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GENERALIZED LOWER SETS OF TRANSITIVE BE-ALGEBRAS

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Abstract: The notion of generalized lower sets is introduced in transitive BE-algebras. Some properties of generalized lower sets are investigated in transitive BE-algebras. Furthermore, a sufficient condition is derived for every generalized lower set BE-algebra to become an ideal.

1 Introduction

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [6]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [5]. Some properties of filters of BE-algebras were studied by S.S. Ahn, Y.H. Kim and J.M. Ko in [2] and by B.L. Meng in [8]. In [11], A. Walendziak discussed some relationships between congruence relations and normal filters of a BE-algebra. In 2012, A. Rezaei and A. Borumand Saeid [9] stated and proved the first, second and third isomorphism theorems in self distributive BE-algebras. Later, these authors in [10] introduced the notion of commutative ideals in a BE-algebra. In 2013, A. Borumand Saeid, A. Rezaei and R.A. Borzooei [3] extensively studied the properties of some types of filters of BE-algebras. In [1], S.S. Ahn and K.S. So generalized the notion of upper sets in BE-algebras and discuss properties of the characterizations of generalized upper sets. In [7], H.S. Kim and K.J. Lee investigated several properties of upper and extended upper sets of BE-algebras.

In this paper, the concept of generalized lower sets is introduced in transitive BE-algebras as a dual of generalized upper sets. We discuss some significant properties of these generalized lower sets of transitive BE-algebras. It is observed that a generalized lower set of a transitive BE-algebra is not an ideal in general. However, a sufficient condition is derived for every generalized lower set to become an ideal. An equivalent condition is derived in terms of generalized lower sets for a subset of a transitive BE-algebra to become an ideal.

2 Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [2], [4], [6] and [8] for the ready reference of the reader.

Definition 2.1. [6] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,
(2) $x * 1 = 1$,
(3) $1 * x = x$,
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called transitive if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra X is called implicative if $(x * y) * x = x$ for all $x, y \in X$. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$.

We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive and symmetric. If X is commutative, then $\leq$ is anti-symmetric and hence a partial order on X.

Theorem 2.2. [8] *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ implies $x = 1$,

(2) $y \leq z$ implies $x * y \leq x * z$ and $z * x \leq y * x$.

Definition 2.3. [2] A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,

(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any $a \in X$, $\langle a \rangle = \{x \in X \mid a^n * x = 1$ for some $n \in \mathbb{N}\}$ is called the principal filter generated a. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. A BE-algebra X is called bounded [4], if there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$. Define an unary operation N on a bounded BE-algebra X by $xN = x * 0$ for all $x \in X$.

Theorem 2.4. [4] *Let X be a transitive BE-algebra and $x, y \in X$. Then*

(1) $1N = 0$ *and* $0N = 1$,

(2) $x \leq xNN$,

(3) $x * yN = y * xN$.

3 Generalized Lower Sets of Transitive BE-algebras

In this section, the notion of generalized lower sets is introduced in transitive BE-algebras. Properties of generalized lower sets are investigated. We state a condition for this generalized lower set to become an ideal. Using this generalized lower set, we establish an equivalent condition of an ideal. We prove that an ideal can be represented by the union of such special sets.

Lemma 3.1. *Let X be a transitive BE-algebra. For any $x, y, z \in X$, we have*

(1) $xNNN \leq xN$,

(2) $x * y \leq yN * xN$,

(3) $(x * yNN)NN \leq x * yNN$,

(4) $x \leq y$ *implies* $yN \leq xN$,

(5) $x \leq y$ *implies* $y * zN \leq x * zN$.

Proof. (1). Let $x \in X$. Then $1 = (x*0)*(x*0) = x*((x*0)*0) = x*xNN \leq x*xNNNN = xNNN * xN$. Hence $xNNN * xN = 1$, which gives $xNNN \leq xN$.
(2). Let $x, y \in X$. Since X is transitive, we get $yN = y*0 \leq (x*y)*(x*0) = (x*y)*xN$. Hence, $1 = yN*yN \leq yN*((x*y)*xN) = (x*y)*(yN*xN)$. Thus, we get $(x*y)*(yN*xN) = 1$. Therefore $x * y \leq yN * xN$.
(3). Let $x, y \in X$. Clearly $(x * yNN)N \leq (x * yNN)NNN$. Since X is transitive, we get $yN * (x * yNN)N \leq yN * (x * yNN)NNN$ and so $x * (yN * (x * yNN)N) \leq x * (yN * (x * yNN)NNN)$. Hence, we get

$$\begin{aligned}
1 &= (x * yNN) * (x * yNN) \\
&= x * ((x * yNN) * yNN) \\
&= x * (yN * (x * yNN)N) \\
&\leq x * (yN * (x * yNN)NNN) \\
&= x * ((x * yNN)NN * yNN) \\
&= (x * yNN)NN * (x * yNN).
\end{aligned}$$

Thus $(x * yNN)NN * (x * yNN) = 1$. Therefore $(x * yNN)NN \leq (x * yNN)$.

(4). Let $x, y \in X$ be such that $x \leq y$. Then by(2), we get $1 = x * y \leq yN * xN$. Hence, $yN * xN = 1$. Therefore $yN \leq xN$.
(5). Let $x, y \in X$ be such that $x \leq y$. Then by (4), we get $yN \leq xN$. Since X is transitive, we get $z * yN \leq z * xN$. Therefore $y * zN \leq x * zN$. □

Definition 3.2. A non-empty subset I of a BE-algebra X is called an *ideal* of X if it satisfies the following conditions for all $x, y \in X$:

(I1) $0 \in I$,
(I2) $x \in I$ and $(xN * yN)N \in I$ imply that $y \in I$.

Obviously the single-ton set $\{0\}$ is an ideal of a BE-algebra X. For, suppose $x \in \{0\}$ and $(xN * yN)N \in \{0\}$ for $x, y \in X$. Then $x = 0$ and $yNN = (0N * yN)N \in \{0\}$. Hence $y \leq yNN = 0 \in \{0\}$. Thus $\{0\}$ is an ideal of X. In the following example, we observe non-trivial ideals of a BE-algebra.

Example 3.3. Let $X = \{1, x, y, z, w, 0\}$. Define an operation $*$ on X as follows:

$*$	1	x	y	z	w	0
1	1	x	y	z	w	0
x	1	1	x	z	z	w
y	1	1	1	z	z	z
z	1	x	y	1	x	y
w	1	1	x	1	1	x
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. It can be easily verified that the set $I = \{0, z, w\}$ is an ideal of X. However, the set $J = \{0, x, y, w\}$ is not an ideal of X, because of $x \in J$ and $(xN * zN)N = (w * y)N = xN = w \in J$ but $z \notin J$.

Here after, by a BE-algebra X we mean a bounded BE-algebra $(X, *, 0, 1)$ unless and otherwise mentioned.

Proposition 3.4. *Let I be an ideal of a transitive BE-algebra X. Then we have:*

(1) *For any $x, y \in X, x \in I$ and $y \leq x$ imply $y \in I$,*
(2) *For any $x \in X, x \in I$ if and only if $xNN \in I$.*

Proof. (1). Let $x, y \in X$. Suppose $x \in I$ and $y \leq x$. By Lemma 3.1(4), we get $xN \leq yN$, which implies $xN * yN = 1$. Hence $(xN * yN)N = 0 \in I$. Since $x \in I$, we get $y \in I$.
(2). Let $x \in X$. Suppose $x \in I$. Then we get $(xN * xNNN)N = (xN * (xNN * 0))N = (xNN * (xN * 0))N = (xNN * xNN)N = 1N = 0 \in I$. Since $x \in I$, it yields $xNN \in I$. Conversely, let $xNN \in I$ for any $x \in X$. Since $x \leq xNN$, by property (1) we get that $x \in I$. □

For any elements a and b of a BE-algebra X and $n \in N$, we use the notation $a^n * b$ instead of $a * (a * (\cdots * (a * (b * x)) \cdots))$ in which a occurs n times.

Definition 3.5. Let X be a BE-algebra. For any $a, b \in X$ and $n \in \mathbb{N}$, we define

$$[a^n; b] = \{x \in X \mid \underbrace{aN * (aN * (\cdots * (aN}_{\text{n times}} * (bN * xN)) \cdots)) = 1\}.$$

in which a occurs n times. We call $[a^n; b]$ is a generalized lower set of a and b in the BE-algebra X.

If $n = 1$ then we have $[a; b] = \{x \in X \mid aN * (bN * xN) = 1\}$, also if $n = 2$ then we have $[a^2; b] = \{x \in X \mid aN * (aN * (bN * xN)) = 1\}$. Similarly we can write $[a^n; b]$ for any $n \in \mathbb{N}$. It is obvious that $0, a, b \in [a^n; b]$ for all $a, b \in X$ and $n \in \mathbb{N}$.

Example 3.6. The BE-algebra $(X, *, 0, 1)$ given in Example 3.3, some of the lower sets are $[x; y] = \{0, x, y, w\}, [x^2; y] = X, [x^2; z] = \{0, x, y, z, w\}$.

Lemma 3.7. *Let X be a transitive BE-algebra. For any $x, y \in X$ and $n \in \mathbb{N}$, we have*

(1) $x \in [a^n; b]$ *implies* $(yN * xN)N \in [a^n; b]$,
(2) $x \in [a^n; b]$ *and* $y \leq x$ *imply that* $y \in [a^n; b]$,
(3) $xN = yN$ *and* $x \in [a^n; b]$ *imply that* $y \in [a^n; b]$,
(4) $x \in [a^n; b]$ *if and only if* $xNN \in [a^n; b]$.

Proof. (1). Let $x \in [a^n; b]$. Then we get $(aN)^n * (bN * xN) = 1$ for some $n \in \mathbb{N}$. Since $xN \leq yN * xN$, we get $bN * xN \leq bN * (yN * xN) \leq bN * (yN * xN)NN$. Hence $1 = (aN)^n * (bN * xN) \leq (aN)^n * (bN * (yN * xN)NN)$.
Therefore $(aN)^n * (bN * (yN * xN)NN) = 1$. Hence $(yN * xN)N \in [a^n; b]$.
(2). Let $x \in [a^n; b]$ and $y \leq x$. Then $(aN)^n * (bN * xN) = 1$ for some $n \in \mathbb{N}$. Since $y \leq x$ then by Lemma 3.1(4), we get $xN \leq yN$. Hence $1 = (aN)^n * (bN * xN) \leq (aN)^n * (bN * yN)$. Therefore $y \in [a^n; b]$.
(3). Suppose $xN = yN$ and $x \in [a^n; b]$. Then we get $(aN)^n * (bN * xN) = 1$ for some $n \in \mathbb{N}$. Hence $(aN)^n * (bN * yN) = 1$. Therefore $y \in [a^n; b]$.
(4). Let $x \in [a^n; b]$. Then $(aN)^n * (bN * xN) = 1$ for some $n \in \mathbb{N}$. Since $xN \leq xNNN$, we get $1 = (aN)^n * (bN * xN) \leq (aN)^n * (bN * xNNN)$. Therefore $(aN)^n * (bN * xNNN) = 1$ which implies $xNN \in [a^n; b]$.
Conversely, let $xNN \in [a^n; b]$. Then $1 = (aN)^n * (bN * xNNN) \leq (aN)^n * (bN * xN)$. Therefore $(aN)^n * (bN * xN) = 1$. Hence $x \in [a^n; b]$. □

Proposition 3.8. *Let X be a transitive BE-algebra and $b \in X$ satisfy the equality $(bN*xN)N = 0$ for all $1 \neq x \in X$. Then $[a^n; b] = X = [b^n; a]$ for all $a \in X$ and $n \in \mathbb{N}$.*

Proof. Let $b \in X$ satisfy the equality $(bN * xN)N = 0$ for all $1 \neq x \in X$. Let $a \in X$ and $n \in \mathbb{N}$. For any $x \in X$, we get $1 = (aN)^n * 1 = (aN)^n * (bN * xN)NN \leq (aN)^n * (bN * xN)$. Therefore $(aN)^n * (bN * xN) = 1$. Hence $x \in [a^n; b]$. Thus $X \subseteq [a^n; b]$. Therefore $[a^n; b] = X$. Again, for any $x \in X$,

$$\begin{aligned} 1 &= (bN)^{n-1} * (aN * 1) \\ &= (bN)^{n-1} * (aN * (bN * xN)NN) \\ &\leq (bN)^{n-1} * (aN * (bN * xN)) \quad \text{(By Lemma 3.1(3))} \\ &= (bN)^{n-1} * (bN * (aN * xN)) \\ &= (bN)^n * (aN * xN) \end{aligned}$$

Therefore $(bN)^n * (aN * xN) = 1$, which implies $x \in [b^n; a]$. Thus $[b^n; a] = X$. Hence $[a^n; b] = X = [b^n; a]$ for all $a \in X$ and $n \in \mathbb{N}$. □

Example 3.9. Let $X = \{1, a, b, c, 0\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	0
1	1	a	b	c	0
a	1	1	1	1	c
b	1	c	1	1	b
c	1	b	c	1	a
0	1	1	1	1	1

If we observe here, for $c \in X$ and $(cN * xN)N = 0$ for all $1 \neq x \in X$. Also, $[c^2; a] = \{1, a, b, c, 0\} = [a^2; c]$ and $[c^2; b] = \{1, a, b, c, 0\} = [b^2; c]$. Therefore $[c^2; a] = X = [a^2; c]$ for all $a \in X$.
However for $b \in X, (bN * cN)N = (b * a)N = cN = a \neq 0$ and $[a^2; b] = [b^2; a] = \{a, b, c, 0\} \neq X$.

In general, a generalized lower set $[a^n; b]$ of a BE-algebra X is not an ideal of X. It can be seen in the following example:

Example 3.10. The BE-algebra $(X, *, 0, 1)$ given in Example 3.3, it can be easily verified that the lower set $[x; y] = \{0, x, y, w\}$ is not an ideal of the BE-algebra X, because of $x \in [x; y]$ and $(xN * zN)N = (w * y)N = xN = w \in [x; y]$ but $z \notin [x; y]$.

Though every generalized lower set $[a^n; b]$ of a BE-algebra X is need not be an ideal. In the following theorem, we derive a sufficient condition for every generalized lower set of a BE-algebra X to become an ideal.

Proposition 3.11. *If X is a self-distributive BE-algebra, then $[a^n; b]$ is an ideal for any $a, b \in X$ and $n \in \mathbb{N}$.*

Proof. Let $a, b \in X$ and $n \in N$. Clearly $0 \in [a^n; b]$. Let $x \in [a^n; b]$ and $(xN * yN)N \in [a^n; b]$. Then $(aN)^n * (bN * xN) = 1$ and $(aN)^n * (bN * ((xN * yN)NN)) = 1$. Hence

$$\begin{aligned} 1 &= (aN)^n * (bN * ((xN * yN)NN)) \\ &\leq (aN)^n * (bN * (xN * yN)) \quad \text{(By Lemma 3.1(3))} \\ &= (aN)^n * ((bN * xN) * (bN * yN)) \\ &= ((aN)^n * (bN * xN)) * ((aN)^n * (bN * yN)) \\ &= 1 * ((aN)^n * (bN * yN)) \\ &= (aN)^n * (bN * yN) \end{aligned}$$

Therefore $(aN)^n * (bN * yN) = 1$, which implies $y \in [a^n; b]$. Hence $[a^n; b]$ is an ideal of X. □

Theorem 3.12. *Let I be a non-empty subset of a transitive BE-algebra X. Then I is an ideal of X if and only if $[a^n; b] \subseteq I$ for every $a, b \in I$ and $n \in \mathbb{N}$.*

Proof. Assume that I is an ideal of X. Let $a, b \in I$. Suppose $x \in [a^n; b]$.Then $(aN)^n * (bN * xN) = 1$. Hence $((aN)^n * (bN * xN)NN)N \leq ((aN)^n * (bN * xN))N = 0 \in I$. By Proposition 3.4(1), we get$((aN)^n * (bN * xN)NN)N \in I$. Since $a \in I$ and I is an ideal, we get $(bN * xN)N \in I$. Again since $b \in I$ and I is an ideal, we get $x \in I$. Hence $[a^n; b] \subseteq I$.

Conversely, suppose that $[a^n; b] \subseteq I$ for all $a, b \in I$ and $n \in \mathbb{N}$. Obviously $0 \in [a^n; b] \subseteq I$. Let $x, y \in X$ be such that $x \in I$ and $(xN * yN)N \in I$. Then

$$\begin{aligned} 1 &= (xN)^{n-1} * 1 \\ &= (xN)^{n-1} * ((xN * yN) * (xN * yN)) \\ &= (xN)^{n-1} * (xN * ((xN * yN) * yN)) \\ &\leq (xN)^n * ((xN * yN)NN * yN). \quad \text{(By Lemma 3.1(5))} \end{aligned}$$

Therefore $(xN)^n * ((xN * yN)NN * yN) = 1$. Thus $y \in [x; (xN * yN)N] \subseteq I$. Therefore I is an ideal of X. □

Since every self-distributive BE-algebra is transitive, the following corollary is an easy consequence:

Corollary 3.13. *Let X be a self-distributive BE-algebra. Every non-empty subset I of X containing $[a^n; b]$ for all $a, b \in I$ and $n \in \mathbb{N}$ is an ideal of X.*

Definition 3.14. Let X be a bounded BE-algebra. A non-empty subset S of X is called a *bounded subalgebra*, if S is closed under the operations $*$ and N. In particular $(xN * yN)N \in S$ whenever $x, y \in S$.

Lemma 3.15. *Every ideal of a transitive BE-algebra is a bounded subalgebra.*

Proof. Suppose I be an ideal of a transitive BE-algebra X. Let $x, y \in I$. Since $y \in I$ then by Proposition 3.4(2), we get $yNN \in I$. Clearly $yN \leq xN * yN \leq (xN * yN)NN$. Hence $(xN * yN)N \leq (xN * yN)NNN \leq yNN \in I$. Then by Proposition 3.4(1), weget $(xN * yN)N \in I$. Therefore I is a bounded subalgebra of X. □

The following two propositions are direct consequences of Lemma 3.15 and Theorem 3.12.

Proposition 3.16. *Let X be a transitive BE-algebra. Every non-empty subset I of X containing $[a^n; b]$ for all $a, b \in I$ and $n \in \mathbb{N}$ is a bounded subalgebra of X.*

Proposition 3.17. *Let X be a self-distributive BE-algebra. Every non-empty subset I of X containing $[a^n; b]$ for all $a, b \in I$ and $n \in \mathbb{N}$ is a bounded subalgebra of X.*

Theorem 3.18. *Let X be a transitive BE-algebra and I be an ideal of X. Then $I = \bigcup_{a,b\in I} [a^n; b]$ for every $n \in \mathbb{N}$.*

Proof. Assume that I is an ideal of X. Let $a, b \in I$ and $n \in \mathbb{N}$. Then by Theorem 3.12, $[a^n; b] \subseteq I$. Hence $\bigcup_{a,b\in I} [a^n; b] \subseteq I$. Again, let $x \in I$. Since $x \in [1^n; x]$, it follows that $I \subseteq \bigcup_{x\in I} [1^n; x] \subseteq \bigcup_{a,b\in I} [a^n; b]$. Hence $I = \bigcup_{a,b\in I} [a^n; b]$ for every $n \in \mathbb{N}$. □

Corollary 3.19. *Let X be a self-distributive BE-algebra and I be an ideal of X. Then $I = \bigcup_{a,b\in I} [a^n; b]$ for every $n \in \mathbb{N}$.*

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