$\pm\begin{pmatrix} 15 & 14 \\ 10 & 31 \end{pmatrix}, \pm\begin{pmatrix} 15 & 22 \\ 2 & 3 \end{pmatrix}, \pm\begin{pmatrix} 15 & 32 \\ 28 & 31 \end{pmatrix}, \pm\begin{pmatrix} 16 & 15 \\ 13 & 28 \end{pmatrix}, \pm\begin{pmatrix} 16 & 33 \\ 31 & 28 \end{pmatrix},$$

$$\pm\begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}, \pm\begin{pmatrix} 17 & 6 \\ 22 & 29 \end{pmatrix}, \pm\begin{pmatrix} 17 & 10 \\ 26 & 9 \end{pmatrix}, \pm\begin{pmatrix} 17 & 14 \\ 14 & 1 \end{pmatrix}, \pm\begin{pmatrix} 17 & 18 \\ 18 & 17 \end{pmatrix},$$

$$\pm\begin{pmatrix} 17 & 24 \\ 4 & 29 \end{pmatrix} \pm\begin{pmatrix} 17 & 28 \\ 8 & 9 \end{pmatrix}, \pm\begin{pmatrix} 17 & 32 \\ 32 & 1 \end{pmatrix}, \pm\begin{pmatrix} 18 & 1 \\ 35 & 10 \end{pmatrix}, \text{ and } \pm\begin{pmatrix} 18 & 17 \\ 19 & 26 \end{pmatrix}.$$

Also $L \cap K = \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix} \right\}$ and so $|LK| = 20736$. Now $LK \subseteq H \leq SL(2, \mathbb{Z}_{36})$ and by Corollary 2.4, $|SL(2, \mathbb{Z}_{36})| = 31104$. Thus $|H| > \frac{1}{2}|SL(2, \mathbb{Z}_{36})|$, which implies that $H = SL(2, \mathbb{Z}_{36})$.

Let $p = d^{-1}c^3 = \begin{pmatrix} 5 & 0 \\ 0 & 11 \end{pmatrix}$, $q = c^{-1}d^{-1}c^{-2} = \begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}$, $r = c^{-1}d^{-1}c^{-1}dcb =$

$\begin{pmatrix} 31 & 0 \\ 0 & 23 \end{pmatrix}$ and $s = d^{-1}c^6b = \begin{pmatrix} 17 & 0 \\ 0 & 19 \end{pmatrix}$. Then $o(p) = o(q) = o(r) = 6$ and $o(s) = 2$. Hence $P = \langle p, q, r, s\rangle$ has order 144. P consists of the diagonal matrices having diagonal elements in $\mathcal{U}(\mathbb{Z}_{36})$. Also $P \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 29 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 31 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 25 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 0 & 19 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ 0 & 11 \end{pmatrix}, \begin{pmatrix} 25 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 29 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 31 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 35 & 0 \\ 0 & 35 \end{pmatrix} \right\}$ and so $|PH| = 373248$. Now $PH \subseteq G \leq GL(2, \mathbb{Z}_{36})$ and by Corollary 2.2, $|GL(2, \mathbb{Z}_{36})| = 373248$. Therefore $G = GL(2, \mathbb{Z}_{36})$. □

Lemma 2.22. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 37 & 37 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{38})$.

Proof. The proof is clear once we observe that $a = eu + ue$, $b = e'b + be'$ and $c = e'c + ce'$, where $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2, \mathbb{Z}_{38})$ and $u = \begin{pmatrix} 20 & 17 \\ 37 & 1 \end{pmatrix}$ is a unit in $M(2, \mathbb{Z}_{38})$. □

Theorem 2.23. *Presentation of* $GL(2, \mathbb{Z}_{38})$ *is*
$GL(2, Z_{38}) = \langle\ a,\ b,\ c \mid\ a^2,\ b^2,\ c^{36},\ c^2ac^{34}a,\ c^2bc^{34}b,\ a(bcabc)^2ac^{29}bacba(cb)^2, (cb)^6cab(cb)^2c^3(ac)^2b(ca)^3bca, (abc)^4(acb)^4abcac^{27}b, (bca)^2(c(bca)^2)^{11}cbc^{11}abcb(cba)^2c(bc)^3a\rangle$, *where* $a = \begin{pmatrix} 1 & 0 \\ 37 & 37 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a, b, c \in GL(2, \mathbb{Z}_{38})$, so $G \leq GL(2, \mathbb{Z}_{38})$. Let $x = (cb)^8cab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = (cb)^8c = \begin{pmatrix} 0 & 37 \\ 1 & 0 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 38$ and $o(y) = 4$. Thus $H = \langle x, y\rangle \leq SL(2, \mathbb{Z}_{38})$. Let L be the group consisting of all lower triangular matrices in $SL(2, \mathbb{Z}_{38})$. By Proposition 2.5, $|L| = 684$. Let $H_1 = \langle u\rangle$, where $u = xyx = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $|H_1| = 38$. Let $v = x^3yx^{13} = \begin{pmatrix} 3 & 0 \\ 1 & 13 \end{pmatrix}$ and $H_2 = \langle v\rangle$. Then $|H_2| = 18$. Also $H_1 \cap H_2 = 1$, so $|H_1H_2| = 684$. Now $H_1, H_2 \leq L$ and hence $H_1H_2 = L \leq H \leq SL(2, \mathbb{Z}_{38})$.

Let $k = x^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $K = \langle k\rangle$. Then $|K| = 38$. Also $L \cap K = 1$. Thus $|LK| = 25992$. Now $LK \subseteq H \leq SL(2, \mathbb{Z}_{38})$ and by Corollary 2.4, $|SL(2, \mathbb{Z}_{38})| = 41040$. But then $|H| > \frac{1}{2}|SL(2, \mathbb{Z}_{38})|$, which implies that $H = SL(2, \mathbb{Z}_{38})$.

Let $p = bc = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Then $o(p) = 18$ and $p \notin H$. If $P = \langle p\rangle$, then $|PH| = 738720$ as $P \cap H = 1$. Now $PH \subseteq G \leq GL(2, \mathbb{Z}_{38})$ and by Corollary 2.3, $|GL(2, \mathbb{Z}_{38})| = 738720$, which implies that $G = GL(2, \mathbb{Z}_{38})$. □

Lemma 2.24. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 26 & 1 \\ 33 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2, \mathbb{Z}_{40})$.

Proof. The proof is clear once we observe that $a = e_1u_1 + u_1e_1$, $b = e_2u_2 + u_2e_2$, $c = e_2u_3 + u_3e_2$, $d = e_2u_4 + u_4e_2$ where $e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2, \mathbb{Z}_{40})$ and $u_1 = \begin{pmatrix} 1 & 39 \\ 1 & 0 \end{pmatrix}$, $u_2 = b$, $u_3 = c$, $u_4 = \begin{pmatrix} 13 & 1 \\ 33 & 0 \end{pmatrix}$ are units in $M(2, \mathbb{Z}_{40})$. □

Theorem 2.25. *Presentation of $GL(2,\mathbb{Z}_{40})$ is*
$GL(2,Z_{40}) = \langle a,b,c,d \mid a^{40}, b^2, c^8, d^{24}, (cd)^2, c^2ac^6a^{39}, c^2bc^6b, c^2dc^6d^{23}, (bd)^2(cba)^2, cd^4c^7d^4, a^3bca^{39}c^7b, (a^3bc^2d^2)^{60}, a^3bd(abd)^4ad^{23}b, ba^3bc^7a^{39}c, a^{21}ba^{19}ba^{38}c^7d^{23}(ba)^2c^7d^{23}, (abcd)^{30}, a^{37}d^{22}a^3d^2ba^{39}cbcd^{23}ad^{23}, ba^2ba^{39}d^2a^3cacd^{21}a^{38}dc^7ac^7, (bd^2)^2c^7a^{36}ca^4, a^{36}dbacd^{23}a^{39}d^{23}bc^7d, d^{23}a^2d^{23}ac^3a^2bacd^{23}a^2d^{15}b, a^{13}cbabc^7, (da)^2d^2bcaba^2c^7a^{39}c(d^{23}a)^2c, (da^{39}d^{23}bab)^5, (bcd)^{24}, bdba^{38}cbdabd^{23}ada^{38}dba^{39}c^7ba^{39}d, ba^{39}d^2acd^{23}a^{39}bac^7a^{39}d^{23}ad^{23}bdb(a^{39}c^7ad^{23})^2, (abc)^4, bd^6a^{39}d^2bd^{23}ad, abd^7a^{39}d^{23}a^{39}dbad^{21}, bd^3bca^{39}cd^{11}a^{25}, a^2bd^7bd^{23}a^{39}c^7d^{23}a^{39}cd^{23}, (abd)^8(acd)^4, a^{39}c^7ba^{39}dba^{39}cabda^{38}d^2ac^7abc, a^{10}bcd^{10}a^{15}cba^{38}d^{22}a(bc^7)^2, da^{39}bacd^{23}adad^{23}c^5a^{10}d^{14}b, ba^{39}d^{23}a^{39}dbcda^2d^{23}(ab)^2c^7a^{39}d^{23}(ad^{23})^3, (a^2d^2)^{20}, ac^7ad^{23}ada^{36}cbab(a^{39}bd^2)^2d^{20}, dca^7cbdba, d^{22}babd^{23}ba^{39}c(ba)^2cadba^{39}, (a^{38}c)^2(da)^2cbad^{23}abd^{23}c, a^{36}d^{21}ca^3dc^7bd^{22}badad^{22}a^{39}d, d^3a^2d^2bca^{37}cd^3a^3d^{23}a^{39}c^3bcd^{23}a, da^{37}d^2ba^2d^{23}babd^{23}ac^7abd^{23}c, d^{23}ac^7bada^{39}cba^7(d^2a^2)^2, cadba^{39}(dc)^2abdba^{38}c^5a^{39}d^2bd^{22}, bd^7cbd^{23}ad^{23}a^{38}da^{39}c, ada^2d^2bdbcbdba^2bca^{39}d^{22}a^{38}d^{23}b, bcadba^{37}cabd^2baba^{39}c^5da^{39}bc, a^{38}(dca)^2d^{22}a^{39}(da)^2bc^7baba^{39}ca^{39}b, bdba^2bcbda^2d^2abd^{22}ba^{39}ca^2, bcaba^{39}dbac^7d^{23}a^{34}ba^{37}, d(abd)^2(cad)^5(cdb)^{20}dc^7d^{23}a^{39}, abcda^3bcd^3a^5bcd^5a^7bcd^7cda^{39}cdbab\rangle,$
where $a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 26 & 1 \\ 33 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a,b,c,d \in GL(2,\mathbb{Z}_{40})$, so $G \leq GL(2,\mathbb{Z}_{40})$. Let $x = bab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = aba^{-1}ba = \begin{pmatrix} 0 & 39 \\ 1 & 0 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 40$ and $o(y) = 4$. Thus $H = \langle x, y\rangle \leq SL(2,\mathbb{Z}_{40})$. Let L be the group consisting of all lower triangular matrices in $SL(2,\mathbb{Z}_{40})$. By Proposition 2.5, $|L| = 640$. Let $H_1 = \langle u\rangle$, where $u = xyx = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $|H_1| = 40$. Now $y^{-1}x^{-2}yx^6yx^{-6}yx^2 = \begin{pmatrix} 3 & 0 \\ 0 & 27 \end{pmatrix}$, $xyx^6yx^3yx^{-2}yx^2 = \begin{pmatrix} 7 & 0 \\ 0 & 23 \end{pmatrix}$, $(yx^4yx^8)^2 = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$, $yx^{-3}(yx^2)^2yx^{-3}yx^7 = \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$, $x^{-2}yx^6yx^{-6}yx^2y = \begin{pmatrix} 13 & 0 \\ 0 & 37 \end{pmatrix}$, $x^2y^{-1}x^{-2}yx^3yx^6yx = \begin{pmatrix} 17 & 0 \\ 0 & 33 \end{pmatrix}$, $yx^2yx^{-9}yx^2yx^{11} = \begin{pmatrix} 19 & 0 \\ 0 & 19 \end{pmatrix}$, $y^{-1}x^2yx^{-9}yx^2yx^{11} = \begin{pmatrix} 21 & 0 \\ 0 & 21 \end{pmatrix}$, $x^2yx^{-2}yx^3yx^6yx = \begin{pmatrix} 23 & 0 \\ 0 & 7 \end{pmatrix}$, $x^{-2}y^{-1}x^6yx^{-6}yx^2y = \begin{pmatrix} 27 & 0 \\ 0 & 3 \end{pmatrix}$, $y^{-1}x^{-3}(yx^2)^2yx^{-3}yx^7 = \begin{pmatrix} 29 & 0 \\ 0 & 29 \end{pmatrix}$, $y^{-1}x^4yx^8yx^4yx^8 = \begin{pmatrix} 31 & 0 \\ 0 & 31 \end{pmatrix}$, $xy^{-1}x^6yx^3yx^{-2}yx^2 = \begin{pmatrix} 33 & 0 \\ 0 & 17 \end{pmatrix}$, $yx^{-2}yx^6yx^{-6}yx^2 = \begin{pmatrix} 37 & 0 \\ 0 & 13 \end{pmatrix}$, $y^2 = \begin{pmatrix} 39 & 0 \\ 0 & 39 \end{pmatrix}$ and $y^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. All these matrices form a subgroup H_2 in $SL(2,\mathbb{Z}_{40})$ and $|H_2| = 16$. Then $|H_1H_2| = 640$ as $H_1 \cap H_2 = 1$. Now $H_1, H_2 \leq L$, so $H_1H_2 = L \leq H \leq SL(2,\mathbb{Z}_{40})$.

Let $k = x^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $K = \langle k\rangle$. Then $|K| = 40$. Clearly $L \cap K = 1$ and so $|LK| = 25600$. Now $LK \subseteq H \leq SL(2,Z_{40})$ and by Corollary 2.4, $|SL(2,\mathbb{Z}_{40})| = 46080$, so $|H| > \frac{1}{2}|SL(2,\mathbb{Z}_{40})|$, which implies that $H = SL(2,Z_{40})$.

Let $r = ca^{-2}bdc = \begin{pmatrix} 3 & 0 \\ 0 & 19 \end{pmatrix}$, $s = dc^{-1}ab(a^{-1}b)^2 = \begin{pmatrix} 13 & 0 \\ 0 & 33 \end{pmatrix}$, $t = a^{-1}baba^{-1}c^{-1}bc = \begin{pmatrix} 27 & 0 \\ 0 & 37 \end{pmatrix}$, $l = c^{-2}a^{-2}bd = \begin{pmatrix} 11 & 0 \\ 0 & 27 \end{pmatrix}$, $m = ba^{-1}d^{-1}bdbaba^2 = \begin{pmatrix} 19 & 0 \\ 0 & 19 \end{pmatrix}$ and $n = c^{-1}a^{-2}bdb = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}$. Now $o(r) = o(s) = o(t) = o(l) = 4$, whereas $o(m) = o(n) = 2$. So $P = \langle r, s, t, l, m, n\rangle$ has order 256. P consists of the diagonal matrices having diagonal elements in $\mathcal{U}(\mathbb{Z}_{40})$. Thus $|PH| = 737280$ as $P \cap H = H_2$. Now $PH \subseteq G \leq GL(2,\mathbb{Z}_{40})$ and by Corollary 2.2, $|GL(2,\mathbb{Z}_{40})| = 737280$. Thus $G = GL(2,\mathbb{Z}_{40})$. □

Lemma 2.26. *The elements* $a = \begin{pmatrix} 1 & 0 \\ 41 & 41 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 11 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 41 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units in* $M(2,\mathbb{Z}_{42})$.

Proof. The proof is clear once we observe that $a = eu + ue$, $b = e'b + be'$, $c = e'c + ce'$, $d = e'd + de'$, where $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are idempotents in $M(2,\mathbb{Z}_{42})$ and $u = \begin{pmatrix} 22 & 19 \\ 41 & 1 \end{pmatrix}$ is a unit in $M(2,\mathbb{Z}_{42})$. □

Theorem 2.27. *Presentation of* $GL(2,\mathbb{Z}_{42})$ *is*
$GL(2,Z_{42}) = \langle a,b,c,d \mid a^2, b^2, c^{12}, d^4, (bd)^2, ac^2ac^{10}, bc^2bc^{10}, dc^2d^3c^{10}, (ad^2)^2, (bd^2)^2, cd^2c^{11}d^2,$
$cabcacbacdb(adc)^3dc^3abcbadabc(da)^2b,\ (ac)^2(b(ad)^2)^2cacbc^{11}d^3a(cba)^2dac,\ dabc^9abdcab(ac)^2,$
$abc^{11}ad^3a(cb)^3cabadabd(ca)^2cd, c^5(ad)^{10}dcada, bcdbc^{11}d^3\rangle$,
where $a = \begin{pmatrix} 1 & 0 \\ 41 & 41 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 11 \\ 1 & 0 \end{pmatrix}$ *and* $d = \begin{pmatrix} 0 & 41 \\ 1 & 0 \end{pmatrix}$ *are Jordan regular units.*

Proof. Let G be a group having the above presentation. Since $a,b,c,d \in GL(2,\mathbb{Z}_{42})$, so $G \leq GL(2,\mathbb{Z}_{42})$. Let $x = dab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = d^{-1} = \begin{pmatrix} 0 & 1 \\ 41 & 0 \end{pmatrix}$. Then $x, y \in G$, $o(x) = 42$ and $o(y) = 4$. So $H = \langle x,y \rangle \leq SL(2,\mathbb{Z}_{42})$. Let L be the group consisting of all lower triangular matrices in $SL(2,\mathbb{Z}_{42})$. By Proposition 2.5, $|L| = 504$. Let $H_1 = \langle u \rangle$, where $u = xy^{-1}x = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then $|H_1| = 42$. Let $v = x^{-1}yx^2yx^{-2}yxyx^{-3}yx^3yx = \begin{pmatrix} 5 & 0 \\ 1 & 17 \end{pmatrix}$ and $w = yx^3yx^{-4}yx^3yx^{10} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}$. Now $o(v) = 6$, $o(w) = 2$ and $vw = wv$. Hence $H_2 = \langle v,w \rangle$ has 12 elements. Also $H_1 \cap H_2 = 1$, therefore $|H_1H_2| = 504$. Now $H_1, H_2 \leq L$ and so $H_1H_2 = L \leq H \leq SL(2,\mathbb{Z}_{42})$.

Let D be the derived subgroup of $SL(2,\mathbb{Z}_{42})$. Then $D = \langle l,m,n \rangle$, where $l = x^4y^{-1}x^{-2}yx^3yx^{-2} = \begin{pmatrix} 11 & 29 \\ 35 & 16 \end{pmatrix}$, $m = xyx^4yx^{-2}y = \begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix}$, $n = x^{-6}y^{-1}xyx^{-2}yx^6yx = \begin{pmatrix} 41 & 19 \\ 19 & 16 \end{pmatrix}$ and $|D| = 8064$. Since $l,m,n \in H$, so $D \leq H$. Also $L \cap D = \left\{ \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ i & 17 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ i & 23 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ i & 13 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ i & 5 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ i & 31 \end{pmatrix}, \begin{pmatrix} 23 & 0 \\ i & 11 \end{pmatrix}, \begin{pmatrix} 25 & 0 \\ i & 37 \end{pmatrix}, \begin{pmatrix} 29 & 0 \\ i & 29 \end{pmatrix}, \begin{pmatrix} 31 & 0 \\ i & 19 \end{pmatrix}, \begin{pmatrix} 37 & 0 \\ i & 25 \end{pmatrix}, \begin{pmatrix} 41 & 0 \\ i & 41 \end{pmatrix} \right\}$. Here $i \in \{0,6,12,18,24,30,36\}$.
Thus $|LD| = 48384$. Now $LD \subseteq H \leq SL(2,\mathbb{Z}_{42})$ and by Corollary 2.4, $|SL(2,\mathbb{Z}_{42})| = 48384$. Thus $H = SL(2,\mathbb{Z}_{42})$.

Let $p = cd^{-1}c^2 = \begin{pmatrix} 5 & 0 \\ 0 & 11 \end{pmatrix}$, $q = c^{-4}db = \begin{pmatrix} 17 & 0 \\ 0 & 25 \end{pmatrix}$, $r = cd^{-1}cdc^2 = \begin{pmatrix} 13 & 0 \\ 0 & 31 \end{pmatrix}$ and $s = c^{-2}db = \begin{pmatrix} 19 & 0 \\ 0 & 23 \end{pmatrix}$. Now $o(p) = o(q) = o(r) = o(s) = 6$. So $P = \langle p,q,r,s \rangle$ has order 144. P consists of the diagonal matrices having diagonal elements in $\mathcal{U}(\mathbb{Z}_{42})$. Also $P \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 17 \end{pmatrix}, \begin{pmatrix} 11 & 0 \\ 0 & 23 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 17 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 19 & 0 \\ 0 & 31 \end{pmatrix}, \right.$

$$\begin{pmatrix} 23 & 0 \\ 0 & 11 \end{pmatrix}, \begin{pmatrix} 25 & 0 \\ 0 & 37 \end{pmatrix}, \begin{pmatrix} 29 & 0 \\ 0 & 29 \end{pmatrix}, \begin{pmatrix} 31 & 0 \\ 0 & 19 \end{pmatrix}, \begin{pmatrix} 37 & 0 \\ 0 & 25 \end{pmatrix}, \begin{pmatrix} 41 & 0 \\ 0 & 41 \end{pmatrix} \Bigg\}.$$

Thus $|PH| = 580608$. Now $PH \subseteq G \leq GL(2, \mathbb{Z}_{42})$ and by Corollary 2.2, $|GL(2, \mathbb{Z}_{42})| = 580608$, which implies that $G = GL(2, \mathbb{Z}_{42})$. □

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